A Thesis for the Degree of M.E.

A Note on the Curl of the Nonholonomic Vector in V_n

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감 사 의 글

이 논문이 완성되기까지 바쁘신 가운데도 많은 지도를 아끼지 않으신 현진오교수님께 감사 드리며, 아울러 그동안 많은 도움을 주신 수학교육과의 여러 교수님과 동료들에게 심심한 사의를 표합니다. 그리고, 그동안 저에게 사랑과 격려를 하여 주신 가족, 친지 및 주위의 많은 분들께 또한 감사를 드립니다.

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강 성 홍

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ABSTRACT (KOREAN)

I . INTRODUCTION

Introducing a set of 4 linearly independent basic null vectors, V.Hlavaty(Geometry of Einstein's Unified Field Theory. P. Noordhoff Ltd. 1957) introduced the concept of the nonholonomic frames and used it successfully as a tool to develop the algebra of the unified field theory in the space-time X_4.

In the present papers, the curl of the vector a_λ will be proved in a refined way as application of orthogonal nonholonomic frames.

Let V_n be a n-dimensonal Riemannian space refered to a real coordinate system x^ν and defined by fundamental metric tensor $h_{\lambda\mu}$, whose determinant is

$$(1.1) \qquad h \overset{\text{def}}{=} \operatorname{Det}(h_{\lambda\mu}) \neq 0.$$

Then there is the unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ defined by

$$(1.2) \qquad h_{\lambda\mu} h^{\lambda\nu} \overset{\text{def}}{=} \delta_\mu^\nu .$$

Consider a set of n linearly independent vectors $\underset{i}{e}^\nu$ (i = 1,2 ⋯ ., n). There is the unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_\lambda$ (i = 1,2, ⋯⋯ ,n) satisfying

$$(1.3)\qquad \underset{i}{e}^{\nu}\ \overset{i}{e}_{\lambda} = \delta^{\nu}_{\lambda} \quad **$$

With these vectors $\underset{i}{e}^{\nu}$ and $\overset{i}{e}_{\lambda}$, a nonholonomic frame of V_n may be constructed in the following way: If $T^{\nu\cdots\cdots}_{\lambda\cdots\cdots}$ are holonomic components of a tensor, the nonholonomic components of the holonomic tensor $T^{\nu\cdots\cdots}_{\lambda\cdots\cdots}$ are defined by

$$(1.4)\qquad {}^{*}T^{i\cdots\cdots}_{j\cdots\cdots} \stackrel{\text{def}}{=} T^{\nu\cdots\cdots}_{\lambda\cdots\cdots}\ \overset{i}{e}_{\nu}\ \underset{j}{e}^{\lambda}\ \cdots\cdots$$

and

$$(1.5)\qquad T^{\nu\cdots\cdots}_{\lambda\cdots\cdots} \stackrel{\text{def}}{=} {}^{*}T^{i\cdots\cdots}_{j\cdots\cdots}\ \underset{i}{e}^{\nu}\ \underset{j}{e}^{\lambda}\ \cdots\cdots .$$

Ⅱ. PRELIMINARY RESULTS

In this section, for our further discussion, results obtained in our previous papers will be introduced without proof.

THEOREM 2.1. The covariant derivative of the holonomic covariant vector is given by

$$(2.1)\qquad \nabla_{\mu}(a_{\lambda}) = [\partial^{*}_{k} a_j - {}^{*}a_i\ {}^{*}\Gamma^{i}_{jk}]\ \overset{k}{e}_{\mu}\ \overset{j}{e}_{\lambda}$$
$$= \nabla^{*}_{k} a_j\ \overset{k}{e}_{\mu}\ \overset{j}{e}_{\lambda}$$

where $\nabla_{\mu} a_{\lambda} = \partial_{\mu} a_{\lambda} - a_{\nu}\Gamma^{\nu}_{\lambda\mu}$.

**. Throughout the present paper, Greek indices take values 1,2 ……,n unless explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic components of a tensor and run from 1 to n. Roman indices also follow the summation convention.

THEOREM 2.2. The covariant derivative of the nonholonomic covariant vector is equivalent to

$$(2.2) \qquad \overset{*}{\nabla}_k a_j = [\partial_\mu a_\lambda - a_\nu \Gamma^\nu_{\lambda\mu}] \, e_j^\lambda e_k^\mu$$

$$= \nabla_\mu a_\lambda \, e_j^\lambda e_k^\mu .$$

COROLLARY 2.3. We have

$$(2.3) \qquad \nabla_\mu a_\lambda = \partial_\mu a_\lambda - {}^*a_j (\nabla_\mu e^j_\lambda) .$$

THEOREM 2.4. We have

$$(2.4) \qquad \nabla_\mu a_\lambda = \partial^*_k a_j e^j_\lambda e^k_\mu + {}^*a_j (\nabla_\mu e^j_\lambda) .$$

THEOREM 2.5. The covariant derivative of the holonomic covariant tensor $a_{\nu\lambda}$ may be expressed in terms of the nonholonomic components;

$$(2.5) \qquad \nabla_\mu a_{\nu\lambda} = [\overset{*}{\partial}_k a_{ij} - {}^*a_{\ell j} {}^*\Gamma^{\ell *}_{ik} - {}^*a_{i\ell} {}^*\Gamma^\ell_{kj}] \, e^i_\nu e^j_\lambda e^k_\mu .$$

THEOREM 2.6. We have

$$(2.6) \qquad \overset{*}{\nabla}_k a_{ij} = [\partial_\mu a_{\nu\lambda} - a_{\omega\lambda} \Gamma^\omega_{\nu\mu} - a_{\nu\omega} \Gamma^\omega_{\mu\lambda}] \, e_i^\nu e_j^\lambda e_k^\mu .$$

Ⅲ. THE CURL OF THE VECTOR

In this section, we shall reconstruct the curl of a vector and obtain its special properties with holonomic and nonholonomic frames.

THEOREM 3.1. For the curl of the vector a_λ, following four expressions are equal to each other.

$$(3.1)\quad \text{(a)}\quad \nabla_\mu a_\lambda - \nabla_\lambda a_\mu$$

$$\text{(b)}\quad \partial_\mu a_\lambda - \partial_\lambda a_\mu$$

$$\text{(c)}\quad [\partial^*_k a_j - \partial^*_j a_k]\, \overset{j}{e}_\lambda \overset{k}{e}_\mu$$

$$\text{(d)}\quad [\nabla^*_k a_j - \nabla^*_j a_k]\, \overset{j}{e}_\lambda \overset{k}{e}_\mu .$$

PROOF. The equality of (a) and (b) is given by the result

$$(3.2)\quad \nabla_\mu a_\lambda = \partial_\mu a_\lambda - a_\lambda - a_\nu \Gamma^\nu_{\lambda\mu} .$$

The equality of (b) and (c) is given by (2.1). By means of (2.2), (c) is equal to (d).

COROLLARY 3.2. The curl of the nonholonomic vector a_j may be expressed in terms of the components holonomic curl.

PROOF. Using (2.2), we have

$$(3.3)\quad \nabla^*_k a_j - \nabla^*_j a_k = [\nabla_\mu a_\lambda - \nabla_\lambda a_\mu]\, e^\mu_k\, e^\lambda_j .$$

THEOREM 3.3. If A is the curl of a covariant vector, we have the following equations;

$$(3.4)\quad \text{(a)}\quad \nabla_\nu A_{\lambda\mu} + \nabla_\lambda A_{\mu\nu} + \nabla_\mu A_{\nu\lambda} = 0$$

$$\text{(b)}\quad \partial_\nu A_{\lambda\mu} + \partial_\lambda A_{\mu\nu} + \partial_\mu A_{\nu\lambda} = 0 .$$

PROOF . By means of the curl,

$$(3.5) \qquad \nabla_\nu A_{\lambda\mu} = \nabla_\nu [\nabla_\mu a_\lambda - \nabla_\lambda a_\mu]$$

$$\nabla_\lambda A_{\mu\nu} = \nabla_\lambda [\nabla_\nu a_\mu - \nabla_\mu a_\nu]$$

$$\nabla_\mu A_{\nu\lambda} = \nabla_\mu [\nabla_\lambda a_\nu - \nabla_\nu a_\lambda] .$$

Using the properties of the covariant derivative, the sum of the left hand side of three equations is identically zero.

The covariant derivative of holonomic tensor $A_{\lambda\mu}$ is given by

$$\nabla_\nu A_{\lambda\mu} = \partial_\nu A_{\lambda\mu} - A_{\sigma\mu} \Gamma^\sigma_{\lambda\nu} - A_{\lambda\sigma} \Gamma^\sigma_{\nu\mu}$$

$$\nabla_\lambda A_{\mu\nu} = \partial_\lambda A_{\mu\nu} - A_{\sigma\nu} \Gamma^\sigma_{\mu\lambda} - A_{\mu\sigma} \Gamma^\sigma_{\lambda\nu}$$

$$\nabla_\mu A_{\nu\lambda} = \partial_\mu A_{\nu\lambda} - A_{\sigma\lambda} \Gamma^\sigma_{\nu\mu} - A_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} .$$

Since $A_{\nu\lambda}$ is skew-symmetric, we have

$$\nabla_\nu A_{\lambda\mu} + \nabla_\lambda A_{\mu\nu} + \nabla_\mu A_{\nu\lambda} = \partial_\nu A_{\lambda\mu} + \partial_\lambda A_{\mu\nu} + \partial_\mu A_{\nu\lambda} .$$

THEOREM 3.4. Let ${}^*A_{ij}$ be a curl of the nonholonomic covariant vector a_λ. Then the covariant derivative of nonholonomic tensor ${}^*A_{ij}$ may be expressed as following relation;

(3.6) $$\nabla^*_i A_{jk} + \nabla^*_j A_{ki} + \nabla^*_k A_{ij}$$

$$= \partial^*_i A_{jk} + \partial^*_j A_{ki} + \partial^*_k A_{ij} = 0 .$$

PROOF . Making use of (2.5), we obtain

(3.7) $$\nabla^*_i A_{jk} = \partial^*_i {}^*A_{jk} - {}^*A_{\ell k} {}^*\Gamma^{\ell}_{ji} - {}^*A_{j\ell} {}^*\Gamma^{\ell}_{ik} .$$

By similar method

(3.8) $$\nabla^*_j A_{ki} = \partial^*_j A_{ki} - {}^*A_{\ell i} {}^*\Gamma^{\ell}_{kj} - {}^*A_{k\ell} \Gamma^{\ell}_{ji}$$

$$\nabla^*_k A_{ij} = \partial^*_k A_{ij} - {}^*A_{\ell j} {}^*\Gamma^{\ell}_{ik} - {}^*A_{i\ell} {}^*\Gamma^{\ell}_{kj} .$$

Since ${}^*A_{ij}$ is skew-symmetric, the sum of each side of (3.7) and (3.8) is

(3.9) $$\nabla^*_i A_{jk} + \nabla^*_j A_{ki} + \nabla^*_k A_{ij} = \partial^*_i A_{jk} + \partial^*_j A_{ki} \partial^*_k A_{ij} .$$

On the other hand, from (2.2)

(3.10) $$\nabla^*_i A_{jk} + \nabla^*_j A_{ki} + \nabla^*_k A_{ij}$$

$$= [\nabla_\nu A_{\lambda\mu} + \nabla_\lambda A_{\mu\nu} + \nabla_\mu A_{\nu\lambda}] \; e^\nu_i \; e^\lambda_j \; e^\mu_k .$$

But the first term of right side of (3.10), by means of (3.4), is zero.

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< 國 文 抄 錄 >

V_n 空間에서의 NONHOLONOMIC VECTOR의 CURL에 對하여

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本 論文의 重要한 目的은 RIEMANN 空間 V_n 에서 HOLONOMIC 構造를 갖는 VECTOR의 CURL에 對한 몇가지 性質들을 NONHOLONOMIC 構造를 利用하여 보다 새로운 方法으로 證明하는데 있다.