Metrization on a Normal Moore Space

By

Kim, Seongchan

Department of Mathematics
Graduate School of Education
Cheju National University

Supervised By

Assistant Prof. Han, Chulsoon

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指導教授 韓 哲 淳

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副 審

副 審 韓 哲 淳

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Normal Moore 空間의 距離化

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이 論文은 developable 空間과 Moore 空間에 対한 몇 가지의 性質을 다루었다.

또한, "X가 σ-discrete 基를 갖는 T_3-空間이면 X는 距離化 可能이다"라는 "Bing의 距離化" 定理를 利用하여 모든 Collectionwise Normal Moore 空間이 距離化 可能이라는 것을 証明하였다.

1. INTRODUCTION

Recall that a topological space $(X,\mathcal{T})$ is metrizable if there is a metric for X such that the induced metric topology coincides with $\mathcal{T}$. Although metric spaces lead naturally to topological spaces, not all topological spaces are metrizable: for instance, nonnormal spaces are not metrizable (metric spaces are normal, and normality is a topological invariant). This leads one to the problem of finding conditions sufficient to ensure the metrizability of a space.

The purpose of this paper is to provide a fairly representative solution to these problems on a Moore space. Note that a Moore space is not metrizable (there is an its example in [1] PP. 269 - 272); however Jones [1937] has shown that every separable normal Moore space is metrizable.

In our paper we see that every collectionwise normal Moore space is metrizable, using the Bing's Metrization: if X is a T_3-space with a σ-discrete base then X is metrizable (proof: [2] PP. 127 - 129).

Our paper is organized into three sections.

§2 is a preliminary section containing some useful properties for a developable space and a Moore space.

§3 contains the main theorem: every collectionwise normal Moore space is metrizable.

2. PRELIMINARY

In this section we collect some basic definitions and some useful properties for a developable space and a Moore space.

DEFINITION 2-1 Suppose $\mathcal{G}_1, \mathcal{G}_2, \cdots$ is a sequence of open covers of a topological space X. We call a sequence $\{\mathcal{G}_n: n\in Z^+\}$ a development for X iff for each $x\in X$, $\{St(x, \mathcal{G}_n): n\in Z^+\}$ is a base at x, where $St(x, \mathcal{G}_n) = \cup\{G: x\in G\in \mathcal{G}_n\}$. A T_1-space X is developable iff X has a development.

There has been some recent interest in investigating an old concept: Moore spaces. Moore spaces are generalizations of metric spaces.

DEFINITION 2-2 A T_3-space with a development is a Moore space.

Using the above definitions, we have the following result:

PROPOSITION 2-1 Every metrizable space is a Moore space.

Proof. Let $(X,\mathcal{T})$ be a space with a metric d. For $\epsilon>0$ and $x\in X$, define $B(x,\epsilon) = \{y: d(x,y)<\epsilon\}$. Then $\mathcal{B} = \{B(x,\epsilon): x\in X, \epsilon>0\}$ is a base for $\mathcal{T}$.

For each $n\in Z^+$, define $\mathcal{B}_n = \{B(x,\frac{1}{n}): x\in X\}$. Then each $\mathcal{B}_n$ is a cover of X. Hence $\{\mathcal{B}_n: n\in Z^+\}$ is a sequence of covers of X.

Let U be any open set containing $x\in X$, then there is $\epsilon>0$ such that $B(x,\epsilon)\subset U$.

Choose $n\in Z^+$ such that $\frac{1}{n}<\epsilon$. Then $z\in St(x,\mathcal{B}_{2n}) = \cup\{B: x\in B\in\mathcal{B}_{2n}\}$ implies that there is $B(y,\frac{1}{2n})\in\mathcal{B}_{2n}$ such that $x, z \in B(y,\frac{1}{2n})$.

Hence

$$d(x,z) \leq d(x,y) + d(y,z) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

and so $z \in B(x,\frac{1}{n}) \subset B(x,\epsilon) \subset U$.

Thus $St(x,\mathcal{B}_{2n}) \subset U$, that is, $\{St(x,\mathcal{B}_n): n \in Z^+\}$ is a base at x; hence, $\{\mathcal{B}_n: n \in Z^+\}$ is a development.

Furthermore, every metrizable space is a T_3-space.

Therefore, $(X,\mathcal{T})$ is a Moore space.

Clearly, metric spaces are Moore spaces; however, there are examples of Moore spaces that are not metrizable. (See [1] PP. 269 - 272.)

Two useful types of development are defined next.

DEFINITION 2-3 If $\{\mathcal{G}_n: n \in Z^+\}$ is a development for X, and $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for each n, then it is called a refinement development.

DEFINITION 2-4 If $\{\mathcal{G}_n: n \in Z^+\}$ is a development and $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots$, then it is called a nested development.

Using the above definitions, we have

PROPOSITION 2-2 Every developable space has a nested development.

Proof. Let X be a developable space. Then there is a sequence $\{\mathcal{G}_n\}$ of open covers such that for each $x \in X$, $\{St(x, \mathcal{G}_n): n \in Z^+\}$ is a base at x.

For each $n \in Z^+$, define

$$\mathcal{H}_n = \mathcal{G}_1 \cap \cdots \cap \mathcal{G}_n = \{G_1 \cap \cdots \cap G_n: G_i \in \mathcal{G}_i, i = 1, \cdots, n\}.$$

Note that $\mathcal{H}_n$ is a collection of open sets for each n. If $n \in Z^+$ and $x \in X$ then there are $G_1, \cdots, G_n$ in $\mathcal{G}_1, \cdots, \mathcal{G}_n$ containing x respectively, and $x \in G_1 \cap \cdots \cap G_n \in \mathcal{H}_n$; hence each $\mathcal{H}_n$ covers X. Moreover, $\mathcal{H}_{n+1}$ refines $\mathcal{H}_n$ because for each $H_{n+1} \in \mathcal{H}_{n+1}$, there is some $H_n \in \mathcal{H}_n$ such that $H_{n+1} \subset H_n$.

Let U be an open set containing $x \in X$. Then there is $n \in Z^+$ such that $St(x, \mathcal{G}_n) \subset U$. Since $x \in H = G_1 \cap \cdots \cap G_n \subset G_1, \cdots, G_n$, we have

$$\begin{aligned} St(x, \mathcal{H}_n) &= \cup\{H: x \in H \in \mathcal{H}_n\} \\ &\subset \cup\{G: x \in G \in \mathcal{G}_i,\ i \leq n\} \\ &= St(x, \mathcal{G}_i),\ i \leq n. \end{aligned}$$

Thus $St(x, \mathcal{H}_n) \subset St(x, \mathcal{G}_n) \subset U$, and so $\{St(x, \mathcal{H}_n): n \in Z^+\}$ is a base at $x \in X$.

Let $\mathcal{U}_i = \bigcup_{n=i}^{\infty} \mathcal{H}_n$, then $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots$ and each $\mathcal{U}_i$ is a cover of X. Since $\{\mathcal{H}_i\}$ is a refinement development, for each $x \in X$ and $n \in Z^+$ $St(x, \mathcal{H}_i) \subset St(x, \mathcal{H}_n)$ if $n \leq i$.

Let V be any open set containing x. Then there is $n \in Z^+$ such that $St(x, \mathcal{H}_n) \subset V$.

Now, for each $i \geq n$

$$\begin{aligned} St(x, \mathcal{U}_i) &= \cup\{W: x \in W \in \mathcal{U}_i\} \\ &= \bigcup_{j \geq i} St(x, \mathcal{H}_j) \\ &\subset St(x, \mathcal{H}_n) \\ &\subset V, \end{aligned}$$

so that $\{St(x, \mathcal{U}_n): n \in Z^+\}$ is a base at x.

Therefore, $\{\mathcal{U}_n\}$ is a nested development for X.

Note that a collection $\mathcal{F}$ is σ-discrete iff $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ where $\mathcal{F}_n$ is a discrete collection.

Recall that $\mathcal{F}_n$ is a discrete collection if $\{\overline{H}_\alpha: H_\alpha \in \mathcal{F}_n\}$ is nbd-finite and the $\bar{H}_\alpha$'s are mutually disjoint.

DEFINITION 2-5 A space X is subparacompact iff every open cover of X has a σ-discrete closed refinement.

By the above concept, we have an important property.

PROPOSITION 2-3 Every developable space is subparacompact.

Proof. Suppose X is a space with a development $\{\mathcal{G}_n : n \in Z^+\}$. Let $\mathcal{O} = \{O_a : a \in A\}$ be an open cover of X. We assume A is well-ordered and define

$$C_{n,a} = O_a - St(X - O_a, \mathcal{G}_n) - \bigcup_{b<a} O_b$$

for each $a \in A$ and each $n \in Z^+$.

Let $\mathcal{E}_n = \{C_{n,a} : a \in A\}$ for each $n \in Z^+$, and $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$.

(1) We show that $\mathcal{E}$ covers X.

For each $x \in X$, there is the least $a \in A$ such that $x \in O_a$. Note that $x \notin O_b$ for each $b<a$. Since $\{\mathcal{G}_n\}$ is a development, there is an $n \in Z^+$ such that $St(x, \mathcal{G}_n) \subset O_a$. Now, $x \notin X - O_a$ so that $x \notin St(X - O_a, \mathcal{G}_n)$. Hence $x \in C_{n,a}$.

(2) We show that $\mathcal{E}$ refines $\mathcal{O}$.

For each $E \in \mathcal{E}$, there is $O_a \in \mathcal{O}$ such that $E \subset O_a$ since $E = C_{n,a} = O_a - St(X - O_a, \mathcal{G}_n) - \bigcup_{b<a} O_b \subset O_a$.

(3) We show that each $C_{n,a}$ is closed.

Let $x \in \overline{C_{n,a}}$. Then $x \in X$. If $b \in A$ and $x \in O_b$, O_b would meet $C_{n,a}$. Hence $b \nless a$. In other words, $b<a$ implies $x \notin O_b$. On the other hand, $x \notin St(X - O_a, \mathcal{G}_n)$; otherwise, $St(X - O_a, \mathcal{G}_n)$ must meet $C_{n,a}$, a contradiction.

If $x \in O_a$, then $x \in C_{n,a}$ and we are done.

Suppose $x \notin O_a$. Then there is $G \in \mathcal{G}_n$ such that $x \in G$; hence $G \subset St(X-O_a, \mathcal{G}_n)$. Since $x \in \overline{C_{n,a}}$, $\emptyset \neq G \cap C_{n,a} \subset St(X-O_a, \mathcal{G}_n) \cap C_{n,a}$ and a contradiction.

Therefore $C_{n,a}$ is closed.

(4) We show that each $\mathcal{E}_n = \{C_{n,a}: a \in A\}$ is discrete.

The $C_{n,a}$'s are evidently mutually disjoint.

Suppose $\mathcal{E}_n$ is not nbd-finite. Then there is $z \in X$ such that any nbd O of z meets infinitely many members of $\mathcal{E}_n$.

Let $G \in \mathcal{G}_n$ and $z \in G$. Then G must meet two members of $\mathcal{E}_n$, say $C_{n,a}$ and $C_{n,b}$ where $b < a$. Since $G \cap C_{n,b} \neq \emptyset$, $G \not\subset St(X-O_b, \mathcal{G}_n)$ and so $G \subset O_b$ for if $x \notin O_b$ and $x \in G$ then $x \in X-O_b$ implies $x \in St(X-O_b, \mathcal{G}_n)$, that is, $G \subset St(X-O_b, \mathcal{G}_n)$ a contradiction.

Now, G cannot meet $C_{n,a}$, a contradiction.

Therefore each $\mathcal{E}_n$ is discrete.

By (1), (2), (3) and (4), every open cover $\mathcal{O}$ has a σ-discrete closed refinement, and the proof is complete.

3. METRIZATION

In this section, we see that every collectionwise normal Moore space is metrizable.

We begin by observing that one important property of discrete collections in topological spaces is given in the next lemma.

LEMMA 3-1 A collection of subsets $\mathcal{H} = \{H_a: a \in A\}$ of a space X is discrete iff for each $x \in X$, there is an open U containing x such that $U \cap H \neq \emptyset$ for at most one element H of $\mathcal{H}$.

Proof. Choose $x \in X$. Then either $x \in \bar{H}_a$ or $x \notin \bigcup_a \bar{H}_a$ for some a.

If $x\in\bar{H}_a$ then $x\notin\bigcup_{b\neq a}\bar{H}_b$ since $\bar{H}_a$'s are mutually disjoint. Thus $x\in X-\bigcup_{b\neq a}\bar{H}_b$. Let $U=X-\bigcup_{b\neq a}\bar{H}_b$, then U is open containing x so that $U\cap H_a\neq\emptyset$ and $U\cap H_b=\emptyset$ for each $b\neq a$. If $x\in X-\bigcup_a\bar{H}_a$ then $X-\bigcup_a\bar{H}_a$ is open containing x such that $(X-\bigcup_a\bar{H}_a)\cap H_a=\emptyset$ for each a.

Conversely, $\mathcal{H}$ is clearly nbd-finite and so $\{\bar{H}_a: a\in A\}$ is nbd-finite. Suppose $\bar{H}_a\cap\bar{H}_b\neq\emptyset$ for $a\neq b$. Let $x\in\bar{H}_a\cap\bar{H}_b$, then $x\in\bar{H}_a$ and $x\in\bar{H}_b$. Choose any open U containing x. Then $U\cap H_a\neq\emptyset$ and $U\cap H_b\neq\emptyset$ and hence there is no open U containing x such that $U\cap H\neq\emptyset$ for at most one element H of $\mathcal{H}$; a contradiction. Thus $\bar{H}_a$'s are mutually disjoint.

Now, we introduce the concept of collectionwise normality.

DEFINITION 3-1 A topological space X is collectionwise normal iff for each discrete collection of subsets $\{H_a: a\in A\}$, there are mutually disjoint open subsets $\{G_a: a\in A\}$ such that $H_a\subset G_a$ for each $a\in A$.

The next lemma clearly follows from the above definition and lemma.

LEMMA 3-2 Let X be collectionwise normal. If $\mathcal{H}$ is a discrete collection of closed sets and a cover of X, then there is a discrete collection $\mathcal{G}=\{G_H: H\in\mathcal{H}\}$ of open sets with $H\subset G_H$ for each $H\in\mathcal{H}$.

Proof. Clearly, $\mathcal{G}$ is a cover of X and G_H's are mutually disjoint.

Let $x\in X$, then there is $H_a\in\mathcal{H}$ such that $x\in H_a\subset G_{H_a}$. Thus

$x \notin \bigcup_{b \neq a} G_{H_b}$ for each $H_b \in \mathcal{H}$. So $x \in X - \bigcup_{b \neq a} G_{H_b} \subset X - \bigcup_{b \neq a} H_b$.

Let $U = X - \bigcup_{b \neq a} H_b$. Then U is open containing x so that

$$U \cap G_{H_a} \neq \emptyset \text{ and } U \cap G_{H_b} = \emptyset \text{ for each } b \neq a.$$

By Lemma 3-1, $\mathcal{G}$ is a discrete collection.

Using the above lemma, we obtain a useful lemma.

LEMMA 3-3 If X is subparacompact and collectionwise normal then every open cover of X has a σ-discrete open refinement.

Proof. Let $\mathcal{O}$ be an open cover of X. Since X is subparacompact, $\mathcal{O}$ has a σ-discrete closed refinement $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ where $\mathcal{H}_n = \{H_{n,a} : a \in A_n\}$ is a discrete collection for each $n \in Z^+$. Since X is collectionwise normal, by Lemma 3-2, there is a discrete collection $\mathcal{G}_n = \{G_{n,a} : a \in A_n\}$ of open sets such that $H_{n,a} \subset G_{n,a}$ for each $a \in A_n$.

Moreover, there is $O \in \mathcal{O}$ such that $H_{n,a} \subset O$ for each n and each a, since $\mathcal{H}$ refines $\mathcal{O}$. Let $\mathcal{O}_n = \{G_{n,a} \cap O : a \in A_n\}$. Then $\mathcal{O}_n$ is clearly discrete

Therefore, $\mathcal{O}' = \bigcup_{n=1}^{\infty} \mathcal{O}_n$ is a σ-discrete open refinement of $\mathcal{O}$.

Using the above lemma and the next proposition, we have the following main theorem in our paper.

PROPOSITION(BING'S METRIZATION) If X is a T_3-space with a σ-discrete base then X is metrizable.

MAIN THEOREM Every collectionwise normal Moore space is metrizable.

Proof. Let X be a collectionwise normal Moore space with a development $\mathcal{G}_1, \mathcal{G}_2, \cdots$. By Proposition 2-3, X is subparacompact; so by Lemma 3-3, $\mathcal{G}_n$ has a σ-discrete open refinement $\mathcal{B}_n$ for each $n \in Z^+$. Thus $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a σ-discrete collection of open sets.

Let U be open containing $x \in X$. Then there is $n \in Z^+$ such that $St(x, \mathcal{G}_n) \subset U$.

Moreover, we can choose $B \in \mathcal{B}_n$ such that $x \in B$. Since $\mathcal{B}_n$ refines $\mathcal{G}_n$, there is $G \in \mathcal{G}_n$ such that $B \subset G$. It follows

$$x \in B \subset G \subset St(x, \mathcal{G}_n) \subset U,$$

so $\mathcal{B}$ is a base for X.

Therefore, X has a σ-discrete base; and hence X is metrizable by the Bing's metrization.

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ABSTRACT

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Kim, Seongchan
Department of Mathematics
Graduate School of Education
Cheju National University

In our paper, we study some useful properties for a developable space and a Moore space.

Moreover, we see that every collectionwise normal Moore space is metrizable, using the Bing's Metrization: If X is a T_3-space with a σ-discrete base then X is metrizable.