A Thesis for the Degree of M. E.

A Note on the Covariant Differentiating of the Nonholonomic Components V_n

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A Note on the Covariant Differentiating of the Nonholonomic Components in V_n

이를 教育學碩士學位 論文으로 提出함

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감 사 의 글

그동안 많은 지도와 격려를 해주신 현진오교수님께 무한한 감사를 드리며, 지도와 편달을 아끼지 않으신 수학과 여러 교수님과 동료들에게 감사를 드립니다.

그리고 항상 사랑으로 보살펴주시는 부모님과 가족들에게도 감사를 드립니다.

1984년 6월 일

고 한 진

CONTENTS

1. INTRODUCTION

Let $a_{\lambda\mu}$ be the symmetric covariant tensor whose determinant

(1.1) $$a \overset{\text{def}}{=} \det(\!(a_{\lambda\mu})\!) \neq 0$$

and let $\{ \underset{i}{e}^{\lambda} \mid (i = 1, 2, \ldots, n) \}$ be a set of n linearly independent vector in n-dimensional Riemannian space V_n refered to a real coordinate system X^{ν}.

Then there is a unique reciprocal set of n-linearly independent covariant vector $\overset{i}{e}_{\lambda}$ $(i = 1, 2, \ldots, n)$ satisfying

(1.2) a $$\underset{i}{e}^{\nu} \overset{i}{e}_{\lambda} = \delta^{\nu}_{\lambda},\ ^{(**)}$$

(1.2) b $$\underset{j}{e}^{\lambda} \overset{i}{e}_{\lambda} = \delta^{i}_{j}.$$

It is well-known that the quantities defined

(1.3) a $$a^{\lambda\nu} \overset{\text{def}}{=} \frac{\text{cofector of } a_{\lambda\nu} \text{ in } a}{a}$$ satisfying

(1.3) b $$a_{\lambda\mu} a^{\lambda\nu} = \delta^{\nu}_{\mu}.$$

DEFINITION 1.1. If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

(1.4) $$T^{i\cdots}_{j\cdots} \overset{\text{def}}{=} T^{\nu\cdots}_{\lambda\cdots} \overset{i}{e}_{\nu} \underset{j}{e}^{\lambda} \cdots .$$

In this paper, for our further discussion, previous results will be introduced without proof.

(**) Throughout the present paper, Greek indices take the values 1,2,...,n unless explicity stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic components of a tensor and run from 1 to n.
Roman indices also follow the summation convention.

2. PRELIMINARY RESULTS

THEOREM 2.1. The holonomic christoffel symbols of the first and second kinds are used to denote the function;

(2.1) $$[\omega, \lambda\mu] = a_{\omega\nu} \left\{ {\nu \atop \lambda\mu} \right\}.$$

THEOREM 2.2. The covariant differentiation of holonomic symmetric tensor with respect to x^{μ}. that is,

(2.2) $$\partial_{\mu} a^{\lambda\nu} = - a^{\sigma\nu} a^{\lambda\omega} ([\omega, \sigma\mu] + [\sigma, \omega\mu]).$$

THEOREM 2.3. The holonomic component of the christoffel symbol of the second kind may be expressed as follows;

(2.3) $$\left\{ {\nu \atop \lambda\mu} \right\} = - \overset{j}{e}_{\lambda} \overset{\kappa}{e}_{\mu} (\nabla_{\kappa} \underset{j}{e}^{\nu})$$

where ∇_{κ} is the symbol of the covariant derivative with respect to $\left\{ {i \atop jk} \right\}$.

THEOREM 2.4. The holonomic components of the christoffel symbols may be expressed as follows;

(2.4) $$[\omega, \lambda\mu] = [m, j\kappa] \overset{j}{e}_{\lambda} \overset{\kappa}{e}_{\mu} \overset{m}{e}_{\omega}.$$

THEOREM 2.5. The derivative of $\underset{i}{e}^{\lambda}$ is a negative self-adjoint. That is,

(2.5) $$\partial_{\kappa} (\overset{i}{e}_{\lambda}) \underset{j}{e}^{\mu} = - \partial_{\kappa} (\underset{j}{e}^{\mu}) \overset{i}{e}_{\lambda}.$$

3. MAIN RESULTS

In this section, we will be reconstructed some well-known results with refined way as applications of the nonholonomic frames.

THEOREM 3.1. We have

$$[\omega, \lambda\mu] = a_{mj} \, (\nabla_\kappa \overset{j}{e}_\lambda) \, \overset{m}{e}_\omega \, \overset{\kappa}{e}_\mu . \tag{3.1}$$

PROOF. Using (1.2)b, (1.4), (2.1), (2.3), (2.5),

$$\begin{aligned}
[\omega, \lambda\mu] &= a_{\nu\omega} \left\{ {\nu \atop \lambda\mu} \right\} \\
&= a_{mi} \, \overset{m}{e}_\omega \, \overset{i}{e}_\nu \, [- \overset{j}{e}_\lambda \, \overset{\kappa}{e}_\mu \, (\nabla_\kappa e_j^{\;\nu})] \\
&= a_{mi} \, \overset{m}{e}_\omega \, \overset{i}{e}_\nu \, e_j^{\;\nu} \, \overset{\kappa}{e}_\mu (\nabla_\kappa \overset{j}{e}_\lambda) \\
&= a_{mi} \, \overset{m}{e}_\omega \, \overset{\kappa}{e}_\mu \, \delta^i_j \, (\nabla_\kappa \overset{j}{e}_\lambda) \\
&= a_{mj} \, \overset{m}{e}_\omega \, \overset{\kappa}{e}_\mu \, (\nabla_\kappa \overset{j}{e}_\lambda) \\
&= a_{mj} \, (\nabla_\kappa \overset{j}{e}_\lambda) \, \overset{m}{e}_\omega \, \overset{\kappa}{e}_\mu .
\end{aligned}$$

THEOREM 3.2. The covariant differentiation of holonomic contravariant tensor $a^{\lambda\nu}$ may be expressed as follows ;

$$(3.2)\qquad \partial_{\mu}(a^{\lambda\nu}) = a^{\nu\lambda}[(\nabla_{\kappa} e_{l}^{\sigma})\, e_{\sigma}^{l} + (\nabla_{\kappa} e_{m}^{\omega})\, e_{\omega}^{m}]\, e_{\mu}^{\kappa}.$$

PROOF. By means of (1.3)b , (2.2) , (2.5) , (3.1),

$$\begin{aligned}
\partial_{\mu}(a^{\nu\lambda}) &= -a^{\sigma\nu} a^{\lambda\omega}([\omega,\sigma\mu] + [\sigma,\omega\mu]) \\
&= -a^{\sigma\nu} a^{\lambda\omega}[a_{ml}(\nabla_{\kappa} e_{\sigma}^{l})\, e_{\omega}^{m}\, e_{\mu}^{\kappa} + a_{ml}(\nabla_{\kappa} e_{\omega}^{m})\, e_{\sigma}^{l}\, e_{\mu}^{\kappa}] \\
&= -a^{\sigma\nu} a^{\lambda\omega} a_{ml}\, e_{\omega}^{m}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\sigma}^{l}) - a^{\sigma\nu} a^{\lambda\omega} a_{ml}\, e_{\sigma}^{l}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\omega}^{m}) \\
&= -a^{\sigma\nu} a^{\lambda\omega} a_{\lambda\omega}\, e_{m}^{\lambda}\, e_{l}^{\omega}\, e_{\omega}^{m}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\sigma}^{l}) - a^{\sigma\nu} a^{\lambda\omega} a_{\lambda\omega}\, e_{l}^{\lambda}\, e_{m}^{\omega}\, e_{\sigma}^{l}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\omega}^{m}) \\
&= -a^{\sigma\nu} e_{m}^{\lambda}\, \delta_{l}^{m}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\sigma}^{l}) - a^{\sigma\nu} e_{m}^{\omega}\, \delta_{\sigma}^{\lambda}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\omega}^{m}) \\
&= -a^{\sigma\nu} e_{l}^{\lambda}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\sigma}^{l}) - a^{\nu\lambda} e_{m}^{\omega}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\omega}^{m}) \\
&= -a^{li} e_{l}^{\sigma}\, e_{i}^{\nu}\, e_{l}^{\lambda}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\sigma}^{l}) - a^{\nu\lambda} e_{m}^{\omega}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\omega}^{m}) \\
&= -a^{\nu\lambda} e_{l}^{\sigma}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\sigma}^{l}) - a^{\nu\lambda} e_{m}^{\omega}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{\omega}^{m}) \\
&= a^{\nu\lambda} e_{\sigma}^{l}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{l}^{\sigma}) + a^{\nu\lambda} e_{\omega}^{m}\, e_{\mu}^{\kappa}(\nabla_{\kappa} e_{m}^{\omega}) \\
&= a^{\nu\lambda}[(\nabla_{\kappa} e_{l}^{\sigma})\, e_{\sigma}^{l} + (\nabla_{\kappa} e_{m}^{\omega})\, e_{\omega}^{m}]\, e_{\mu}^{\kappa}.
\end{aligned}$$

THEOREM 3.3. The covariant differentiation of holonomic covariant tensor $a_{\theta\omega}$ may be expressed in following manner;

$$(3.3)\quad \partial_\mu (a_{\theta\omega}) = a_{\theta\omega} [(\nabla_\kappa \overset{l}{e}_\sigma) \underset{l}{e}^\sigma + (\nabla_\kappa \overset{m}{e}_\omega) \underset{m}{e}^\omega] \overset{\kappa}{e}_\mu .$$

PROOF. From(2.5), (3.3)a $\partial_\mu (a_{\lambda\omega}) a^{\nu\lambda} = -\partial_\mu (a^{\nu\lambda}) a_{\lambda\omega}$.

Multiplying $a_{\theta\nu}$ to both sides of (3.3)a and by making use of (1.3)b and (3.2). We obtain

$$\partial_\mu (a_{\lambda\omega}) a^{\nu\lambda} a_{\theta\nu} = -\partial_\mu (a^{\nu\lambda}) a_{\lambda\omega} a_{\theta\nu},$$

$$\partial_\mu (a_{\lambda\omega}) \delta^\lambda_\theta = -\partial_\mu (a^{\nu\lambda}) a_{\lambda\omega} a_{\theta\nu},$$

$$\begin{aligned}
\partial_\mu (a_{\theta\omega}) &= -\partial_\mu (a^{\nu\lambda}) a_{\lambda\omega} a_{\theta\nu} \\
&= -a^{\nu\lambda} [(\nabla_\kappa \underset{l}{e}^\sigma) \overset{l}{e}_\sigma + (\nabla_\kappa \underset{m}{e}^\omega) \overset{m}{e}_\omega] \overset{\kappa}{e}_\mu a_{\lambda\omega} a_{\nu\theta} \\
&= -a^{\nu\lambda} a_{\nu\theta} [(\nabla_\kappa \underset{l}{e}^\sigma) \overset{l}{e}_\sigma + (\nabla_\kappa \underset{m}{e}^\omega) \overset{m}{e}_\omega] \overset{\kappa}{e}_\mu a_{\lambda\omega} \\
&= -\delta^\lambda_\theta a_{\lambda\omega} [(\nabla_\kappa \underset{l}{e}^\sigma) \overset{l}{e}_\sigma + (\nabla_\kappa \underset{m}{e}^\omega) \overset{m}{e}_\omega] \overset{\kappa}{e}_\mu \\
&= -a_{\theta\omega} [(\nabla_\kappa \underset{l}{e}^\sigma) \overset{l}{e}_\sigma + (\nabla_\kappa \underset{m}{e}^\omega) \overset{m}{e}_\omega] \overset{\kappa}{e}_\mu \\
&= a_{\theta\omega} [(\nabla_\kappa \overset{l}{e}_\sigma) \underset{l}{e}^\sigma + (\nabla_\kappa \overset{m}{e}_\omega) \underset{m}{e}^\omega] \overset{\kappa}{e}_\mu .
\end{aligned}$$

COROLLARY 3.4. The covariant differentiating of the determinant of holonomic covariant tensor a may be expressed as follows;

$$\partial_{\mu} a = a[(\nabla_{\kappa} \overset{l}{e}_{\sigma}) \underset{l}{e}^{\sigma} + (\nabla_{\kappa} \overset{m}{e}_{\omega}) \underset{m}{e}^{\omega}] \overset{\kappa}{e}_{\mu}. \tag{3.4}$$

PROOF. Making use of (1.1) and (3.3),

$$\begin{aligned} \partial_{\mu} a &= a\, a^{\theta\omega} \partial_{\mu}(a_{\theta\omega}) \\ &= a\, a^{\theta\omega} a_{\theta\omega} [(\nabla_{\kappa} \overset{l}{e}_{\sigma}) \underset{l}{e}^{\sigma} + (\nabla_{\kappa} \overset{m}{e}_{\omega}) \underset{m}{e}^{\omega}] \overset{\kappa}{e}_{\mu} \\ &= a[(\nabla_{\kappa} \overset{l}{e}_{\sigma}) \underset{l}{e}^{\sigma} + (\nabla_{\kappa} \overset{m}{e}_{\omega}) \underset{m}{e}^{\omega}] \overset{\kappa}{e}_{\mu}. \end{aligned}$$

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국 문 초 록

Vn 공간에서의 NONHOLONOMIC 성분들의 공변미분에 대한 소고

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이 논문의 주요한 목적은 HOLONOMIC과 NONHOLONOMIC COMPONENT 사이의 관계를 구명하고, 지금까지 잘 알려진 몇가지 성질들을 새로운 방법으로 재구성하고 증명하여 봄으로써 이들 관계식을 이용하여 다른 각도에서 연구하는데 있다.