博士學位論文

Transversal Conformal fields and Killing forms on foliations

濟州大學校 大學院

數 學 科

金 祐 徹

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Transversal Conformal fields and Killing forms on foliations

Woo Cheol Kim

(Supervised by professor Seoung Dal Jung)

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Department of Mathematics

GRADUATE SCHOOL

JEJU NATIONAL UNIVERSITY

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1 Introduction

Let $(M, \mathcal{F})$ be a smooth manifold with a Riemannian foliation is a foliation $\mathcal{F}$ on a smooth manifold M such that the normal bundle $Q = TM/L$ may be endowed with a metric g_Q whose Lie derivative is zero along leaf directions ([23]). Note that we can choose a Riemannian metric g_M on M such that $g_M|_{T\mathcal{F}^{\perp}} = g_Q$; such a metric is called *bundle – like*. Denote by $(M, g_M, \mathcal{F})$. Recently, S. D. Jung and K. Richardson ([11]) proved the *generalized Obata theorem* which states that: $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where G is the discete subgroup of $O(q)$ acting by isometries on the last q coordinates of the sphere $S^q(1/c)$ of radius $1/c$ if and only if there exists a non-constant basic function f such that

$$\nabla_X \nabla f = -c^2 f X$$

for all foliated normal vectors X, where c is a positive real number and ∇ is the transverse Levi-Civita connection on the normal bundle Q.(See below)

Let R^{∇}, ρ^{∇} and σ^{∇} be the transversal curvature tensor, transverse Ricci operator and transversal scalar curvature with respect to the transversal Levi-Civita connection ∇ on Q ([23]). Let κ_B be the basic part of the mean curvature form of the foliation $\mathcal{F}$ and $\kappa_B^{\sharp}$ its dual vector field (see Section 2). Then we have the following well-known theorem.

Theorem A. ([11]) *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of a nonzero constant transversal scalar curvature σ^{∇}. If*

M admits a transversal nonisometric conformal field Y satisfying one of the following conditions:

(1) *$Y = \nabla h$ for any basic function h, or*

(2) *$\theta(Y)\rho^\nabla = \mu g_Q$ for some basic function μ, or*

(3) *$\rho^\nabla(\nabla f_Y) = \frac{\sigma^\nabla}{q}\nabla f_Y$, $g_Q(\kappa_B^\sharp, \nabla f_Y) = 0$ and $g_Q(A_{\kappa_B^\sharp}\nabla f_Y, \nabla f_Y) \le 0$,*

when $f_Y = \frac{1}{q}\mathrm{div}_\nabla Y$,

then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$.

Now, we recall two tensor fields E^∇ and Z^∇ ([5], [10]) by

$$E^\nabla(Y) = \rho^\nabla(Y) - \frac{\sigma^\nabla}{q}Y, \quad Y \in T\mathcal{F}^\perp, \tag{1.1}$$

$$Z^\nabla(X,Y) = R^\nabla(X,Y) - R_\sigma^\nabla(X,Y), \tag{1.2}$$

where $R_\sigma^\nabla(X,Y)s = \frac{\sigma^\nabla}{q(q-1)}\{g_Q(\pi(Y),s)\pi(X) - g_Q(\pi(X),s)\pi(Y)\}$ for any vector field, $X, Y \in TM$ and $s \in \Gamma Q$. Trivially, if $E^\nabla = 0$ (resp. $Z^\nabla = 0$), then the foliation is transversally Einsteinian (resp. transversally constant sectional curvature). The tensor Z^∇ is called as the transversal concircular curvature tensor, which is a generalization of the concircular curvature tensor on a Riemannian manifold. In an ordinary manifold, the concircular curvature tensor is invariant under a concircular transformation which is a conformal transformation preserving geodesic circles ([25]). Then we have the well-known theorem.

Theorem B. ([5]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of a nonzero constant transversal scalar curvature* σ^∇*. If* M *admits a transversal nonisometric conformal field* Y *such that*

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) \geq 0,$$

then $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$*.*

Theorem C. ([7, 10]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of a nonzero constant transversal scalar curvature* σ^∇*, and suppose that* $\mathcal{F}$ *is minimal. If* M *admits a transversal nonisometric conformal field* Y *such that*

$$(1)\theta(Y)|E^\nabla|^2 = 0, \quad ([7])$$

$$(2)\theta(Y)|Z^\nabla|^2 = 0. \quad ([10])$$

then $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$*.*

Namely, we extend Theorem C as follows: There are many results about the Riemannian foliations admitting a transversal nonisometric conformal field ([5], [7], [10], [11], [21]).

Main Theorem 1. *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of a nonzero constant transversal scalar curvature* σ^∇*, and suppose that* $\mathcal{F}$ *is minimal. If* M *admits a transversal nonisometric conformal field* Y

such that

$$\theta(Y)|E^{\nabla}|^2 = const. \quad \theta(Y)|Z^{\nabla}|^2 = const.$$

then $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$.

Also, we study a generalization of Theorem A (2) and (3) when $\mathcal{F}$ is minimal.

Main Theorem 2. *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of a nonzero constant transversal scalar curvature* σ^{∇}*, and suppose that* $\mathcal{F}$ *is minimal. If* M *admits a transversal nonisometric conformal field* Y *such that*

$$\theta(Y)g_Q(\theta(Y)E^{\nabla}, E^{\nabla}) \le 0,$$

then $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$.

Remark. See also ([26]) for the ordinary manifold.

Main Theorem 3. *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of a nonzero constant transversal scalar curvature* σ^{∇}*. If* M *admits a transversal conformal field* $\bar{Y}$ *such that* $Y = K + \nabla h$*, where* K *is a transversal Killing field and* h *is a basic function, then* $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$.

Remark. Main Theorem 3 is a generalization of Theorem A (1).

On the other hand, a transverse Killing fields and conformal fields are very important objects for studying mathematical and physical problems on foliated manifolds. As their generalizations, transverse Killing forms and conformal Killing forms were studied by many authors ([7], [24]). In 2012, S. D. Jung and K. Richardson ([13]) studied the parallelness of transverse Killing and conformal Killing forms on a compact manifold. Namely, we have the following.

Theorem D. ([13]) *Let $\mathcal{F}$ be a Riemannian foliation on a compact Riemannian manifold M. If the transversal curvature endomorphism is nonpositive, then any transverse conformal Killing r-form $(1 \le r \le q-1)$ is parallel, where $q = \mathrm{codim}\,\mathcal{F}$.*

When $(\mathcal{F}, J)$ is a transverse Kähler foliation, the parallelness of such forms was studied in ([6, 8]), as follows.

Theorem E. *Let $(\mathcal{F}, J)$ be a transverse Kähler foliation on a closed, connected Riemannian manifold. Then*

(1) *if the mean curvature vector is transversally holomorphic, then any transverse Killing r-form $(r \ge 2)$ is parallel* ([6])*;*

(2) *if the foliation is minimal, then for any transverse conformal Killing r-form ϕ $(2 \le r \le q-2)$, $J\phi$ is parallel* ([8])*;*

On a complete Riemannian foliation, the parallelness of L^2-transverse forms was studied in ([4]) and ([12]). Namely,

Theorem F. *Let $\mathcal{F}$ be a Riemannian foliation on a complete foliated Riemannian manifold M. Assume that all leaves are compact and the mean curvature is bounded. If* *the transversal curvature endomorphism is nonpositive, then*

(1) L^2*-transverse Killing forms are parallel* ([4])*;*

(2) L^2*-transverse conformal Killing forms are parallel* ([12])*;*

The parallelness of L^2-transverse conformal Killing forms on a transverse Kähler foliation was studied by S. D. Jung and H. Liu ([12]). That is,

Theorem G. ([12]) *Let* $(\mathcal{F}, J)$ *be a minimal transverse Kähler foliation on a complete Riemannian manifold with all leaves be compact. Then for any* L^2*-transverse conformal Killing* r*-form* ϕ $(2 \leq r \leq q-2)$*,* $J\phi$ *is parallel.*

Remark. Note that any transverse Killing form is a transverse conformal Killing form. Hence from Theorem G, for any L^2-transverse Killing ϕ, $J\phi$ is also parallel. But generally, the parallelness of $J\phi$ does not impty the parallelness of ϕ. Hence we study the parallelness of L^2-transverse Killing forms on a transverse Kähler foliation.

In Section 4, we study the parallelness and vanishing theorem of L^2-transverse Killing forms on a transverse Kähler foliation. That is,

Main Theorem 4. *Let* $(\mathcal{F}, J)$ *be a transverse Kähler foliation on a complete Riemannian manifold such that all leaves be compact. Assume that the mean curvature vector field is transversally holomorphic, coclosed and bounded. Then* L^2*-transverse Killing* r*-forms* $(r \geq 2)$ *are parallel. In addition, if* $vol(M)$ *is infinite, then* L^2*-transverse Killing* r*-forms* $(r \geq 2)$ *are trivial.*

2 Foliations

2.1 Definitions

Let M^{p+q} be a smooth manifold of dimensional $n = p + q$.

Definition 2.1 A family $\mathcal{F} \equiv \{l_\alpha\}_{\alpha \in A}$ of connected subsets of a manifold M^{p+q} is called a p-dimensional (or codimension q) $foliation$ if

(1) $M = \cup_\alpha l_\alpha$,

(2) $l_\alpha \cap l_\beta = \varnothing \quad$ for any $\alpha \neq \beta$,

(3) for any point $p \in M$, there exist a C^r-chart (φ_i, U_i), such that if $U_i \cap l_\alpha \neq \varnothing$, then the connected component of $U_i \cap l_\alpha$ is homeomorphic to A_c, where

$$A_c = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q | y = \text{constant}\}.$$

Here (φ_i, U_i) is called a distinguished (or $foliated$) chart.

Remark. From (3) in definition 2.1, we know that on $U_i \cap U_j \neq \varnothing$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ has the form

$$\varphi_j^{-1} \circ \varphi_i(x, y) = (\varphi_{ij}(x, y), \gamma_{ij}(y)), \tag{2.1}$$

where $\varphi_{ij} : \mathbb{R}^{p+q} \to \mathbb{R}^p$ is a differentiable map and $\gamma_{ij} : \mathbb{R}^q \to \mathbb{R}^q$ is a diffeomorphism.

Let $(M, \mathcal{F})$ be a smooth manifold of dimension $n = p + q$ endowed with a foliation $\mathcal{F}$ given by an integrable subbundle $L \subset TM$ of rank p. The set $\mathcal{F}$ is a partition of M

into immersed submanifolds (*leaves*) such that the transition functions for the local product neighborhoods (foliation charts) are smooth. The subbundle L is the tangent bundle to the foliation; at each $p \in M$, L_p is the tangent space to the leaf through p. We assume throughout the paper that the foliation is *Riemannian*; this means that there is a metric on the local space of leaves - a holonomy-invariant transverse metric g_Q on the normal bundle $Q = TM/L$. The phrase $holonomy-invariant$ means $\theta(X)g_Q = 0$, where $\theta(X)$ is the transversal Lie derivative for all leafwise vector fields $X \in \Gamma L$. This condition is characterized by the existence of a unique metric and torsion-free connection ∇ on Q ([1]).

We often assume that $(M, g_M, \mathcal{F})$ be a Riemannian manifold with the foliation $\mathcal{F}$ and a $bundle-like\ metric$ g_M ([22]), which is exactly a metric on the manifold such that the leaves of the foliation are locally equidistant.

Now, we consider an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0 \tag{2.2}$$

where $\sigma : Q \to L^\perp$ is a bundle map satisfying $\pi \circ \sigma = id$. Let g_Q be the holonomy invariant metric on Q induced by $g_M = g_L + g_{L^\perp}$; that is

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)) \quad \forall s, t \in \Gamma Q. \tag{2.3}$$

Let ∇ be the transverse Levi-Civita connection in Q, which is defined ([4]) by

$$
\nabla_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L \\ \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.4}
$$

where $s \in \Gamma Q$ and $Y_s = \sigma(s) \in \Gamma L^\perp$ correspinding to s under the canonical isomorphism $Q \cong L^\perp$ and ∇^M is the Levi-Civita connection on M. The curvature R^∇ of ∇ is defined by $R^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for $X, Y \in \Gamma TM$. Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$ ([16]), we can define the transversal Ricci operator $\rho^\nabla : \Gamma Q \to \Gamma Q$ by

$$
\rho^\nabla(s) = \sum_{a=1}^{q} R^\nabla(s, E_a)E_a, \tag{2.5}
$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basic frame of Q. And the transversal Ricci curvature Ric^∇ is given by $\mathrm{Ric}^\nabla(s_1, s_2) = g_Q(\rho^\nabla(s_1), s_2)$ for any $s_1, s_2 \in \Gamma Q$. The transversal scalar curvature σ^∇ is given by $\sigma^\nabla = \mathrm{Tr}\rho^\nabla$. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model spase is Einsteinian, that is,

$$
\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot \mathrm{id} \tag{2.6}
$$

with constant transversal scalar curvature σ^∇. The *mean curvature vector* τ of $\mathcal{F}$ is defined by

$$
\tau = \pi\left(\sum_{i=1}^{p} \nabla^M_{E_i} E_i\right), \tag{2.7}
$$

where $\{E_i\}$ is a local orthonomal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* if $\tau = 0$. A differential form $\omega \in \Omega^r(M)$ is *basic* if

$$
i(X)\omega = 0, \quad \theta(X)\omega = 0, \quad \forall X \in \Gamma L. \tag{2.8}
$$

Let $\Omega_B^r(\mathcal{F})$ be the set of all basic r-forms on M. The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$, where κ is a g_Q-dual 1-form of τ. D. Domminguez ([2]) proved that any Riemannian foliation is isoparametric for some bundle-like metric. It is well-known ([9]) that on an isoparametric Riemannian foliation $\mathcal{F}$, the mean curvature form κ is closed, i.e., $d\kappa = 0$. Let d_B be the restriction of d on $\Omega_B(\mathcal{F})$ and δ_B its formal adjoint operator of d_B with respect to the global inner product $\ll \cdot, \cdot \gg$, which is given by

$$\ll \phi, \psi \gg = \int_M \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}} \tag{2.9}$$

for any basic r-form ϕ and ψ, where $\chi_{\mathcal{F}}$ is characteristic form of $\mathcal{F}$ ([20]) and $\bar{*}$ is the transversal star operator on $\Omega_B^*(\mathcal{F})$, that is, $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ defined by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}), \quad \forall \phi \in \Omega_B^r(\mathcal{F}). \tag{2.10}$$

The operator δ_B is given by

$$\delta_B \phi = (-1)^{q(r+1)+1} \bar{*} d_B \bar{*} \phi + i(\kappa_B^\sharp)\phi$$

where $d_T = d - \epsilon(\kappa_B)$. Here $\epsilon(X^*) = X^* \wedge$ is the adjoint operator of $i(X)$, κ_B is the basic part of the mean curvature form κ and $(\cdot)^\sharp$ is a g_Q-dual vector to $(\cdot)$. By a direct calculation, we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,

$$\epsilon(X^*)\bar{*}\phi = (-1)^{r+1} \bar{*} i(X)\phi \tag{2.11}$$

for any vector field $X \in Q$.

Note that the induced connection ∇ on $\Omega_B^*(\mathcal{F})$ from the connection ∇ on Q and Riemannian connection ∇^M on M extends the partial Bott connection, which satisfies

$\nabla_X \phi = \theta(X)\phi$ for any $X \in \Gamma L$ ([8]). Locally, d_B ad δ_B are given by

$$d_B = \sum_{a=1}^{q} \theta^a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_{a=1}^{q} i(E_a)\nabla_{E_a} + i(\kappa_B^\sharp), \tag{2.12}$$

where θ^a is the dual basic 1-form of E_a

The *basic Laplacian* Δ_B acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B.$$

Then for any basic function f, we have

$$\Delta_B f = \delta_B d_B f = -\sum_a \nabla_{E_a}\nabla_{E_a} f + \kappa_B^\sharp(f). \tag{2.13}$$

Now, we recall the generalized maximum principle for foliation ([11]).

Theorem 2.2 ([11]) *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M. For any basic function f, the condition $(\Delta_B - \kappa_B^\sharp)f \geq 0$ implies that f is constant.*

In ([8]), for any bundle-like metric with $\kappa \in \Omega_B^1(\mathcal{F})$, it is proved that there exists another bundle-like metric for which the mean curvature form is basic-harmonic. That is, $\Delta_B \kappa = 0$.

2.2 Infinitesimal automorphisms

Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying $[Y, Z] \in \Gamma L$ for all $Z \in \Gamma L$. An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$ ([12]). Let

$$\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) | Y \in V(\mathcal{F})\}. \tag{2.14}$$

It is trivial that an element s of $\bar{V}(\mathcal{F})$ satisfies $\nabla_X s = 0$ for all $X \in \Gamma L$. Hence the metric defined by (2.3) induces an identification ([17])

$$\bar{V}(\mathcal{F}) \cong \Omega_B^1(\mathcal{F}). \tag{2.15}$$

For the later use, we recall the transversal divergence theorem ([6]) and the tautness theorem ([1, 18]) on a foliated Riemannian manifold.

Theorem 2.3 (Transversal divergence theorem) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a transversally oriented foliation $\mathcal{F}$ and a bundle-like metric g_M. Then*

$$\int_M \operatorname{div}_\nabla \bar{X} = \int_M g_Q(\bar{X}, \tau) \tag{2.16}$$

for all $X \in V(\mathcal{F})$, where $\operatorname{div}_\nabla X$ denotes the transversal divergence of X with respect to the connection ∇ defined by (2.4).

Theorem 2.4 (Tautness theorem) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. If the transversal Ricci operator ρ^∇ is positive definite, then $\mathcal{F}$ is taut, i.e., there exists a bundle-like metric $\bar{g}_M$ for which all leaves are minimal submanifolds.*

We define an operator $A_Y : \Gamma Q \to \Gamma Q$ for any vector field $Y \in V(\mathcal{F})$ by

$$A_Y s = \theta(Y)s - \nabla_Y s. \tag{2.17}$$

Then it is proved ([13]) that, for any vector field $Y \in V(\mathcal{F})$,

$$A_Y s = -\nabla_{Y_s} \bar{Y} \tag{2.18}$$

where $Y_s = \sigma(s) \in \Gamma TM$. So A_Y depends only on $\bar{Y} = \pi(Y)$ and is a linear operator. Moreover, A_Y extends in an obvious way to tensors of any type on Q (see [13] for details). In particular, for any basic 1-form $\phi \in \Omega_B^1(\mathcal{F})$, the operator A_Y is given by

$$(A_Y \phi)(s) = -\phi(A_Y s) \quad \forall s \in \Gamma Q. \tag{2.19}$$

Now, we introduce the operator $\nabla_{tr}^* \nabla_{tr} : \Omega_B^*(\mathcal{F}) \to \Omega_B^*(\mathcal{F})$ as

$$\nabla_{tr}^* \nabla_{tr} \phi = -\sum_a \nabla^2_{E_a, E_a} \phi + \nabla_\tau \phi, \tag{2.20}$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in TM$. It is well-known that $\nabla_{tr}^* \nabla_{tr}$ is a non-negative and formally self-adjoint operator ([3]). Then we have the following generalized Weitzenböck formula.

Theorem 2.5 ([3]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then the generalized Weizenböck formula is given by the following : for any basic form $\phi \in \Omega_B^r(\mathcal{F})$*

$$\triangle_B \phi = \nabla_{tr}^* \nabla_{tr} \phi + F(\phi) + A_\tau \phi, \tag{2.21}$$

where $F(\phi) = \sum_{a,b} E^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi$*. In particular, if* ϕ *is a basic 1-form, then* $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$*. Here for any basic form* ϕ*,* $\phi^\sharp$ *means that* $\phi(X) = g_Q(\phi^\sharp, \pi(X))$ *for all* $X \in TM$*.*

Corollary 2.6 *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* $q \geq 2$ *and a bundle-like metric* g_M*. Then, for any basic r-form* ϕ*,*

$$-\frac{1}{2} \triangle_B |\phi|^2 = -g_Q(\triangle_B \phi, \phi) + |\nabla_{tr}\phi|^2 + g_Q(F(\phi), \phi) + g_Q(A_\tau \phi, \phi), \tag{2.22}$$

where $|\nabla_{tr}\phi|^2 = \sum_a g_Q(\nabla_{E_a}\phi, \nabla_{E_a}\phi)$*.*

For any vector filed $X \in V(\mathcal{F})$, if we put $\triangle_B \bar{X} = (\triangle_B \phi)^\sharp$, where $\phi^\sharp = \bar{X}$, then we have the following corollary.

Corollary 2.7 *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* $q \geq 2$ *and a bundle-like metric* g_M*. Then, for any vector field* $X \in V(\mathcal{F})$*,*

$$\triangle_B \bar{X} = \nabla_{tr}^* \nabla_{tr} \bar{X} + \rho^\nabla(\bar{X}) - A_\tau^t \bar{X}, \tag{2.23}$$

where A^t *is an adjoint operator of* A*.*

2.3 Transversal Killing and conformal fields

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M such that $\kappa \in \Omega_B^1(\mathcal{F})$. If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 0$, then Y is called a *transversal Killingfield* of $\mathcal{F}$. If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 2f_Y g_Q$ for a basic function f_Y depending on Y, then Y is called a *transversal conformal field* of $\mathcal{F}$; in this case, we have

$$f_Y = \frac{1}{q}\mathrm{div}_\nabla Y. \tag{2.24}$$

Now we put, for any vector field $Y \in V(\mathcal{F})$,

$$\square_B Y = \triangle_B Y - 2\rho^\nabla(Y). \tag{2.25}$$

Trivially, $\square_B$ is a formally self-adjoint operator.

For any vector field $Y \in V(\mathcal{F})$, let $B_Y^\mu : \Gamma Q \to \Gamma Q (\mu \in \mathbb{R})$ be given by

$$B_Y^\mu := A_Y + A_Y^t + \mu \cdot \mathrm{div}_\nabla Y \cdot \mathrm{id}. \tag{2.26}$$

It is well-known ([13,19]) that Y is transversal conformal (resp. transversal Killing) if and only if

$$B_Y^{2/q} = 0 \quad (\mathrm{resp.} B_Y^0 = 0). \tag{2.27}$$

Theorem 2.8 ([10]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation of codimension q and a bundle-like metric g_M. Then, for any vector field $Y \in V(\mathcal{F})$, the following holds:*

(1) *Y is a transversal Killing field if and only if*

$$(i)\ \Box_B Y + A_\tau^t Y + A_Y \tau = 0, \quad \mathrm{div}_\nabla Y = 0,$$

$$(ii)\ \int_M g_Q(B_Y^0 Y, \tau) \geq 0.$$

(2) *Y is a transversal conformal field if and only if*

$$(i)\ \Box_B Y + A_\tau^t \bar{Y} + A_Y \tau = \left(1 - \frac{2}{q}\right) d_B \mathrm{div}_\nabla Y,$$

$$(ii)\ \int_M g_Q(B_Y^{2/q} Y, \tau) \geq 0.$$

Theorem 2.9 (cf. [19]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation of codimension q and a bundle-like metric g_M and $Y \in V(\mathcal{F})$. If a transversal conformal field Y satisfies*

$$(i)\ \int_M g_Q(B_Y^0 Y, \kappa^\sharp) \geq 0 \quad \textit{and} \quad (ii)\ d_B \mathrm{div}_\nabla Y = 0,$$

then Y is the transversal Killing field.

Lemma 2.10 ([7]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. If Y is a transversal conformal field, i.e., $\theta(Y) g_Q = 2 f_Y g_Q$, then we have*

$$g_Q((\theta(Y)\nabla)(E_a, E_b), E_c) = \delta_b^c f_a + \delta_a^c f_b - \delta_a^c f_c, \tag{2.28}$$

$$(\theta(Y) R^\nabla)(E_a, E_b) E_c = (\nabla_a \theta(Y)\nabla)(E_b, E_c) - (\nabla_b \theta(Y)\nabla)(E_a, E_c), \tag{2.29}$$

$$g_Q((\theta(Y) R^\nabla)(E_a, E_b) E_c, E_d) = \delta_b^d \nabla_a f_c - \delta_b^c \nabla_a f_d - \delta_a^d \nabla_b f_d + \delta_a^c \nabla_b f_d, \tag{2.30}$$

$$(\theta(Y) Ric^\nabla)(E_a, E_b) = -(q-2)\nabla_a f_b + \delta_a^b (\triangle_B f_Y - \kappa^\sharp(f_Y)), \tag{2.31}$$

where $\nabla_a = \nabla_{E_a}$ *and* $f_a = \nabla_a f_Y$.

From (2.31), we have the following lemma.

Lemma 2.11 ([10]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented Riemannian manifold with a foliation of codimension* q *and a bundle-like metric* g_M*, if* Y *is the transversal conformal field, i.e.,* $\theta(Y)g_Q = 2f_Y g_Q$*, then*

$$\theta(Y)\sigma^\nabla = 2(q-1)(\triangle_B f_Y - \kappa^\sharp(f_Y)) - 2f_Y\sigma^\nabla. \tag{2.32}$$

Proposition 2.12 ([10]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* q *and a bundle-like metric* g_M *such that* $\kappa \in \Omega^1_B(\mathcal{F})$ *and* $\delta_B\kappa = 0$*. Assume that the transversal scalar curvature is constant. If* Y *is a transversal conformal field with* $\theta(Y)g_Q = 2f_Y g_Q$*,* $f_Y \neq 0$*, then*

$$\triangle_B f_Y = \frac{\sigma^\nabla}{q-1} f_Y + \kappa^\sharp(f_Y) \tag{2.33}$$

and consequently if $\sigma^\nabla \neq 0$*, then*

$$\int_M f_Y = -\frac{q-1}{\sigma^\nabla}\int_M \kappa^\sharp(f_Y) = 0. \tag{2.34}$$

3 Riemannian foliations admitting transversal conformal fields

Precisely, see [14] for this chapter.

3.1 Generalized Lichnerowicz-Obata theorem

Let $(M, g_M, \mathcal{F})$ be a connected and oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M such that $\kappa \in \Omega^1_B(\mathcal{F})$.

Definition 3.1 *Let G be a discrete group. Then $(M, \mathcal{F})$ is transversally isometric to (N, G), the isometric action of G on a Riemannian manifold N if there exists a smooth, sujective map $\phi : M \to N$ such that :*

(1) *The function ϕ induces a homeomorphism between the leaf space $M/\mathcal{F}$ and the orbit space N/G.*

(2) *For each $x \in M$, the push forward ϕ_* restricts to an isometry $\phi_* : Q_x \to T_{\phi(x)}N$, where Q is the normal bundle of the foliation and TN is the tangent bundle of N.*

Now, we recall the generalized Obata Theorem for foliations which was proved by Lee and Richardson ([17]).

Theorem 3.2 (Generalized Obata theorem, ([11])) *Let $(M, g_M, \mathcal{F})$ be a connected, complete Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M, and let c be a positive real number. Then the following are equivalent:*

(1) *There exists a non-constant basic funtion f such that $\nabla_X \nabla f = -c^2 f X$ for all vectors $X \in T\mathcal{F}^{\perp}$.*

(2) *$(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where the discrete subgroup G of the ortogonal group $O(q)$ acts by isometries on the last q coodinates of the q-sphere $S^q(1/c)$ of radius $1/c$ in Euclidean space $\mathbb{R}^{q+1}$.*

Now, we recall well-known facts for characterizing the Riemannian foliation admitting the transversal conformal field.

Theorem 3.3 ([7]) *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like mertic g_M and $\rho^\nabla(X) \geq \frac{\sigma^\nabla}{q} X (\sigma^\nabla \neq 0)$ for any $X \in \Gamma Q$. If M admits a transversal conformal field $\bar{Y} \in \Gamma Q$ such that $\theta(Y) g_Q = 2 f_Y g_Q (f_Y \neq 0)$, then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, with $c = \frac{\sigma^\nabla}{q(q-1)}$.*

Theorem 3.4 ([11]) *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of a nonzero constant transversal scalar curvature σ^∇. If M admits a transversal nonisometric conformal field Y satisfying one of the following conditions:*

(1) *$Y = \nabla h$ for any basic function h,*

(2) *$\theta(Y) \rho^\nabla = \mu g_Q$ for some basic function μ,*

(3) *$\rho^\nabla(\nabla f_Y) = \frac{\sigma^\nabla}{q} \nabla f_Y$, $g_Q(\kappa_B^\sharp, \nabla f_Y) = 0$ and $g_Q(A_{\kappa_B^\sharp} \nabla f_Y, \nabla f_Y) \leq 0$,*

then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$.

Now, we prove the generalization of Theorem 3.4 (1).

Theorem 3.5 *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of a nonzero constant transversal scalar curvature σ^∇. If M admits a transversal conformal field Y such that $Y = K + \nabla h$, where K is a transversal Killing field and h is a basic function, then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$.*

Proof. Let Y be a transversal conformal field such that $\theta(Y)g_Q = 2f_Y g_Q$ and $Y = K + \nabla h$, where K is a transversal Killing field and h is a basic function. Then

$$g_Q(\nabla_X Y, Z) + g_Q(\nabla_Z Y, X) = 2f_Y g_Q(X, Z)$$

for any normal vector field $X, Z \in \Gamma Q$. On the other hand, since the transversal scalar curvature σ^∇ is constant, from Proposition 2.12, we have

$$(\triangle_B - \kappa_B^\sharp) f_Y = \frac{\sigma^\nabla}{q-1} f_Y. \tag{3.1}$$

Since $Y = K + \nabla h$, we have $\theta(Y)g_Q = \theta(\nabla h)g_Q = 2f_Y g_Q$. That is,

$$g_Q(\nabla_X \nabla h, Z) + g_Q(\nabla_Z \nabla h, X) = 2f_Y g_Q(X, Z). \tag{3.2}$$

On the other hand, $(\nabla\nabla h)(X, Z) = g_Q(\nabla_X \nabla h, Z)$ is symmetric. Therefore, from (3.2)

$$(\nabla\nabla h)(X, Z) = f_Y g_Q(X, Z). \tag{3.3}$$

Hence from (2.12) and (3.3), we have

$$(\triangle_B - \kappa_B^\sharp) h = -q f_Y. \tag{3.4}$$

From (3.1) and (3.4), we get

$$(\triangle_B - \kappa_B^\sharp)\left(f_Y + \frac{\sigma^\nabla}{q(q-1)}h\right) = 0.$$

By the generalized maximum principle (Theorem 2.2), we have

$$f_Y + \frac{\sigma^\nabla}{q(q-1)}h = const,$$

which implies

$$\nabla\nabla f_Y + \frac{\sigma^\nabla}{q(q-1)}\nabla\nabla h = 0. \tag{3.5}$$

from (3.3) and (3.5), we have

$$\nabla_X \nabla f_Y = -\frac{\sigma^\nabla}{q(q-1)} f_Y X.$$

for any $X \in \Gamma Q$.

By the generalized Obata theorem(Theorem 3.2), (M, $\mathcal{F}$) is transversally isometric to $(S^q(1/c), G)$ with $c^2 = \frac{\sigma^\nabla}{q(q-1)}$. □

3.2 Transversal Einstein tensor

In this section, we characterize the Riemannian foliation by the Einstein tensor. First,we define the tensor $E^\nabla : \Gamma Q \to \Gamma Q$ by

$$E^\nabla(Y) = \rho^\nabla(Y) - \frac{\sigma^\nabla}{q}Y, \qquad Y \in \Gamma Q,$$

which is called the *transversal Einstein tensor* of $\mathcal{F}$. Trivially, if $E^\nabla = 0$, then $\mathcal{F}$ is transversal Einsteinian.

Let $\{E_a\}$ be a local orthonormal basic frame on Q.

Lemma 3.6 ([10]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. If Y is a transversal conformal field, i.e., $\theta(Y)g_Q = 2f_Y g_Q$ Then*

$$(i) \quad tr_Q E^\nabla = 0, \quad |E^\nabla|^2 = |\rho^\nabla|^2 - \frac{(\sigma^\nabla)^2}{q}, \tag{3.6}$$

$$(ii) \quad \mathrm{div}_\nabla E^\nabla = \frac{q-2}{2q} \nabla\sigma^\nabla, \tag{3.7}$$

$$(iii) \quad (\theta(Y)E^\nabla)(E_a, E_b) = -(q-2)\left\{\nabla_a f_b + \frac{1}{q}(\triangle_B f - \kappa_B^\sharp(f))\delta_a^b\right\},$$

where $tr_Q E^\nabla = \sum_{a=1}^q g_Q(E^\nabla(E_a), E_a)$ and $E^\nabla(X,Y) = g_Q(E^\nabla(X), Y)$ for all vector foeld $X, Y \in \Gamma Q$. If σ^∇ is constant , then $\mathrm{div}_\nabla E^\nabla = 0$.

Theorem 3.7 ([5]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M such that $\delta_B\kappa_B = 0$. Assume that the transversal scalar curvature σ^∇ is non-zero constant. If M admits a transversal nonisometric conformal field Y, i.e., $\theta(Y)g_Q = 2f_Y g_Q$ $(f_Y \neq 0)$, such that*

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) \geq 0,$$

then $\mathcal{F}$ is transversally isometric to $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

For our main theorem, we prepare some lemmas.

Lemma 3.8 *If a transversal conformal field Y satisfies $\theta(Y)E^\nabla = \mu g_Q$ for some basic function μ, then*

$$\theta(Y)E^\nabla = 0.$$

Proof. Let Y be the transversal conformal field such that $\theta(Y)g_Q = 2f_Y g_Q$. From (2.31), we have

$$-(q-2)\nabla_a f_b + (\Delta_B f_Y - \kappa_B^\sharp(f_Y))\delta_a^b = \mu\delta_a^b. \tag{3.8}$$

From (2.12) and (3.8), we have

$$\mu = \frac{2(q-1)}{q}(\Delta_B f_Y - \kappa_B^\sharp(f_Y)). \tag{3.9}$$

From (3.8) and (3.9), we have

$$-(q-2)\Big\{\nabla_a f_b + \frac{1}{q}(\Delta_B f_Y - \kappa_B^\sharp(f_Y))\delta_a^b\Big\} = 0. \tag{3.10}$$

Therefore, the proof follows from Lemma 3.6 (iii). □

Lemma 3.9 *If Y is a transversal conformal field, then*

$$\theta(Y)|E^\nabla|^2 = 2g_Q(\theta(Y)E^\nabla, E^\nabla).$$

Proof. Let $\{E_a\}$ be a local orthonormal basic frame on Q such that $(\nabla E_a)_x = 0$ at a point x. Let Y be the transversal conformal field such that $\theta(Y)g_Q = 2f_Y g_Q$. Then at x, we have

$$\begin{aligned}\theta(Y)|E^Q|^2 &= \sum_a \theta(Y)g_Q(E^\nabla(E_a), E^\nabla(E_a))\\ &= \sum_a(\theta(Y)g_Q)(E^\nabla(E_a), E^\nabla(e_a)) + 2\sum_a g_Q((\theta(Y)E^\nabla)(E_a), E^\nabla(E_a))\\ &\quad + 2\sum_a g_Q(E^\nabla(\theta(Y)E_a), E^\nabla(E_a))\\ &= 2f_Y|E^\nabla|^2 + 2g_Q(\theta(Y)E^\nabla, E^\nabla) + 2\sum_a g_Q(E^\nabla(\theta(Y)E_a), E^\nabla(E_a)). \end{aligned} \tag{3.11}$$

Now, we calculate the last term in the above equation. That is,

$$\begin{aligned}
&\sum_a g_Q(E^\nabla(\theta(Y)E_a), E^\nabla(E_a)) \\
&\quad = \sum_{a,b} g_Q(E^\nabla(\theta(Y)E_a), E_b) g_Q(E^\nabla(E_a), E_b) \\
&\quad = \sum_{a,b} g_Q(E^\nabla(E_b), \theta(Y)E_a) g_Q(E^\nabla(E_b), E_a) \\
&\quad = \frac{1}{2}\sum_{a,b} \theta(Y)\{g_Q(E^\nabla(E_b), E_a) g_Q(E^\nabla(E_b), E_a)\} - 2f_Y|E^\nabla|^2 \\
&\qquad - \sum_a g_Q((\theta(Y)E^\nabla)(E_a), E^\nabla(E_a)) - \sum_a g_Q(E^\nabla(\theta(Y)E_a), E^\nabla(E_a)).
\end{aligned}$$

Hence we have

$$\begin{aligned}
2\sum_a g_Q(E^\nabla(\theta(Y)E_a), E^\nabla(E_a)) &= \frac{1}{2}\theta(Y)|E^\nabla|^2 - 2f_Y|E^\nabla|^2 \\
&\quad - g_Q(\theta(Y)E^\nabla, E^\nabla). \qquad (3.12)
\end{aligned}$$

From (3.11) and (3.12), the proof is completed. □

Proposition 3.10 *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Assume that the transversal scalar curvature is constant. If Y is a transversal nonisometric conformal field with $\theta(Y)g_Q = 2f_Y g_Q$ $(f_Y \neq 0)$, then*

$$\begin{aligned}
2(q-2)\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) &= \int_M \{4f_Y^2|E^\nabla|^2 + f_Y\theta(Y)|E^\nabla|^2\} \qquad (3.13) \\
&\quad + 2(q-2)\int_M g_Q(E^\nabla(f_Y\nabla f_Y), \kappa_B^\sharp).
\end{aligned}$$

Theorem 3.11 ([5]) *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of a nonzero constant transversal scalar curvature σ^∇, and* *suppose that $\mathcal{F}$ is minimal. If M admits a transversal nonisometric conformal field Y such that*

$$\theta(Y)|E^{\nabla}|^2 = 0,$$

then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$.

From the above lemmas, we prove our main theorem.

Theorem 3.12 *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of a nonzero constant transversal scalar curvature σ^{∇}, and suppose that $\mathcal{F}$ is minimal. If M admits a transversal nonisometric conformal field Y such that*

$$\theta(Y)|E^{\nabla}|^2 = const,$$

then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$.

Proof. Let Y be the transversal nonisometric conformal field such that $\theta(Y)g_Q = 2f_Y g_Q$ ($f_Y \neq 0$). From Proposition 2.12 , we have

$$\int_M f_Y = 0. \tag{3.14}$$

Assume that $\mathcal{F}$ is minimal. Since $\theta(Y)|E^{\nabla}|^2 = const$, from (3.14) and Proposition 3.10, we have

$$2(q-2)\int_M g_Q(E^{\nabla}(\nabla f_Y), \nabla f_Y) = 4\int_M f_Y^2 |E^Q|^2 \geq 0$$

Hence from Theorem 3.7, the proof is completed. □

Lemma 3.13 *Let Y be a transversal conformal field such that $\theta(Y)g_Q = 2f_Y g_Q$. Then for any basic function h,*

$$\int_M hf_Y = -\frac{1}{q}\int_M \theta(Y)h + \frac{1}{q}\int_M \mathrm{div}_\nabla(hY).$$

Proof. Let $\omega = Y^b$ be the dual basic 1-form of the transversal conformal form Y. Then

$$\int_M h(\delta_B\omega) = \int_M g_Q(\omega, d_B h) = \int_M i(Y)d_B h = \int_M \theta(Y)h.$$

Since $\delta_B = \delta_T + i(\tau_B)$ and $\delta_T\omega = -\mathrm{div}_\nabla(Y) = -qf_Y$, we have

$$\begin{aligned} q\int_M hf_Y &= -\int_M h(\delta_T\omega) \\ &= -\int_M h(\delta_B\omega) + \int_M hi(\tau_B)\omega) \\ &= -\int_M \theta(Y)h + \int_M g_Q(hY, \tau_B) \\ &= -\int_M \theta(Y)h + \int_M \mathrm{div}_\nabla(hY). \end{aligned}$$

Last equality in above follows from the transversal divergence theorem (Theorem 2.3).□

From Lemma 3.13, we prove the followy theorem.

Theorem 3.14 *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of a nonzero constant transversal scalar curvature σ^∇, and suppose that $\mathcal{F}$ is minimal. If M admits a transversal nonisometric conformal field Y such that*

$$\theta(Y)g_Q(\theta(Y)E^\nabla, E^\nabla) \le 0,$$

then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$.

Proof. Let Y be a transversal nonisometric conformal field, i.e., $\theta(Y)g_Q = 2f_Y g_Q (f_Y \neq 0)$. From (2.13) and Proposition 3.10, if we put $h = g_Q(\theta(Y)E^\nabla, E^\nabla)$, then from Lemma 3.13, we have

$$\begin{aligned}(q-2)&\int_M g_Q(E(\nabla f_Y), \nabla f_Y)\\ &= 2\int_M f_Y^2|E^\nabla|^2 + \int_M hf_Y + (q-2)\int_M g_Q(E(f_Y\nabla f_Y), \kappa_B^\sharp)\\ &= 2\int_M f_Y^2|E^\nabla|^2 - \frac{1}{q}\int_M \theta(Y)h + \frac{1}{q}\int_M g_Q(hY, \kappa_B^\sharp)\\ &\quad + (q-2)\int_M g_Q(E^\nabla(f_Y\nabla f_Y), \kappa_B^\sharp).\end{aligned}$$

Since $\mathcal{F}$ is minimal, we have

$$(q-2)\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) = 2\int_M f_Y^2|E^\nabla|^2 - \frac{1}{q}\int_M \theta(Y)g_Q(\theta(Y)E^\nabla, E^\nabla).$$

Hence by the condition $\theta(Y)g_Q(\theta(Y)E^\nabla, E^\nabla) \leq 0$, we have

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) \geq 0.$$

From Theorem 3.7, the proof of Theorem 3.14 is completed. □

Corollary 3.15 *Theorem 3.14 is a generalization of Theorem 3.4 (3).*

Proof. Assume that $E^\nabla(\nabla f_Y) = \frac{\sigma^Q}{q}\nabla f_Y$, that is, $E^\nabla(\nabla f_Y) = 0$. By differentiation, we have

$$(\nabla_{e_a}E^\nabla)(\nabla f_Y) + E^\nabla(\nabla_a \nabla f_Y) = 0. \tag{3.15}$$

from (3.15), we have

$$0 = \sum_a g_Q((\nabla_{e_a} E^\nabla)(\nabla f_Y) + E^\nabla(\nabla_a \nabla f_Y), e_a)$$
$$= g_Q(\nabla f_Y, \mathrm{div}_\nabla(E^\nabla)) + \sum_a g_Q(E^\nabla(\nabla_a \nabla f_Y), e_a)$$
$$= \sum_a g_Q(\nabla_a \nabla f_Y, E^\nabla(e_a)). \tag{3.16}$$

From (3.6), (3.7), $\mathrm{div}_\nabla E^\nabla = 0$ and so the last equality in the above follows. Hence from Lemma 3.9 (iii) and (3.16), we have

$$g_Q(\theta(Y)E^\nabla, E^\nabla) = \sum_a g_Q((\theta(Y)E^\nabla)(e_a), E^\nabla(e_a))$$
$$= -(q-2)\sum_a g_Q(\nabla_a \nabla f_Y, E^\nabla(e_a))$$
$$- \frac{q-2}{q}(\triangle_B f_Y)\sum_a g_Q(e_a, E^\nabla(e_a))$$
$$= -(q-2)\sum_a g_Q(\nabla_a \nabla f_Y, E^\nabla(e_a)) - \frac{q-2}{q}(\triangle_B f_Y)\mathrm{tr}_Q E^\nabla$$
$$= 0.$$

The last equality follows from $\mathrm{tr}_Q E^\nabla = 0$. Hence the conditions of Theorem 3.4 (3) implies that $g_Q(\theta(Y)E^\nabla, E^\nabla) = 0$. That is, by Theorem 3.14, $(M, \mathcal{F})$ is transversally isometric to the sphere. □

3.3 Transversal concircular curvature tensor

In this section, we study the Riemannian foliation by the some condition of the transversal concircular curvature tensor Z^∇.

Definition 3.16 *The transversal concircular curvature tensor* Z^∇ *of* $\mathcal{F}$ *is define by*

$$Z^\nabla(X,Y) = R^\nabla(X,Y) - R^\nabla_\sigma(X,Y), \tag{3.17}$$

where

$$R^\nabla_\sigma(X,Y)s = \frac{\sigma^\nabla}{q(q-1)}\{g_Q(\pi(Y),s)\pi(X) - g_Q(\pi(X),s)\pi(Y)\}$$

for any $X,Y \in TM$ *and* $s \in \Gamma Q$.

Trivially, if $Z^\nabla = 0$, then $\mathcal{F}$ is a foliation of transversally constant sectional curvature. In an ordinary manifold, the concircular curvature tensor is invariant under a concirclar transformation which is a conformal transformation preserving geodesic circles ([25]).

It is well-known that for any $s \in \Gamma Q$,

$$\sum_a Z^\nabla(\sigma(s), E_a)E_a = E^\nabla(s). \tag{3.18}$$

Also, we have ([10])

$$|Z^\nabla|^2 = |R^\nabla|^2 - \frac{2(\sigma^\nabla)^2}{q(q-1)}. \tag{3.19}$$

Proposition 3.17 ([10]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* $q \geq 2$ *and a bundle-like metric* g_M. *Assume that* *the transversal scalar curvature* σ^{∇} *is constant. If* Y *is a transversal nonisometric conformal field with* $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$, *then*

$$\int_M g_Q(E^{\nabla}(\nabla f_Y), \nabla f_Y) = \frac{1}{2}\int_M \{{f_Y}^2|Z^{\nabla}|^2 + \frac{1}{4}f_Y\theta(Y)|Z^{\nabla}|^2\} + \int_M g_Q(\rho^{\nabla}(f_Y\nabla f_Y), \kappa_B^{\sharp}).$$

From Proposition 3.17 and Theorem 3.7, we have

Theorem 3.18 ([10]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of a nonzero constant transversal scalar curvature* σ^{∇}*, and suppose that* $\mathcal{F}$ *is minimal. If* M *admits a transversal nonisometric conformal field* Y *such that*

$$\theta(Y)|Z^{\nabla}|^2 = 0,$$

then $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$.

Now, we extend theorem 3.18. Namely, we have the following.

Theorem 3.19 *Let* $(M, g_M, \mathcal{F})$ *be a closed, connected Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of a nonzero constant transversal scalar curvature* σ^{∇}*, and suppose that* $\mathcal{F}$ *is minimal. If* M *admits a transversal nonisometric conformal field* Y *such that*

$$\theta(Y)|Z^{\nabla}|^2 = const.,$$

then $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$.

Proof. Let Y be the transversal nonisometric conformal field such that $\theta(X)g_Q = 2f_Y g_Q (f_Y \neq 0)$. From Proposition 2.12, we have

$$\int_M f_Y = 0. \tag{3.20}$$

Assume that $\mathcal{F}$ is minimal. Since $\theta(Y)|Z^\nabla|^2 = const$, from (3.20) and Proposition 3.24, we have

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) = \frac{1}{2}\int_M f_Y^2 |Z^\nabla|^2.$$

Hence from Theorem 3.7, the proof is completed. □

Corollary 3.20 *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a minimal foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Assume that the transversal scalar curvature is nonzero constant and either $\theta(Y)|\rho|^2$ or $\theta(Y)|R|^2$ is constant. If M admits a transversal nonisometric conformal field, then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$.*

Proof. The proof followe from (3.6), (3.19), Theorem 3.11 and Theorem 3.18. □

4 L^2-transverse Killing form

Precisely, see [15] for this chapter.

4.1 Basic facts

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M.

Definition 4.1 *A basic r-form $\phi \in \Omega_B^r(\mathcal{F})$ is called a transverse conformal Killing r-form if for any vector field $X \in \Gamma Q$,*

$$\nabla_X \phi = \frac{1}{r+1} i(X) d_B \phi - \frac{1}{q-r+1} X^* \wedge \delta_T \phi, \tag{4.1}$$

where X^* is the g_Q-dual 1-form to X ans $\delta_T = \delta_B - i(\kappa_B^\sharp)$. In addition, a basic r-form ϕ satisfying (4.1) with $\delta_T \phi = 0$ is called a transverse Killing r-form. That is, a transverse Killing r-form ϕ satisfies

$$\nabla_X \phi = \frac{1}{r+1} i(X) d_B \phi. \tag{4.2}$$

Trivially, a basic form ϕ is a transverse Killing form if and only if

$$i(X) \nabla_X \phi = 0 \tag{4.3}$$

for any vector field $X \in \Gamma Q$.

Proposition 4.2 *On a Riemannian foliation of codimension q, we have that*

(1) *ϕ is a transverse conformal Killing r-form if and only if $\bar{*}\phi$ is a transverse conformal Killing $(q-r)$-form.*

(2) *ϕ is a transverse Killing r-form if and only if $\bar{*}\phi$ is a closed transverse conformal Killing $(q-r)$-form.*

Proof. (1) is proved ([13]). For the proof of (2), let ϕ be a transverse Killing r-form. Since $[\nabla, \bar{*}] = 0$, from (2.12) and (4.2), we have

$$\nabla_X(\bar{*}\phi) = (-1)^r \frac{1}{r+1}\epsilon(X^*)\bar{*}d_B\phi. \tag{4.4}$$

On the other hand, from (2.11) we have

$$\delta_T\bar{*}\phi = (-1)^{r^2-1}\bar{*}d_B\phi. \tag{4.5}$$

Hence from (4.4) and (4.5), we have

$$\nabla_X(\bar{*}\phi) = -\frac{1}{r+1}\epsilon(X^*)\delta_T\bar{*}\phi. \tag{4.6}$$

Since $\delta_T\phi = 0$, $d\bar{*}\phi = 0$, i.e., $\bar{*}\phi$ is closed. Hence from (4.1), $\bar{*}\phi$ is a closed transverse conformal Killing $(q-r)$-form, which completes the proof of (2). □

Remark. A transverse conformal Killing 1-form (resp. transverse Killing 1-form) is a g_Q-dual 1-form of a transverse conformal Killing vector field (resp. transverse Killing vector field).

Proposition 4.3 ([13]) *On a Riemannian foliation, a basic r-form ϕ is a transverse Killing form if and only if*

$$\Delta_B\phi = \frac{r+1}{r}F(\phi) + \theta(\kappa_B^\sharp)\phi, \tag{4.7}$$

or

$$F(\phi) = \frac{r}{r+1}\delta_T d_B\phi. \tag{4.8}$$

Then we obtained the parallelness of the transverse conformal Killing form on compact manifold.

Theorem 4.4 ([8]) *Let $\mathcal{F}$ be a Riemannian foliation on a compact Riemannian manifold M. If the transversal curvature endomorphism is nonpositive, then any transverse conformal Killing r-form $(1 \le r \le q-1)$ is parallel, where $q = \mathrm{codim}\mathcal{F}$.*

Definition 4.5 *A basic form ϕ is said to be a L^2-basic form if $\phi \in L^2\Omega_B^*(\mathcal{F})$, i.e., $\|\phi\|_B^2 < \infty$.*

We recall the generalized maximum principle on a complete foliated Riemannian manifold, that is, a Riemannian foliation with a complete bundle-like metric.

Theorem 4.6 ([8]) *(Generalized maximum principle) Let $\mathcal{F}$ be a Riemannian foliation on a complete foliated Riemannian manifold with all leaves be compact. Assume that κ_B is coclosed and bounded. Then a nonnegative basic function f such that $(\Delta_B - \kappa_B^\sharp) f \le 0$ with $\int_M f^p < \infty$ (for some $p > 1$) is constant.*

By using the generalized maximum principle, we proved the parallelness of the L^2-transverse on complete manifolds.

Theorem 4.7 ([4, 12]) *Let $\mathcal{F}$ be a Riemannian foliation on a complete foliated Riemannian manifold M. Assume that all leaves are compact and the mean curvature is bounded. If the transversal curvature endomorphism is nonpositive, then*

(1) *all L^2-transverse Killing forms are parallel* ([4])*;*

(2) *all L^2-transverse conformal Killing forms are parallel* ([12]);

4.2 Transverse conformal Killing form on Kähler foliation

In this section, we study the parallelness of the transverse conformal Killing form on a transverse Kähler foliation.

Let $(\mathcal{F}, J)$ be a transverse Kähler foliation of codimension $q = 2m$ on a Riemannian manifold (M^{p+q}, g_M). That is, the normal bundle $Q = TM/L$ admits a holonomy invariant almost complex structure $J : Q \to Q$ such that $\nabla J = 0$ and $g_Q(JX, JY) = g_Q(X, Y)$ for any $X, Y \in Q$ ([16]). Note that for any $X, Y \in \Gamma Q$, the 2-form ω defined by

$$\omega(X, Y) = g_Q(X, JY) \tag{4.9}$$

is a basic 2-form, which is closed as consequence of $\nabla_{g_Q} = 0$ and $\nabla J = 0$. Let $\{E_a\}(a = 1, \cdots, q)$ be a local orthonormal basic frame on Q and $\{\theta^a\}$ be its dual basic. Then

$$\omega = -\frac{1}{2}\sum_{a=1}^{2m} \theta^a \wedge J\theta^a. \tag{4.10}$$

Definition 4.8 *Let $\Lambda : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-2}(\mathcal{F})$ and $\widetilde{J} : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$, which are given by*

$$\Lambda(\phi) = i(\omega)\phi, \quad \widetilde{J}(\phi) = \sum_{a=1}^{2m} J\theta^a \wedge i(E_a)\phi, \tag{4.11}$$

respectively, where $i(\phi_1 \wedge \phi_2) = i(\phi_2^\sharp)i(\phi_1^\sharp)$ and $(J\phi)(X) = -\phi(JX)$ for any basic 1-forms $\phi_i (i = 1, 2)$ and ϕ. Trivially, if $\phi \in \Omega_B^1(\mathcal{F})$, then $\widetilde{J}\phi = J\phi$. From now on, we write $\widetilde{J} = J$ unless any confusion.

Then we have the following lemma.

Lemma 4.9 ([6]) *On a transverse Kähler foliation* $(\mathcal{F}, J)$*, we have*

$$[F, \Lambda] = 0, \quad [\delta_B, \Lambda] = 0, \quad [d_B, \Lambda] = -\delta_T^c, \tag{4.12}$$

$$[\Delta_B, \Lambda] = \delta_T i(J\kappa_B^\sharp) + i(J\kappa_B^\sharp)\delta_T, \tag{4.13}$$

$$[A_{\kappa_B^\sharp}, \Lambda] = [\theta(\kappa_B^\sharp), \Lambda] = -(\delta_T^c i(\kappa_B^\sharp) + i(\kappa_B^\sharp)\delta_T^c), \tag{4.14}$$

where $\delta_T^c = -\sum_{a=1}^{2m} i(JE_a)\nabla_{E_a}$*. If* $\kappa_B^\sharp$ *is transversally holomorphic, i.e.,* $\theta(\kappa_B^\sharp)J = 0$*, then*

$$\delta_T i(J\kappa_B^\sharp) + i(J\kappa_B^\sharp)\delta_T = -(\delta_T^c i(\kappa_B^\sharp) + i(\kappa_B^\sharp)\delta_T^c),$$

which means

$$[\Delta_B, \Lambda] = [\theta(\kappa_B^\sharp), \Lambda]. \tag{4.15}$$

Proposition 4.10 ([8]) *On a transverse Kähler foliation, if* ϕ *is a transverse Killing* r*-form* $(r \geq 2)$*, then* $\Lambda\phi$ *is a transverse Killing* $(r-2)$*-form.*

Now, we give some facts about a transverse Killing form on transverse Kähler foliation.

Lemma 4.11 *Let* $(\mathcal{F}, J)$ *be a transverse Kähler foliation and* $\kappa_B^\sharp$ *be transversally holomorphic. Then any transverse Killing* r*-form* ϕ *satisfies*

$$F(\Lambda\phi) = 0, \quad \Delta_B\Lambda\phi = \theta(\kappa_B^\sharp)(\Lambda\phi).$$

Proof. Let ϕ be a transverse Killing r-form. Since $\Lambda\phi$ is also a transverse Killing $(r-2)$-form, from (4.7)

$$\Delta_B\Lambda\phi = \frac{r-1}{r-2}F(\Lambda\phi) + \theta(\kappa_B^\sharp)(\Lambda\phi). \tag{4.16}$$

On the other hand, since $\kappa_B^\sharp$ is transversally holomorphic, from (4.15) and (4.7) we have

$$\Delta_B \Lambda\phi = \frac{r+1}{r} F(\Lambda\phi) + \theta(\kappa_B^\sharp)(\Lambda\phi). \tag{4.17}$$

From (4.16) and (4.17), if $r > 2$, then

$$F(\Lambda\phi) = 0. \tag{4.18}$$

For $r = 2$, it is trivial that $F(\Lambda\phi) = 0$. therefore, the proof of the second formula follows from (4.16). □

Proposition 4.12 *Let $(\mathcal{F}, J)$ be a transverse Kähler foliation on a closed Riemannian manifold. Assume that $\kappa_B^\sharp$ is transversally holomorphic. Then for any transverse Killing r-form ϕ $(r \ge 2)$, $\Lambda\phi$ is parallel.*

Proof. Let ϕ be a transverse Killing r-form $(r \ge 2)$. From Lemma 4.11 and (2.21), we have

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp)|\Lambda\phi|^2 = -|\nabla_{tr}\Lambda\phi|^2. \tag{4.19}$$

That is, $(\Delta_B - \kappa_B)|\Lambda\phi|^2 \le 0$. Hence by the generalized maximum principle ([11]), $|\Lambda\phi|$ is constant. Therefore, from (4.19) again, $\Lambda\phi$ is parallel. □

Now we define the operators $R_\pm^\nabla(X) : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r\pm1}(\mathcal{F})$ for any $X \in TM$ by

$$R_+^\nabla(X)\phi = \sum_{a=1}^{2n} \theta^a \wedge R^\nabla(X, E_a)\phi, \tag{4.20}$$

$$R_-^\nabla(X)\phi = \sum_{a=1}^{2n} i(E_a) R^\nabla(X, E_a)\phi. \tag{4.21}$$

Then we have the following lemmas directly.

Lemma 4.13 *On a transverse Kähler foliation* $(\mathcal{F}, J)$, *we have*

$$[R_+^\nabla(X), \Lambda] = R_-^\nabla(JX)$$

for any vector field $X \in Q$.

Proof. This is easy by using (4.20) and (4.21) □

Lemma 4.14 ([8]) *On a transverse Kähler foliation, if* ϕ *is a transverse Killing* r-*form, then*

$$\nabla_{X,Y}^2 \phi = \frac{1}{r} i(Y) R_+^\nabla(X)\phi$$

for any $X, Y \in TM$.

Proposition 4.15 *Let* $(\mathcal{F}, J)$ *be a transverse Kähler foliation on a closed Riemannian manifold. Assume that* $\kappa_B^\sharp$ *is transversally holomorphic. Then for any transverse Killing* r-*form* $(r \geq 2)$,

$$F(\phi) = 0.$$

Proof. Let ϕ be a transverse killing r-form $(r \geq 2)$. Since $\Lambda\phi$ is parallel, from Lemma 4.13 we have that

$$R_-^\nabla(JX)\phi = -\Lambda R_+^\nabla(X)\phi. \tag{4.22}$$

On the other hand, from Lemma 4.14 we have

$$\begin{aligned} i(Y)\Lambda R_+^\nabla(X)\phi &= \Lambda i(Y) R_+^\nabla(X)\phi \\ &= r\Lambda\nabla_{X,Y}^2\phi = r\nabla_{X,Y}^2\Lambda\phi = 0. \end{aligned}$$

Since Y is arbitrary, we have

$$\Lambda R_+^{\nabla}(X)\phi = 0. \tag{4.23}$$

Hence from (4.22) we get

$$R_-^{\nabla}(JX)\phi = 0,$$

Since $X \in Q$ is also arbitrary, we have

$$R_-^{\nabla}(X)\phi = 0, \tag{4.24}$$

which implied $F(\phi) = -\sum_a \theta^a \wedge R_-^{\nabla}(E_a)\phi = 0$. So the proof follows. □

Theorem 4.16 ([8]) *Let $(\mathcal{F}, J)$ be a transverse Kähler foliation on a closed Riemannian manifold. Assume that $\kappa_B^{\sharp}$ is transversally holomorphic.. Then any transverse Killing r-form ϕ $(r \geq 2)$ is parallel.*

4.3 L^2-transverse Killing form

Lemma 4.17 *Let $(\mathcal{F}, J)$ be a transverse Kähler foliation on a complete foliated Riemannian manifold. Then for any L^2-basic form ϕ, $\Lambda\phi$ is also a L^2-basic form.*

Proof. Let ϕ be a basic form. Then for any vector field $X \in Q$,

$$|i(X)\phi|^2 + |X^* \wedge \phi|^2 = |X|^2|\phi|^2.$$

If we choose $X = E_a$, then $|i(E_a)|^2 + |\theta^a \wedge \phi|^2 = |\phi|^2$. Hence

$$|i(E_a)\phi|^2 \leq |\phi|^2.$$

Hence if $\phi \in L^2\Omega_B^*(\mathcal{F})$, then $i(E_a)\phi \in L^2\Omega_B^*(\mathcal{F})$ for any $a = 1, \ldots, q$. And so $i(JE_a)i(E_a) \in L^2\Omega_B^*(\mathcal{F})$, which implies that $\sum_a i(JE_a)i(E_a)\phi \in L^2\Omega_B^*(\mathcal{F})$, that is, $\Lambda\phi$ is L^2. □

Proposition 4.18 *Let $(\mathcal{F}, J)$ be a transverse Kähler foliation on a complete foliated Riemannian manifold and all leaves be compact. Assume that κ_B is transversally holomorphic, coclosed and bounded. Then for any L^2-transverse Killing r-form ϕ $(r \geq 2)$, $\Lambda\phi$ is a parallel L^2-transverse Killing $(r-2)$-form.*

Proof. Let ϕ be a L^2-transverse Killing r-form. From Proposition 4.10 and Lemma 4.11, the generalized Weitzenböck formula (2.21) implies

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp)|\Lambda\phi|^2 = -|\nabla_{tr}\Lambda\phi|^2 \leq 0. \tag{4.25}$$

Since $\Lambda\phi$ is a L^2-form (Lemma 4.17), by the generalized maximum principle (Theorem 4.6), Eq. (4.25) implies that $|\Lambda\phi|$ is constant. So from (4.25) again, $\nabla_{tr}\Lambda\phi = 0$, i.e., $\Lambda\phi$ is parallel. □

Proposition 4.19 *Let $(\mathcal{F}, J)$ be a transverse Kähler foliation on a complete foliated Riemannian manifold and all leaves be compact. Assume that κ_B is transversally holomorphic, coclosed and bounded. Then for a L^2-transverse Killing r-form ϕ $(r \geq 2)$,*

$$F(\phi) = 0. \tag{4.26}$$

Proof. Let ϕ be a L^2-transverse Killing r-form. Since $\Lambda\phi$ is a parallel L^2-transverse Killing form (Proposition 4.18), the proof is the same to one of Proposition 4.15. □

Theorem 4.20 *Let $(\mathcal{F}, J)$ be a transverse Kähler foliation on a complete Riemannian manifold with all leaves be compact. Assume that the mean curvature vector field is transversally holomorphic, coclosed and bounded. Then all L^2-transverse Killing r-forms*

$(r \geq 2)$ are parallel. In addition, if $vol(M)$ is infinite, then all L^2-transverse Killing r-forms $(r \geq 2)$ are trivial.

Proof. Let ϕ be a L^2-transverse Killing r-form. From Proposition 4.3 and Proposition 4.19, we have

$$\Delta_B \phi = \theta(\kappa_B^\sharp)\phi.$$

Hence by the generalized Weitzenböck formula (2.21), we have

$$(\Delta_B - \kappa_B^\sharp)|\phi|^2 = -|\nabla_{tr}\phi|^2 \leq 0. \tag{4.27}$$

Therefore, by Theorem 4.6, $|\phi|$ is constant, which implies that $\nabla_{tr}\phi = 0$. That is, ϕ is parallel. which proves the first statement in the Theorem 4.20.

If $vol(M)$ is infinite, it is trivial that $\int_M |\phi|^2 < \infty$ implies $|\phi| = 0$. Hence, the prove is complete. □

Corollary 4.21 *Let $(\mathcal{F}, J)$ be a minimal transverse Kähler foliation on a complete Riemannian manifold with all leaves be compact. Then L^2-transverse Killing r-forms $(r \geq 2)$ are parallel. In addition, if $vol(M)$ is infinite, then all L^2-transverse Killing r-forms $(r \geq 2)$ are trivial.*

Corollary 4.22 *On a complete Kähler manifold, all L^2-Killing r-forms $(r \geq 2)$ are parallel.*

References

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. 10 (1992), 179-194.

[2] D. Dominguez, *A tenseness theorem for Riemannian foliations*, C. R. Acad. Sci., Ser. I 320 (1995), 1331-1335.

[3] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. 39 (2001), 253-264.

[4] S. D. Jung, *Transverse Killing forms on complete foliated Riemannian manifolds*, Honam Math. J. 36 (2014), 731-737.

[5] S. D. Jung, *Riemannian foliations admitting transversal conformal field II*, Geom Dedicata. 175 (2015), 257-266.

[6] S. D. Jung, *Transverse conformal Killing forms on Kähler foliations*, J. Geom. Phys. 90 (2015), 29-41.

[7] S. D. Jung, M. J. Jung, *Riemannian foliations admitting transversal conformal fields*, Geom Dedicata. 133 (2008), 155-168.

[8] S. D. Jung and M. J. Jung, *Transverse Killing forms on a Kähler foliation*, Bull. Korean Math. Soc. 49 (2012), 445-454.

[9] M. J. Jung and S. D. Jung, *Liouville type theorem for transversally harmonic map and biharmonic maps*, J. Korean Math. Soc. 54 (2017), 763-772.

[10] S. D. Jung and K. R. Lee, *The properties of riemannian foliations admitting transversal conformal fields*, Bull.Korean Math. Soc. 55 (2018), 1273-1283.

[11] S. D. Jung, K. R. Lee and K. Richardson, *Generalized Obata theorem and its applications on foliations*, J. Math. Anal. Appl. 376 (2011), 129-135.

[12] S. D. Jung and H. Liu, *L^2-transverse conformal Killing forms on complete foliated manifolds*, Math. Z. 288 (2018),665-677.

[13] S. D. Jung and K. Richardson, *Transverse conformal Killing forms and a Gallot-Meyer theorem for foliations*, Math. Z. 270 (2012), 337-350.

[14] W. C. Kim and S. D. Jung, *On Riemannian foliations admitting transversal conformal fields*, Hiroshima Math. J. 50 (2020), 59-72.

[15] W. C. Kim and S. D. Jung, *L^2-transverse Killing forms on a transverse Kähler foliation*, J. Geom. 111 (2020), 1-12.

[16] F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, in Harmonic maps (New Orleans, La., 1980), 87-121, Lecture Notes in Math., 949, Springer, Berlin, 1982

[17] J. Lee and K. Richardson, *Lichnerowicz and Obata theorems for foliations.* Pacific J. Math. 206 (2002), 339-357.

[18] P. March, M. Min-Oo and E.A. Ruh, *Mean curvature of Riemannian foliations.*, Can. Math. Bull. 39 (1996), 95-105.

[19] M. Min-Oo, M, E.A. Ruh and Ph. Tondeur, *Transversal curvature and tautness for Riemannian foliations* , Lecture Notes in Mathematics, 1481 (1991), 145-146.

[20] H.K. Pak and J.H. Park, *Transversal harmonic transformations for Riemannian foliations* , Ann. Global Anal. Geom. 30 (2006), 97-105.

[21] H. K. Pak and J. H. Park., *A note on generalized Lichnerowica-Obata theorem for Riemannian foliations*, Bull. Korean Math. Soc. 48 (2011), 769-777.

[22] B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. 69 (1959), 119-132.

[23] Ph. Tondeur, *Geometry of foliations*, Birkhäuser-Verlag, Basel; Boston: Berlin, (1997).

[24] Ph. Tondeur and G. Toth, *On transversal infinitesimal automorphisms for harmonic foliations* , Geometriae Dedicata. 24 (1987), 229-236.

[25] K. Yano, *Concircular geometry I, Concircular transformations*, Proc. Imp. Acad. Tokyo 16 (1940), 195-200.

[26] K. Yano, *On Riemannian manifolds with constant scalar curvature admitting a conformal transformation group*, Proc. Nat. Acad. Sci. U.S. 55 (1966), 472-476.