碩士學位 請求論文

Some Vector Fields on a C^{∞} − Manifold

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본 論文에서는 C^∞ − 多樣體(C^∞ − Manifold)上에서의 接空間(Tangent space) $T_p(M)$을 定義하고 單位球(Unit sphere) S^2와 楕圓體(Ellipsoid) N 上에서의 接空間을 구하여 S^2의 接空間上에서의 構造(Frame)를 求하고 $F(x, y, z) = (\frac{x}{2}, y, z)$로 定義되는 函數 $F : S^2 \to N$가 C^∞ − 寫像(C^∞− mapping)임을 보인다.

다음은 리이환(環) (Lie algebra) 槪念을 紹介하고 아래와 같은 性質을 밝힌다. 첫째, 3 − 次元 接空間(3 − dimensional tangent vector space)에서 두 개의 백터의 內積(Vector product)을 一般的인 백터의 內積으로 定義하면 3 − 次元 接空間이 리이환 構造를 滿足한다. 둘째, $n \times n$ 行列 ($n \times n$ matrix) X, Y들의 內積 XY, YX에 대하여 [X, Y]를 [X, Y] = XY − YX로 定義하면 모든 $n \times n$ 行列들의 集合에서 [X, Y]가 리이환 構造를 한다. 세째, 두 개의 C^∞ − 백터場(C^∞ − vector field) X, Y에 대하여 [X, Y] = XY − YX로 定義하면 모든 C^∞ − 多樣體 백터場들의 集合 $\mathfrak{X}(M)$는 리이환 構造를 한다.

I. INTRODUCTION

In this paper, we introduce some properties of the most basic tools used in the study of differentiable manifolds, and we also examine the basic tools for the special two-dimensional smooth manifolds, e.g. the unit sphere $S^2 = \{(x, y, z)\,;\, x^2 + y^2 + z^2 = 1\}$ and the ellipsoid $N = \{(x, y, z)\,;\, 4x^2 + y^2 + z^2 = 1\}$.

In chapter II, for a n-dimensional smooth manifold M, we define the tangent space $T_p(M)$ attached to each $p \in M$. Each element X_p of $T_p(M)$ can be considered as an operator on C^∞ – functions defined by some neighborhood about p, and we calculate the tangent vector on the unit sphere S^2 in $\mathbf{R}^3$ and on the ellipsoid N in $\mathbf{R}^3$ in the view of the definition of the tangent space. With the computation of the frame on the tangent space about the unit sphere S^2, we can explicitly represent the tangent space of S^2.

On the other hand, we introduce the fact that a C^∞–mapping $F : M \to N$ induces a linear map $F_* : T_p(M) \to T_{F(p)}N$ on the tangent space at each point. About the unit sphere S^2 and the ellipsoid N, using proper coordinate neighborhoods(U, φ) and (V, ψ) on S^2 and N, respectively, the author shows that the function $F : S^2 \to N$ defined by $F(x, y, z) = (\frac{x}{2}, y, z)$ is a C^∞– mapping, and by the differentiation of function of several variables, he also calculates the exact formular F_* and F.

In chapter III, assigning a vector X_p to each $p \in M$, we obtain a vector field on M, and study some properties of the vector field on a C^∞– manifold M. Defining Lie Bracket between two vector fields, the author introduces the concept of Lie algebra, and show that the vector space V^3 with dimension 3 is a Lie algebra with the Lie Bracket as the usual vector product and show that the space of all $n \times n$ matrices is also Lie algebra with the Lie Bracket $[X, Y] = XY - YX$.

II. TANGENT VECTOR SPACES

In this thesis, if we put (U, φ) a coordinate neighborhood then for any $p \in U$, $\varphi : U \rightarrow R^n$ defined by $\varphi(p) = (x^1, x^2, \cdots, x^n)$ is a homeomorphism on U.

Definition 2.1. Let f be a real-valued function on an open set U of a n-dimensional manifold M. Then $f : U \rightarrow R$ is a C^∞– function if each $p \in U$ lies in a coordinate neighborhood (U, φ) such that $f\ \varphi^{-1}(x^1, \cdots, x^n)$ is C^∞ on $\varphi(U)$.

Example 2.2 The unit sphere $M = S^2 = \{(x, y, z) ; x^2 + y^2 + z^2 = 1\}$ is a nontrivial two-dimensional manifold realized as a surface in R^3.

Let $U_1 = S^2 \setminus \{(0,0,1)\}$, $U_2 = S^2 \setminus \{(0, 0, -1)\}$ be the subsets obtained by deleting the north and south poles respectively. Let

$$\varphi_\alpha ; U_\alpha \rightarrow R^2 \simeq \{(x, y, o)\}, \ (\alpha = 1, 2)$$

be streographic projections from the respective poles, so

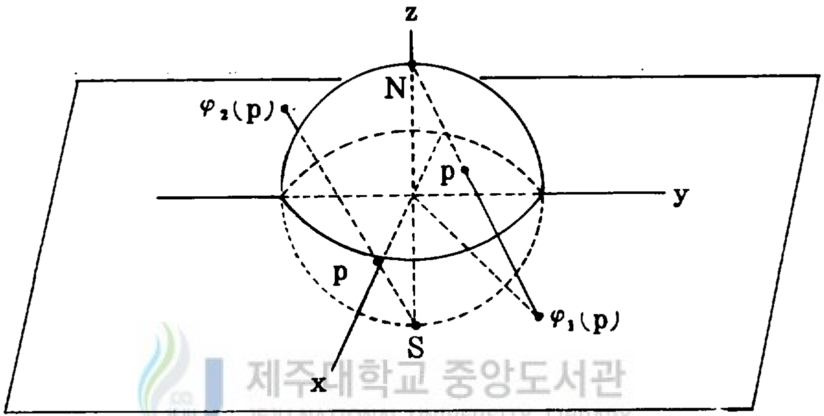
$$\varphi_1(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right), \ \varphi_2(x, y, z) = \left(\frac{x}{1 + z}, \frac{y}{1 + z} \right).$$

It can be easily checked that on the overlap $U_1 \cap U_2$

$$\varphi_1 \circ \varphi_2^{-1} : R^2 \setminus \{(0,0)\} \to R^2 \setminus \{(0,0)\}$$

is a smooth diffeomorphism, given by the inversion

$$\varphi_1 \circ \varphi_2^{-1}(x, y) := \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$



Stereographic projection

The Hausdorff separation property follows easily from that of R^3, so S^2 is a differentiable, indeed analytic, two-dimensional manifold.

Let f be a real-valued function on S^2 defined by $f(x, y, z) = x+y+z$.

For any $p \in S^2$, say $p \in U_1$,

$$f \circ \varphi_1^{-1}(x, y) = \frac{2x}{x^2+y^2+1}$$

By similar computation, on the other case, we can see easily that f is a C^∞-function on the two-dimensional manifold S^2.

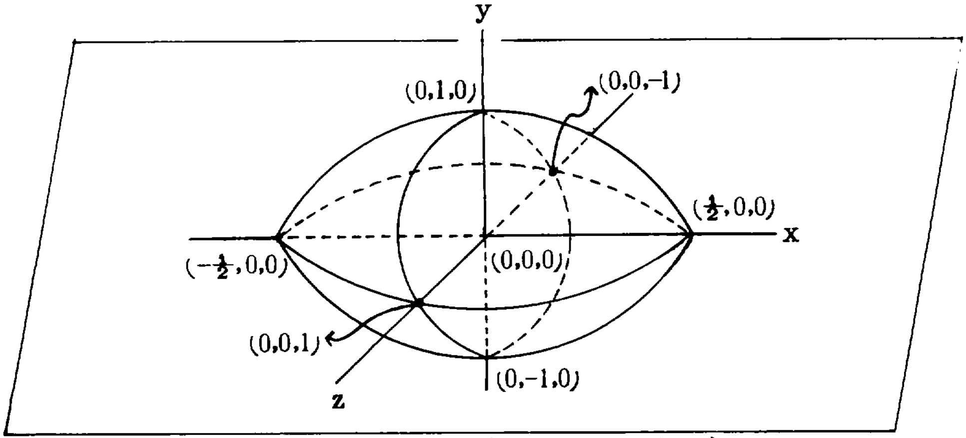
Definition 2.3. Let W and N be C^∞-manifolds. A function F is a C^∞-mapping of W into N, if for every $p \in W$ there exist (U, φ) of p and (V, ψ) of F(p) with $F(U) \subset V$ such that

$$\psi \circ F \circ \varphi^{-1}(U) : \varphi(U) \rightarrow \psi(V)$$

is the C^∞-function in Euclidean sense.

Furthermore We call F homeomorphism if $\psi \; F \; \varphi^{-1}$ is homeomorphism. A C^∞ mapping $F: M \rightarrow N$ between C^∞-manifolds is called a diffeomorphism if it is a homeomorphism and F and F^{-1} are C^∞-mappings.

Example 2.4. The Ellipsoid $N = \{(x, y, z) ; 4x^2 + y^2 + z^2 = 1\}$ is a nontrivial two-dimensional manifold realized as a surface in R^3.



The Ellipsoid $N=\{(x,y,z):4x^2+y^2+z^2=1\}$

Let

$V_1=\{(x,y,z):Z=\sqrt{1-4x^2-y^2}\}$

$V_2=\{(x,y,z):Z=-\sqrt{1-4x^2-y^2}\}$

$V_3=\{(x,y,z):y=\sqrt{1-4x^2-z^2}\}$

$V_4=\{(x,y,z):y=-\sqrt{1-4x^2-z^2}\}$

$V_5=\{(x,y,z):x=\frac{1}{2}\sqrt{1-y^2-z^2}\}$

$V_6=\{(x,y,z):x=-\frac{1}{2}\sqrt{1-y^2-z^2}\}$

Then $\bigcup_{i=1}^{6}V_i=N$. Let $\psi_\alpha:V_\alpha\to R^2$ be defined by

$\psi_\alpha(x,y,z) = (x,y), (\alpha=1, 2).$

$\psi_\beta(x,y,z) = (y,z), (\beta=3, 4).$

$\psi_\gamma(x,y,z) = (x,z), (\gamma=5, 6).$

It can be easily checked that on the overlap

$$\psi_\alpha \circ \psi_\beta^{-1} : \{(y, z) : y^2 + z^2 < 1\} \to \{(x, y) : 4x^2 + y^2 < 1\}$$

given by $\psi_\alpha \circ \psi_\beta^{-1}(y, z) = (\pm\frac{\sqrt{1-y^2+z^2}}{2}, y)$ (The sign depends on α and β),

$$\psi_\beta \circ \psi_\gamma^{-1} : \{(x, z) : 4x^2 + z^2 < 1\} \to \{(y, z) : y^2 + z^2 < 1\}$$

given by $\psi_\beta \circ \psi_\gamma^{-1}(x, z) = (\pm\sqrt{1-4x^2-z^2}, z)$ (The sign depends on β and γ),

and $\psi_\gamma \circ \psi_\alpha^{-1} : \{(x, y) : 4x^2 + y^2 < 1\} \to \{(x, z) : 4x^2 + z^2 < 1\}$

given by $\psi_\gamma \circ \psi_\alpha^{-1}(x, y) = (x, \pm\sqrt{1-4x^2-y^2})$ (The sign depends on γ and α).

By similar computation, on the other case N is a differentiable two-dimensional manifold.

Let F be a function from S^2 into N defined by $F(x, y, z) = (\frac{x}{2}, y, z)$. For any point $p = (x, y, z) \in S^2$, say $p \in U_1$ and $F(p) \in V_1$.

$$\psi_1 \ F \ \varphi_1^{-1}(x, y) = \left(\frac{x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}\right).$$

By the same method, on the other case, we can show that F is a C^∞-mapping, homeomorphism and diffeomorphism, since

$$\varphi_1 \circ F^{-1} \circ \psi_1^{-1}(x, y) = \left(\frac{2x}{1-\sqrt{1-4x^2-y^2}}, \frac{y}{1-\sqrt{1-4x^2-y^2}} \right)$$

is C^∞.

Given any point $p \in M$, we define $C^\infty(p)$ as the algebra of C^∞-functions whose domain of definition includes some open neighborhood of p, with functions identified if they agree on any neighborhood of p. The objects so obtained are called "germs" of C^∞–functions.

Definition 2.5. We define the tangent space $T_p(M)$ to M at p to be the set of all mapping $X_p : C^\infty(p) \to R$ satisfying for all α , $\beta \in R$, and $f, g \in C^\infty(p)$ the two conditions;

(i) $X_p(\alpha f + \beta g) = \alpha (X_p f) + \beta (X_p g)$

(ii) $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$,

with the vector space operations in $T_p(M)$ defined by

$$(X_p + Y_p)f = X_p f + Y_p f, \quad (\alpha X_p)f = \alpha (X_p f).$$

Any $X_p \in T_p(M)$ is called a tangent vector to M at p.

Example 2.6. Let $M=\{(x, y, z): x^2+y^2+z^2=1\}$ be a two-dimensional manifold, and let $P=(x_0, y_0, z_0)$ and $\varphi_1(P)=(u_0, v_0)=(\frac{x_0}{1-z_0}, \frac{y_0}{1-z_0})$.

For any C^∞-function f defined by some open neighborhood of P, consider any C^∞-curve $(u(t), v(t))$ $(-1<t<1)$ and $(u(0), v(0))=(u_0, v_0)$ passing through the point (u_0, v_0) in R^2. Then

$\varphi_1^{-1}(u(t), v(t))=(x(t), y(t), z(t))$ is a curve on the unit sphere S^2 passing through the point P. Also $X_p=(x'(t_0), y'(t_0), z'(t_0))$ is a tangent vector at P to the unit sphere, because

$$\lim_{\triangle t \downarrow 0} \frac{f(x(t), y(t), z(t)) - f(x(t_0), y(t_0), z(t_0))}{\triangle t}, \quad (\triangle t = t - t_0)$$

$$= (\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p)) \cdot (x'(t_0), y'(t_0), z'(t_0))$$

$$= Df(P) \cdot X_p.$$

If we denote the limit by $X_p(f)=Df(P)\cdot X_p$ then X_p is a mapping from $C^\infty(p)$ into R satisfying the conditions (i) and (ii) in the Definition 2.5..

To show that, we let $f, g \in C^\infty(p)$ and $\alpha, \beta \in R$. Then

$$X_p(\alpha f+\beta g) = D(\alpha f+\beta g)(P)\cdot X_p$$

$$= \lim_{\triangle t \downarrow o} \frac{(\alpha f+\beta g)(x(t),\ y(t),\ z(t)) - (\alpha f+\beta g)(P)}{\triangle t},\ (\triangle t = t - t_o)$$

$$= \lim_{\triangle t \downarrow o} \alpha \frac{[f(x(t), y(t), z(t)) - f(P)]}{\triangle t} + \lim_{\triangle t \downarrow o} \beta \frac{[g(x(t), y(t), z(t)) - g(P)]}{\triangle t}$$

$$= \alpha Df(P)\cdot X_p + \beta Dg(P)\cdot X_p$$

$$= \alpha X_p(f) + \beta X_p(g),$$

$$X_p(fg) = D(fg)(P)\cdot X_p$$

$$= \lim_{\triangle t \downarrow o} \frac{(fg)(x(t),\ y(t),\ z(t)) - (fg)(P)}{\triangle t},\ (\triangle t = t - t_o)$$

$$= \lim_{\triangle t \downarrow o} \frac{f(x(t),\ y(t),\ z(t))\ g(x(t),\ y(t),\ z(t)) - f(P)\cdot g(P)}{\triangle t}$$

$$= \lim_{\triangle t \downarrow o} \frac{[f(x(t),\ y(t),\ z(t)) - f(P)]g(x(t),\ y(t),\ z(t)) + f(P)[g(x(t), y(t),\ z(t)) - g(P)]}{\triangle t}$$

$$= [Df(P)\cdot X_p]g(P) + f(P)[Dg(P)\cdot X_p]$$

$$= (X_p f)g + f(X_p g).$$

Hence X_p is a tangent vector at P on M.

Theorem 2.8. Let $F : M \rightarrow N$ be a C^{∞}-map of manifolds. Then for $p \in M$ the map $F^{*} : C^{\infty}(F(p)) \rightarrow C^{\infty}(p)$ defined by $F^{*}(f) = f \circ F$ is a homomorphism of algebras and induces a dual vector space homomorphism $F_{*} : T_p(M) \rightarrow T_{F(p)}(N)$, defined by $F_{*}(X_p)f = X_p(F^{*}f)$, which gives $F_{*}(X_p)$ as a map of $C^{\infty}(F(p))$ to R.

proof. The proof consists of routinely checking the statements against definitions. We omit the verification that F^{*} is a homomorphism and consider F_{*} only. Let $X_p \in T_p(M)$ and $f, g \in C^{\infty}(F(p))$: we must prove that the map $F_{*}(X_p) : C^{\infty}(F(p)) \rightarrow R$ is a vector at F(p), that is, a linear map. we have

$$\begin{aligned} F_{*}(X_p)(fg) &= X_p F^{*}(fg) \\ &= X_p[(f \circ F)(g \circ F)] \\ &= X_p(f \circ F)g(F(p)) + f(F(p))X_p(f \circ G), \end{aligned}$$

and so we obtain

$$F_{*}(X_p)(fg) = (F_{*}(X_p)f)g(F(p)) + f(F(p))F_{*}(X_p)g.$$

Thus $F_* : T_p(M) \rightarrow T_{F(P)}(N)$. Further, F_* is a homeomorphism

$$F_*(\alpha X_p + \beta Y_p) f = (\alpha X_p + \beta Y_p)(F \circ f)$$

$$= \alpha X_p(F \circ f) + \beta Y_p(F \circ f)$$

$$= \alpha F_*(X_p) f + \beta F_*(Y_p) f$$

$$= [\alpha F_*(X_p) + \beta F_*(Y_p)] f.$$

Corollary 2.9. If $F : M \rightarrow N$ is a diffeomorphism of M onto an open set $U \subset N$ and $p \in M$, then $F_* : T_p(M) \rightarrow T_{F(P)}(N)$ is an isomorphism onto.

Example 2.10. Let F be the diffeomorphism from S^2 into N defined by $F(x, y, z) = (\frac{x}{2}, y, z)$. Here we calculate $F_* : T_p(S^2) \rightarrow T_{F(P)}(N)$.

Let $(x(t), y(t), z(t))$ $(a \le t \le b)$ be a curve on the unit sphere S^2 and $P = (x(t_0), y(t_0), z(t_0))$ $(a \le t_0 \le b)$. Then $F(x(t), y(t), z(t)) = (\frac{x(t)}{2}, y(t), z(t))$ is a curve on the ellipsoid $N = \{(x, y, z) : 4x^2 + y^2 + z^2 = 1\}$.

Furthermore, by Example 2.6.,

$(x'(t_0), y'(t_0), z'(t_0))$ is a tangent vector at P on the unit sphere S^2, and similarly,

$$\left.\frac{d}{dt}\right|_{t=t_0} F(x(t),\ y(t),\ z(t))$$

is a tangent vector at F(P) on the ellipsoid N.

From the calculation of

$$\left.\frac{d}{dt}\right|_{t=t_0} F(x(t),\ y(t),\ z(t))$$

$$= \begin{pmatrix} \frac{\partial}{\partial x}(\frac{x}{2}) & \frac{\partial}{\partial y}(\frac{x}{2}) & \frac{\partial}{\partial z}(\frac{x}{2}) \\ \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) & \frac{\partial}{\partial z}(y) \\ \frac{\partial}{\partial x}(z) & \frac{\partial}{\partial y}(z) & \frac{\partial}{\partial z}(z) \end{pmatrix} \begin{pmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{pmatrix},$$

$(\frac{1}{2}x'(t_0),\ y'(t_0),\ z'(t_0))$ is a tangent vector at F(P) on the manifold N.

Hence, we can think

$$F_* = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark We see that if (U, φ) is a coordinate on M, from corollary 2.9. the coordinate map φ then induces an isomorphism $\varphi_* : T_p(M) \to T_{\varphi(p)}(R^n)$ of the tangent space at each point $p \in U$ onto $Ta(R^n)$, $a = \varphi(p)$.

To establish a coordinate frame at a point p on a manifold, we first investigate a basis of the tangent space in the manifold R^n. We note that any vector $X_p = (x_1, x_2, \cdots, x_n)$ with the initial point $P = (x_1(0), x_2(0), \cdots, x_n(0))$ is a tangent vector at P on R^n, since we think that

$$X_p(f) = \lim_{\triangle t \downarrow 0} \frac{f(x_1(t), x_2(t), \cdots, x_n(t)) - f(P)}{\triangle t}$$

$$= Df(P) \cdot X_p, \ (\triangle t = t - t_0).$$

We note that

$$Df(P) \cdot X_p = \left(\frac{\partial f}{\partial x_1}(P), \frac{\partial f}{\partial x_2}(P), \cdots, \frac{\partial f}{\partial x_n}(P)\right) \cdot (x_1, x_2, \cdots, x_n)$$

$$= x_1 \frac{\partial f}{\partial x_1}(P) + x_2 \frac{\partial f}{\partial x_2}(P) + \cdots + x_n \frac{\partial f}{\partial x_n}(P).$$

Hence we can rewrite

$$X_p(f) = x\frac{\partial f}{\partial x}(P) + y\frac{\partial f}{\partial y}(P) + z\frac{\partial f}{\partial z}(P) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)_P(f).$$

So it is reasonable to write that

$$X_p = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \text{ .}$$

Hence X_p is a linear combination of x, y, z with a kind of frame

$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$. So we usually think that $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ is a basis of R^n.

From corollary 2.9., we may put $E_{ip} = \varphi_*^{-1}(\frac{\partial}{\partial x_i})$.

Example 2.10. In the two-dimensional manifold $M=S^2$, we calculate the coordinate frame at $P=(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$. since $\varphi_1 : U_1 \to R^2$, we think $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ is a basis at $\varphi_1(P)=(\frac{2+\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2})$

$$\varphi_{1*}^{-1}(\frac{\partial}{\partial u})(f) = \frac{\partial}{\partial u}(f\ \varphi_1^{-1})\Big|_{\varphi_1(P)}.$$

Let $f(x, y, z)=x$, $g(x, y, z)= y$ and $h(x,y,z)=Z$. Then

$$E_{1p}(f) = \varphi_{1*}^{-1}(\frac{\partial}{\partial u})(f) = \frac{\partial}{\partial u}(f\circ\varphi_1^{-1})\Big|_{\varphi_1(P)} = \frac{3-2\sqrt{2}}{4}$$

$$E_{1p}(g) = \varphi_{1*}^{-1}(\frac{\partial}{\partial u})(g) = \frac{\partial}{\partial u}(g\circ\varphi_1^{-1})\Big|_{\varphi_1(P)} = -\frac{1}{4}$$

$$E_{1p}(h) = \varphi_{1*}^{-1}(\frac{\partial}{\partial u})(h) = \frac{\partial}{\partial u}(h\circ\varphi_1^{-1})\Big|_{\varphi_1(P)} = \frac{2-\sqrt{2}}{4}$$

$$E_{2p}(f) = \varphi_{1*}^{-1}(\frac{\partial}{\partial v})(f) = \frac{\partial}{\partial v}(f\circ\varphi_1^{-1})\Big|_{\varphi_1(P)} = -\frac{1}{4}$$

$$E_{2p}(g) = \varphi_{1*}^{-1}(\frac{\partial}{\partial v})(g) = \frac{\partial}{\partial v}(g\circ\varphi_1^{-1})\Big|_{\varphi_1(P)} = \frac{3-2\sqrt{2}}{4}$$

$$E_{2p}(h) = \varphi_{1*}^{-1}(\frac{\partial}{\partial v})(h) = \frac{\partial}{\partial v}(h\circ\varphi_1^{-1})\Big|_{\varphi_1(P)} = \frac{2-\sqrt{2}}{4} \text{ .}$$

So

$$E_{1p} = \left(\frac{3-2\sqrt{2}}{4}, -\frac{1}{4}, \frac{2-\sqrt{2}}{4}\right),$$

$$E_{2p} = \left(-\frac{1}{4}, \frac{3-2\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}\right).$$

Hence

$$\left\{\left(\frac{3-2\sqrt{2}}{4}, -\frac{1}{4}, \frac{2-\sqrt{2}}{4}\right), \left(-\frac{1}{4}, \frac{3-2\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}\right)\right\}$$

is a frame at $P = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ on the unit sphere in R^3.

III. SOME PROPERTIES OF THE VECTOR FIELD ON A C^∞ - MANIFOLD M

Definition 3.1. A Vector field X of class C^∞ on M is a function assigning to each point p of M a vector $X_p \in T_p(M)$ whose components in the frame of any local coordinate (U, φ) are functions of class C^∞ on the domain U of the coordinates. Throughout this thesis, we will use a vector field to mean a C^∞ - vector field.

Lemma 3.2. If X is a C^∞ - vector field on U and f is a C^∞ - function on U, then $f \mapsto X_f$ map $C^\infty(U) \rightarrow C^\infty(U)$.

Proof. Let X_f be defined by $(X_f)(p) = X_p f$, and let the components of X be the functions $\alpha^1(p), \cdots, \alpha^n(p)$ so that $X = \sum_{i=1}^{n} \alpha^i E_{ip}$.

Then we have

$$\begin{aligned}(Xf)(p) &= X_p f \\ &= [\sum_{i=1}^{n} \alpha^i(p) E_{ip}](f) \\ &= \sum_{i=1}^{n} \alpha^i(p) E_{ip}(f)\end{aligned}$$

$$= \sum_{i=1}^{n} \alpha^i(p)\, \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)(f)$$

$$= \sum_{i=1}^{n} \alpha^i(p) \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})\Big|_{\varphi(P)}$$

$$= \sum_{i=1}^{n} \alpha^i(p)\left(\frac{\partial f}{\partial x^i}\right)(p),$$

$\alpha^i(p) \in C^\infty(U)$ and $\frac{\partial f}{\partial x^i} \in C^\infty(U)$ induce that the Xf is C^∞ – function on U.

Lemma 3.3. A vector field X is a linear map of $C^\infty(U)$ to $C^\infty(U)$.

Proof. For all $\alpha, \beta \in R$, and C^∞ – function f, g on U,

$$
\begin{aligned}
[X(\alpha f+\beta g)](p) &= X_p(\alpha f + \beta g) \\
&= \alpha (X_p f) + \beta (X_p g) \\
&= \alpha (Xf)(p) + \beta (Xg)(p), \\
[X(fg)](p) &= (X_p f)g(p) + f(p)(X_p g) \\
&= [(Xf)(p)]g(p) + f(p)\,[(Xg)(p)].
\end{aligned}
$$

Definition 3.4. If X and Y are C^∞ -vector fields, then the product of X and Y defined by $[X, Y] = XY - YX$ is called the bracket of X and Y, where XY is an operator on C^∞ -function on M.

Remark Here $XY(f) = X(Yf)$ is a C^∞ - function by **Lemma** 3.3..

We denote by $\mathfrak{X}(M)$ the set of all C^∞ -vector fields defined on the C^∞ -manifold M. It is itself a vector space over **R**, since if X and Y are C^∞ -vector field on M so is any linear combination of them with constant coefficients. In fact any linear combinaton with coefficients which are C^∞ — functions on M is again a C^∞ -vector field. For $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ implies that the vector field $Z = fX + gY$, with the obvious definition

$Z_p = f(p)X_p + g(p)Y_p$ for each $p \in M$, is a C^∞ -vector field.

Definition 3.5. We shall say that a vector space $\mathscr{L}$ over **R** is a Lie algebra if in addition to its vector space structure it possesses a product, that is, a map $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$, taking the pair (X, Y) to the element [X, Y] of $\mathscr{L}$ which has the following properties :

(i) it is bilinear over R :

$$[\alpha_1 X_1 + \alpha_2 X_2 Y] = \alpha_1[X_1, Y] + \alpha_2[X_2, Y],$$

$$[X, \alpha_1 Y_1 + \alpha_2 Y_2] = \alpha_1[X, Y_1] + \alpha_2[X, Y_2]$$

(ii) it is skew commutative :

$$[X, Y] = -[Y, X]$$

(iii) it satisfies the Jacobi identity :

$$[X, [Y, Z]]+[Y, [Z, X]]+[Z, [X, Y]] = 0.$$

Example 3.6. A vector space V^3, of dimension 3 over R with the usual vector product of vector calculus is a Lie algebra.

To show that, we let $X=(x^1, x^2, x^3)$, $Y=(y^1, y^2, y^3)$ and $Z=(z^1, z^2, z^3)$ be vectors in a vector space V^3. Then

(i) $[\alpha_1 X + \alpha_2 Y, Z]$

$$= [\alpha_1(x^1, x^2, x^3) + \alpha_2(y^1, y^2, y^3), (z^1, z^2, z^3)]$$

$$= [(\alpha_1 x^1 + \alpha_2 y^1, \alpha_1 x^2 + \alpha_2 y^2, \alpha_1 x^3 + \alpha_2 y^3), (z^1, z^2, z^3)]$$

$$= (\alpha_1 x^1 + \alpha_2 y^1, \alpha_1 x^2 + \alpha_2 y^2, \alpha_1 x^3 + \alpha_2 y^3) \times (z^1, z^2, z^3)$$

$$= \{(\alpha_1 x^2 + \alpha_2 y^2) z^3 - (\alpha_1 x^3 + \alpha_2 y^3) z^1\} e_1 - \{(\alpha_1 x^1 + \alpha_2 y^1) z^3$$

$$- (\alpha_1 x^3 + \alpha_2 y^3 z^1\} e_2 + \{(\alpha_1 x^1 + \alpha_2 y^1) z^3 - (\alpha_1 x^3 + \alpha_2 y^3) z^1\} e_3$$

$$= \alpha_1\{(x^2 z^3 - x^3 z^2) e_1 - (x^1 z^3 - x^3 z^1) e_2 + (x^1 z^3 - x^3 z^1) e_3\}$$

$$+ \alpha_2\{(y^2 z^3 - y^3 z^2) e_1 - (y^1 z^3 - y^3 z^1) e_2 + (y^1 z^3 - y^3 z^1) e_3\}$$

$$= \alpha_1[X, Z] + \alpha_2[Y, Z].$$

By similar method,

$$[X, \alpha_1 Y + \alpha_2 Z] = \alpha_1[X, Y] + \alpha_2[X, Z].$$

(ii) $[X, Y]$

$$= [(x^1, x^2, x^3), (y^1, y^2, y^3)]$$

$$= (x^1, x^2, x^3) \times (y^1, y^2, y^3)$$

$$= (x^2 y^3 - x^3 y^2) e_1 - (x^1 y^3 - x^3 y^1) e_2 + (x^1 y^2 - x^2 y^1) e_3$$

$$= -(y^2 x^3 - y^3 x^2) e_1 + (y^1 x^3 - y^3 x^1) e_2 - (y^1 x^2 - y^2 x^1) e_3$$

$$= -\{(y^2 x^3 - y^3 x^2) e_1 - (y^1 x^3 - y^3 x^1) e_2 + (y^1 x^2 - y^2 x^1) e_3\}$$

$$= -(y^1, y^2, y^3) \times (x^1, x^2, x^3)$$

$$= -[(y^1, y^2, y^3), (x^1, x^2, x^3)]$$

$$= -[Y, X]$$

(iii) [X, [Y, Z]]

$$= [\, (x^1,\ x^2,\ x^3),\ (y^1,\ y^2,\ y^3) \times (z^1,\ z^2,\ z^3) \,]$$

$$= [\, (x^1,\ x^2,\ x^3),\ (y^2z^3 - y^3z^2,\ y^1z^3 - y^3z^1,\ y^1z^2 - y^2z^1) \,]$$

$$= \{x^2(y^1z^2 - y^2z^1) - x^3(y^1z^3 - y^3z^1)\}\, e_1$$

$$- \{x^1(y^1z^2 - y^2z^1) - x^3(y^2z^3 - y^3z^2)\}\, e_2$$

$$+ \{x^1(y^1z^3 - y^3z^1) - x^2(y^2z^3 - y^3z^2)\}\, e_3 \cdots\cdots (1)$$

By the same computation,

[Y, [Z, X]]

$$= \{y^2(z^1x^2 - z^2x^1) - y^3(z^1x^3 - z^3x^1)\}\, e_1$$

$$- \{y^1(z^1x^2 - z^2x^1) - y^3(z^2x^3 - z^3x^2)\}\, e_2$$

$$+ \{y^1(z^1x^3 - z^3x^1) - y^2(z^2x^3 - z^3x^2)\}\, e_3 \cdots\cdots (2)$$

[Z, [X, Y]]

$$= \{z^2(x^1y^2 - x^2y^1) - z^3(x^1y^3 - x^3y^1)\}\, e_1$$

$$- \{z^1(x^1y^2 - x^2y^1) - z^3(x^2y^3 - x^3y^2)\}\, e_2$$

$$+ \{z^1(x^1y^3 - x^3y^1) - z^2(x^2y^3 - x^3y^2)\}\, e_3 \cdots\cdots (3)$$

Adding both sides of (1), (2) and (3),

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Example 3.7. Let $M_n(R)$ denote the algebra of $n \times n$ matrices over R with XY denoting the usual matrix product of X and Y.

Then $[X, Y] = XY - YX$, the commutator of X and Y, defines a Lie algebra structure on $M_n(R)$: For, let X, Y and Z be $n \times n$ matrices in $M_n(R)$, and let $\alpha, \beta \in R$. Then

(i) $$\begin{aligned}[\alpha X + \beta Y, Z] &= (\alpha X + \beta Y)Z - Z(\alpha X + \beta Y) \\ &= \alpha(XZ) + \beta(YZ) - \alpha(ZX) - \beta(ZY) \\ &= \alpha(XZ - ZX) + \beta(YZ - ZY) \\ &= \alpha[X, Z] + \beta[Y, Z],\end{aligned}$$

$$\begin{aligned}[X, \alpha Y + \beta Z] &= X(\alpha Y + \beta Z) - (\alpha Y + \beta Z)X \\ &= \alpha(XY) + \beta(XZ) - \alpha(YX) - \beta(ZX) \\ &= \alpha(XY - YX) + \beta(XZ - ZX) \\ &= \alpha[X, Y] + \beta[X, Z].\end{aligned}$$

(ii) $[X, Y] = XY - YX = -(YX - XY) = -[Y, X].$

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$

$$= X[Y, Z] - [Y, Z]X + Y[Z, X] - [Z, X]Y + Z[X, Y] - [X, Y]Z$$

$$= X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) - (ZX - XZ)Y$$

$$+ Z(XY - YX) - (XY - YX)Z$$

$$= X(YZ) - X(ZY) - (YZ)X + (ZY)X + Y(ZX) - Y(XZ) - (ZX)Y$$

$$+ (XZ)Y + Z(XY) - Z(YX) - (XY)Z + (YX)Z$$

$$= 0.$$

Now suppose that X and Y denote C^{∞}-vector fields on a manifold M, that is, X, $Y \in \mathfrak{X}(M)$. Then, in general, the operator $f \longmapsto X_p(Yf)$ defined on $C^{\infty}(p)$ does not define a vector at p. Thus XY, considered as an operator on C^{∞}-functions on M, does not determine a C^{∞}-vector field. However, oddly enough, $XY - YX$ does ; it defines a vector field $Z \in \mathfrak{X}$ according to the prescription

$$Z_pf = (XY - YX)_pf = X_p(Yf) - Y_p(Xf), \; f \in C^{\infty}(p).$$

Theorem 3.8. $Z \in \mathfrak{X}(M)$ is a C^{∞}-vector field on a manifold M.

Proof : If $f \in C^{\infty}(p)$, then Xf and Yf are C^{∞} on a neighborhood of p, and the prescription above determines a linear map of $C^\infty(p) \to \mathbf{R}$.

Therefore if the property (ii) of **Definition** 2.5. holds for Z_p, then Z_p is an element of $T_p(M)$ at each $p \in M$. Consider f, $g \in C^\infty(p)$.

Then f, $g \in C^\infty(U)$ for some open set U containing p. We have the relations :

$$
\begin{aligned}
(XY - YX)_p\ (fg) &= X_p(Yfg) - Y_p(Xfg) \\
&= X_p(fYg - gYf) - Y_p(fXg - gXf) \\
&= (X_pf)\ (Yg)_p + f(p)X_p(Yg) - (X_pg)\ (Yf)_p - g(p)X_p\ (Yf) \\
&\quad - (Y_pf)\ (Xg)_p - f(p)Y_p(Xg) + (Y_pg)\ (Xf)_p + g(p)(Y_pXf),
\end{aligned}
$$

so that

$$
\begin{aligned}
Z_p(fg) &= (XY - YX)_p\ (fg) \\
&= f(p)\ (XY - YX)_pg - g(p)\ (XY - YX)_pf \\
&= f(p)Z_pg + g(p)Z_pf.
\end{aligned}
$$

Finally, if f is C^∞ on any open set $U \subset M$, then so is $(XY - YX)f$, and therefore Z is a C^∞ − vector field on M.

Theorem 3.9. $\mathfrak{X}(M)$ with the product $[X, Y]$ is a Lie algebra.

Proof : If $\alpha, \beta \in R$ and X_1, X_2, Y are C^∞ – vector fields, then it is straightforward to verify that

$$[\alpha X_1 + \beta X_2, Y]f = \alpha[X_1, Y]f + \beta[X_2, Y]f.$$

Thus $[X, Y]$ is linear in the first variable. Since the skew commutativity $[X, Y] = -[Y, X]$ is immediate from the definition, we see that linearity in the first variable implies linearity in the second. Therefore $[X, Y]$ is bilinear and skew-commutative.

Using the definition, we obtain

$$\begin{aligned}[X, [Y, Z]]f &= X(([Y, Z]f) - [Y, Z](Xf) \\ &= X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf) + Z(Y(Xf)).\end{aligned}$$

If we evaluate $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$ applied to a C^∞ – function f, then the Jacobi identity follows immediately.

Theorem 3.10. For any C^∞ – vector field $X, Y \in \mathfrak{X}(M)$ and C^∞ – function f on M, we have the relation :

$$[X, fY] = (Xf)Y + f[X, Y].$$

Proof. By means of bracket of X and fY, at each point p on the C^{∞} −manifold M,

$$(X(fY) - (fY)X)\ (p) = X_p(fY) - (fY)_pX$$

$$= (X_pf(p))Y + f(p)X_pY - f(p)Y_pX$$

$$= (X_pf(p))Y + f(p)\,(X_pY - Y_pX)$$

$$= (Xf)_pY + f(p)\ (XY - YX)_p.$$

Theorem 3.11. Let $F: M \to N$ be a C^{∞} − mapping and suppose that X_1, X_2 and Y_1, Y_2 are vector fields on N, M respectively, which are F-related, that is, for $i = 1, 2$, $F_*(X_i) = Y_i$. Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related, that is, $F_*[X_1, X_2] = [F_*(X_1),\ F_*(X_2)]$.

Proof. Before proving the theorem we note the following necessary and sufficient condition for X on N and Y on M to be F-related : for any g which is C^{∞} on some open set $V \subset M$,

(*) $$(Y_g)\ F = X(g\ F)$$

on $F^{-1}(V)$. This is essentially a restatement of the definition of F-related, for if $p \in F^{-1}(V)$, then $F_*(X_p)g = X_p(g \circ F) = X(g \circ F)(p)$; and $Y_{F(p)}g$ is the value of the C^∞ − function Yg at $F(p)$, that is, $((Yg) \circ F)(p)$. Thus the condition holds if and only if $F_*(X_p) = Y_{F(p)}$ for all $p \in M$.

Returning to the proof we consider $f \in C^\infty(V)$, $V \subset M$, so that $Y_1 f$ and $Y_2 f \in C^\infty(V)$ also. Apply (∗) first with $g = Y_2 f$ and then with $g = f$ giving the equalities

$$[Y_1(Y_2 f)] \circ F = X_1((Y_2 f) \circ F) = X_1[X_2(f \circ F)].$$

Interchanging the roles of Y_1, Y_2 and X_1, X_2 and substracting, we obtain

$$([Y_1, Y_2]f) \circ F = [X_1, X_2](f \circ F),$$

which according to (∗) is eguivalent to $[X_1, X_2]$ and $[Y_1, Y_2]$ being F- related.

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