ON SOME RELATIONS OF TWO NONHOLONOMIC CHRISTOFFEL SYMBOLS IN Vn

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감 사 의 글

이 논문이 완성되기까지 연구에 바쁘신 가운데도 자상한 지도를 하여 주신 현진오 교수님께 무한한 감사를 드리오며 그동안 많은 도움을 주신 수학교육과의 모든 교수님께 심심한 사의를 표합니다.

그리고 그동안 저에게 사랑과 격려를 아끼지 않으신 주위의 여러분들께 감사 드립니다.

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오 봉 림

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국 문 초 록

Vn 공간에서 두개의 NONHOLONOMIC CHRIS - TOFFEL 기호의 어떤 관계에 대하여

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본 논문의 목적은 일반적인 대칭공변 TENSOR $a_{\lambda\mu}$에 의하여 정의된 CHRISTOFFEL 기호에 대한 HOLONOMIC 과 NONHOLONOMIC 성분들 사이의 관계를 연구하고, 제1과 제2 NONHOLONOMIC CHRISTOFFEL 기호들에 대한 유용한 표현 형식을 유도하였다.

1. INTRODUCTION

Let V_n be a n-dimensional Riemannian space referred to a real coordinate system x^ν and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determinant

(1.1) $$h \overset{\text{def}}{=} \text{Det}\,((h_{\lambda\mu})) \neq 0.$$

According to (1.1), there is a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ defined by

(1.2) $$h_{\lambda\mu}\, h^{\lambda\nu} \overset{\text{def}}{=} \delta_\mu^\nu.$$

Let $\underset{i}{e}^\nu$ $(i = 1, 2, \cdots, n)$ be a set of n linearly independent vectors. Then there is a unique reciprocal set of n linearly independent covariant vector $\overset{i}{e}_\lambda$ $(i = 1, 2, \cdots, n)$, satisfying

(1.3) $$\underset{i}{e}^\nu\, \overset{i}{e}_\lambda = \delta_\lambda^\nu \quad (*)$$

With these vectors $\underset{i}{e}^\nu$ and $\overset{i}{e}_\lambda$ a nonholonomic frame of V_n may be constructed in the following way:

If $A_\lambda^\nu{}^{\cdots}_{\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

(1.4) $$A_j^i{}^{\cdots\cdots}_{\cdots\cdots} = A_\mu^\nu{}^{\cdots\cdots}_{\cdots\cdots}\, \overset{i}{e}_\nu\, \underset{j}{e}^\mu$$

From the above definition, we obtain that

(1.5) $$A_\lambda^\nu{}^{\cdots\cdots}_{\cdots\cdots} = A_j^i{}^{\cdots\cdots}_{\cdots\cdots}\, \underset{i}{e}^\nu\, \overset{j}{e}_\lambda \cdots\cdots$$

(*) Throughout the present paper, Greek indices take the values $1, 2, \cdots, n$ unless explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic components of a tensor and run from 1 to n. Roman indices also follow the summation convention.

With respect to orthogonal nonholonomic frame of V_n constructed by an orthogonal ennuple $\underset{i}{e}^{\nu}$ ($i = 1,2, \cdots, n$) it was shown by Chung. K.T.& Hyun,J.O. 1976 that

(1.6) $$h_{ij} = \delta_{ij} , h^{ij} = \delta^{ij}$$

(1.7) $$\underset{i}{e}^{\nu} = \overset{i}{e}^{\nu} , \overset{j}{e}_{\lambda} = \underset{j}{e}_{\lambda} .$$

In this paper, studying the relationships between holonomic and nonholonomic components of the Christoffel symbols defined by a general symmetric covariant tensor $a_{\lambda\mu}$, we derive a useful representation of the first and second nonholonomic Christoffel symbols.

2. PRELIMINARY RESULTS

Consider a symmetric covariant tensor a whose determinant $a \stackrel{\text{def}}{=} \text{Det}(a_{\lambda\mu}) \neq 0$. It is well known that the quantities $a^{\lambda\nu}$ defined by

$a^{\lambda\nu} \stackrel{\text{def}}{=} \dfrac{\text{cofactor of } a_{\lambda\nu} \text{ in } a}{a}$ is symmetric contravariant tensor satisfying

(2.1) $$a_{\lambda\mu} a^{\lambda\nu} = \delta^{\nu}_{\mu}.$$

Take a coordinate system y^i for which we have at a point P of V_n

(2.2) $$\frac{\partial y^i}{\partial x^\lambda} = \overset{i}{e}_\lambda, \quad \frac{\partial x^\nu}{\partial y^i} = \underset{i}{e}^\nu$$

If $a_{\lambda\mu}$ and a_{ij} are holonomic and nonholonomic components of the tensor defined above, it follows that

(2.3) $$a_{jk} = a_{\lambda\mu} \underset{j}{e}^\lambda \underset{k}{e}^\mu.$$

In this section, for our further discussions, results obtained in the previous paper will be introduced without proof.

THEOREM 2.1. The derivative of $\underset{i}{e}^\lambda$ is a negative self-adjoint. That is

(2.4) $$\partial_k(\overset{i}{e}_\lambda)\, \underset{j}{e}^\mu = -\partial_k(\underset{j}{e}^\mu)\, \overset{i}{e}_\lambda.$$

THEOREM 2.2. The derivative of the tensor $a_{\lambda\mu}$ is a negative self-adjoint.

(2.5) $$a^{\lambda\mu}\, \partial_k(a_{\lambda\mu}) = -a_{\lambda\mu}\, \partial_k(a^{\lambda\mu}).$$

THEOREM 2.3. The holonomic and nonholonomic components of the Christoffel symbols satisfy

(2.6) $$[m, jk]_a = [\omega, \lambda\mu]_a \underset{j}{e}^{\lambda} \underset{k}{e}^{\mu} \underset{m}{e}^{\omega} + a_{\lambda\mu}(\partial_\gamma \underset{j}{e}^{\lambda}) \underset{k}{e}^{\gamma} \underset{m}{e}^{\mu}.$$

(2.7) $$\left\{ {i \atop jk} \right\}_a = \left\{ {\nu \atop \lambda\mu} \right\}_a \overset{i}{e}_{\nu} \underset{j}{e}^{\lambda} \underset{k}{e}^{\mu} + \overset{i}{e}_{\nu} \underset{k}{e}^{\mu} (\partial_\mu \underset{j}{e}^{\nu}).$$

Here, $[m, jk]$ and $\left\{ {i \atop jk} \right\}$ are the first and second Christoffel symbol of nonholonomic frame, respectively, defined by $a_{\lambda\mu}$.

THEOREM 2.4 The nonholonomic components of the Christoffel symbols of the second kind may be expressed as

(2.8) $$\begin{aligned} \left\{ {i \atop jk} \right\}_a &= \overset{i}{e}_{\nu} \underset{k}{e}^{\mu} (\partial_\mu \underset{j}{e}^{\nu} + \left\{ {\nu \atop \lambda\mu} \right\}_a \underset{j}{e}^{\lambda}) \\ &= \overset{i}{e}_{\nu} \underset{k}{e}^{\mu} (\nabla_\mu \underset{j}{e}^{\nu}) \\ &= - \underset{j}{e}^{\nu} \underset{k}{e}^{\mu} (\nabla_\mu \overset{i}{e}_{\nu}) \end{aligned}$$

Where ∇_μ is the symbol of the covariant derivative with respect to $\left\{ {\nu \atop \lambda\mu} \right\}_a$

THEOREM 2.5 The holonomic components of the Christoffel symbols, as follows

(2.9) $$[\omega, \lambda\mu]_a = [m, jk]_a \overset{m}{e}_{\omega} \overset{j}{e}_{\lambda} \overset{k}{e}_{\mu} + a_{jk} (\partial_\mu \overset{j}{e}_{\lambda}) \overset{k}{e}_{\omega}.$$

(2.10) $$\begin{aligned} \left\{ {\alpha \atop \beta\gamma} \right\}_a &= \left\{ {i \atop jk} \right\}_a \underset{i}{e}^{\alpha} \overset{j}{e}_{\beta} \overset{k}{e}_{\gamma} - (\partial_\gamma \underset{j}{e}^{\alpha}) \overset{j}{e}_{\beta} \\ &= \left\{ {i \atop jk} \right\}_a \underset{i}{e}^{\alpha} \overset{j}{e}_{\beta} \overset{k}{e}_{\gamma} + (\partial_\gamma \overset{j}{e}_{\beta}) \underset{j}{e}^{\alpha} \\ &= - \overset{j}{e}_{\beta} \overset{k}{e}_{\gamma} (\nabla_k \underset{j}{e}^{\alpha}) \\ &= \overset{k}{e}_{\gamma} \underset{j}{e}^{\alpha} (\nabla_k \overset{j}{e}_{\beta}). \end{aligned}$$

3. MAIN RESULTS.

THEOREM 3.1 The nonholonmic components of the Christoffel symbols of the first and second kinds satisfy

(3.1) $\left\{ {i \atop jk} \right\}_a = a^{im} [m,jk]_a .$

PROOF. From(2.7),(2.9)

$$\left\{ {p \atop qr} \right\}_a = \left\{ {\lambda \atop \mu\omega} \right\}_a \overset{p}{e}_\lambda \underset{q}{e}^\mu \underset{r}{e}^\omega + \overset{p}{e}_\lambda \underset{r}{e}^\omega (\partial_\omega \underset{q}{e}^\lambda)$$

$$= a^{\lambda\sigma}[\sigma,\mu\omega]_a \overset{p}{e}_\lambda \underset{q}{e}^\mu \underset{r}{e}^\omega + \overset{p}{e}_\lambda \underset{r}{e}^\omega (\partial_\omega \underset{q}{e}^\lambda)$$

$$= a^{\lambda\sigma}([i,jk]_a \overset{i}{e}_\sigma \overset{j}{e}_\mu \overset{k}{e}_\omega + a_{jk} (\partial_\omega \overset{j}{e}_\mu) \overset{k}{e}_\sigma)$$

$$\overset{p}{e}_\lambda \underset{q}{e}^\mu \underset{r}{e}^\omega + \overset{p}{e}_\lambda \underset{r}{e}^\omega (\partial_\omega \underset{q}{e}^\lambda).$$

By means of the (1.3),(2.3),(2.4)

$$\left\{ {p \atop qr} \right\}_a = a^{ip}[i,jk] \underset{q}{\delta}^j \underset{r}{\delta}^k + a^{pk} a_{jk} (\partial_\omega \overset{j}{e}_\mu) \underset{q}{e}^\mu \underset{r}{e}^\omega$$

$$+ \overset{p}{a}_\lambda \underset{r}{e}^\omega (\partial_\omega \underset{q}{e}^\lambda)$$

$$= a^{ip}[i,qr]_a + (\partial_\omega \overset{p}{e}_\mu) \underset{q}{e}^\mu \underset{r}{e}^\omega - (\partial_\omega \overset{p}{e}_\lambda) \underset{q}{e}^\lambda \underset{r}{e}^\omega$$

$$= a^{ip}[i,qr]_a .$$

By the theorem 3.1,

COROLLARY 3.2. We have

$$a_{il} \left\{ {i \atop jk} \right\}_a = [l,jk]_a .$$

PROOF. Multiplying both sides of (3.1) by a_{il}, using (2.1) as required.

THEOREM 3.3 The nonholonomic components of the Christoffel symbols of the second kind may be expressed as the symbols of the covariant derivative with respect to the second kind.

PROOF. Using(2.7),(2.10),

$$
\begin{aligned}
\left\{ {i \atop jk} \right\}_a &= \left\{ {\lambda \atop \mu\omega} \right\}_a \overset{i}{e}_\lambda \underset{j}{e}^\mu \underset{k}{e}^\omega + \overset{i}{e}_\lambda \underset{k}{e}^\omega (\partial_\omega \underset{j}{e}^\lambda) \\
&= \overset{m}{e}_\omega \underset{l}{e}^\lambda (\nabla_m \overset{l}{e}_\mu)\, \overset{i}{e}_\lambda \underset{j}{e}^\mu \underset{k}{e}^\omega + \overset{i}{e}_\lambda \underset{k}{e}^\omega\, (\partial_\omega \underset{j}{e}^\lambda) \\
&= (\nabla_k \overset{i}{e}_\mu) \underset{j}{e}^\mu + \overset{i}{e}_\lambda \underset{k}{e}^\omega (\partial_\omega \underset{j}{e}^\lambda)
\end{aligned}
$$

where ∇_p is the symbol of the covariant derivative with respect to $\left\{ {i \atop jk} \right\}_a$.

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ABSTRACT

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The purpose of the present paper is to study the relationships between holonomic and nonholonomic components of the Christoffel symbols defined by general symmetric covariant tensor $a_{\lambda\mu}$ and we derive a useful representation of the first and second nonholonomic Christoffel symbols.