LINEAR OPERATORS ON A HILBERT SPACE ℓ^2

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CHAPTER I
INTRODUCTION

The most important single operator which plays a vital role in all parts of Hilbert space theory is the unilateral shift. Perhaps the simplest way to define it is to consider the Hilbert space ℓ^2 of square summable sequences. The theory of linear operators and problems in Hilbert space were introduced by P. R. Halmos, I. D. Berd, D. A. Herrero etc, in term of shifts on Hilbert space. Also the theory of shifts on Hilbert space was used by Sz-Nagy and Foias (1970) in the study of the geometry of space of minimal unitary dilations of contractions.

In this paper we study properties of some Hilbert spaces analogous to ℓ^2 space, and investigate the consequences for spectra of shifts, compact operators related to shifts. The outline of the present paper is as follows. In chapter II, we define several Hilbert spaces analogous to ℓ^2 space, and study properties (completeness, separability etc) of these spaces. Also we give two examples of subspaces according to completeness. In chapter III, we investigate characterizations of linear functionals on several Hilbert spaces, by Riesz Representation theorem, and obtain results for spectra (point spectrum, approximate point spectrum etc) of unilateral shifts and weighted shifts. In chapter IV, we study the topological concepts and properties of the space of bounded linear operators on ℓ^2 space. Using these concepts we obtain the equivalent condition of compact operator on ℓ^2 space, and investigate the compactness of operator corresponding to infinite matrix.

CHAPTER II
SOME HILBERT SPACES

LEMMA 2.1. [7]. If $\xi_1, \xi_2, \cdots$ and $\eta_1, \eta_2, \cdots$ are complex numbers such that $\sum_{k=1}^{\infty} |\xi_k|^2 < \infty$, $\sum_{k=1}^{\infty} |\eta_k|^2 < \infty$.

(*a*) $\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq (\sum_{k=1}^{\infty} |\xi_k|^2)^{\frac{1}{2}} (\sum_{m=1}^{\infty} |\eta_m|^2)^{\frac{1}{2}}$

(CAUCHY-SCHWARZ INEQUALITY)

(*b*) $(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^2)^{\frac{1}{2}} \leq (\sum_{k=1}^{\infty} |\xi_k|^2)^{\frac{1}{2}} + (\sum_{m=1}^{\infty} |\eta_m|^2)^{\frac{1}{2}}$

(MINKOWSKI INEQUALITY)

PROOF. (a) Let $(\tilde{\xi}_j)$ and $(\tilde{\eta}_j)$ be such that $\Sigma |\tilde{\xi}_i|^2 = 1$, $\sum |\tilde{\eta}_j|^2 = 1$. Since $|\tilde{\xi}_j \tilde{\eta}_j| \leq \frac{1}{2}(|\tilde{\xi}_j|^2 + |\tilde{\eta}_j|^2)$, summing over j we obtain $\sum |\tilde{\xi}_j \tilde{\eta}_j| \leq \frac{1}{2} + \frac{1}{2} = 1$. (1)

We now take any nonzero $x = (\xi_j)$ and $y = (\eta_j)$ such that $\sum |\xi_k|^2 < \infty$ and $\sum |\eta_k|^2 < \infty$ and let

$$\tilde{\xi}_j = \frac{\xi_j}{(\sum |\xi_k|^2)^{\frac{1}{2}}}, \quad \tilde{\eta}_j = \frac{\eta_j}{(\sum |\eta_k|^2)^{\frac{1}{2}}} \qquad (2)$$

Then $\Sigma |\tilde{\xi}_j|^2 = \Sigma |\tilde{\eta}_j|^2 = 1$. Substituting (2) into (1), we obtain the desired inequality.

(b) Write $\xi_j + \eta_j = \omega_j$. The triangle inequality for numbers gives

$$|\omega_j|^2 = |\xi_j + \eta_j| \; |\omega_j| \leqq (|\xi_j| + |\eta_j|)|\omega_j| .$$

Summing over j from 1 to any fixed n, we obtain

$$\sum_{j=1}^{n} |\omega_j|^2 \leq \sum_{j=1}^{n} |\xi_j||\omega_j| + \sum_{j=1}^{n} |\eta_j||\omega_j| .$$

Applying the Cauchy-Schwarz inequality to the sums on the right, we obtain

$$\sum_{j=1}^{n} |\omega_j|^2 \leq \left\{ [\sum_{k=1}^{n} |\xi_k|^2]^{\frac{1}{2}} + [\sum_{k=1}^{n} |\eta_k|^2]^{\frac{1}{2}} \right\} (\sum_{m=1}^{n} |\omega_m|^2)^{\frac{1}{2}} .$$

Dividing by the last factor on the right, we obtain the Minkowski inequality with n instead of ∞. Letting $n \to \infty$, the series on the left converges since on the right this inequality yields two converging series, and (b) is proved.

DEFINITION 2.2. [1, 2, 3]. The set of all sequences $x=(\xi_n)=(\xi_1, \xi_2, \cdots)$ of complex numbers for which $\sum_{n=1}^{\infty} |\xi_n|^2 < \infty$ is denoted by $\ell^2(\mathbb{C})$.

Define the addition and scalar multiplication by $x+y=(\xi_n + \eta_n)$, $\alpha x = (\alpha \xi_n)$ where $x=(\xi_n)$, $y=(\eta_n) \in \ell^2(\mathbb{C})$ and $\alpha \in \mathbb{C}$. Then by Lemma 2.1.(b), $x+y \in \ell^2(\mathbb{C})$ and also $\alpha x \in \ell^2(\mathbb{C})$. Hence $\ell^2(\mathbb{C})$ is a vector space. Also by Lemma 2.1.(a), the series $\sum_{n=1}^{\infty} |\xi_n \eta_n|$ converges. With the inner product defined by $(x, y) = \sum_{n=1}^{\infty} \xi_n \bar{\eta}_n$, $\ell^2(\mathbb{C})$ is a complex inner product space. The norm $\| x \|$, on $\ell^2(\mathbb{C})$ is defined by $\| x \| = (x, x)^{\frac{1}{2}} = (\sum_{n=1}^{\infty} |\xi_n|^2)^{\frac{1}{2}}$. Then $\ell^2(\mathbb{C})$ is a normed space.

DEFINITION 2.3. [7]. A space X is said to be complete if every Cauchy sequence in X converges.

LEMMA 2.4. [7, 8]. $\ell^2(\mathbb{C})$ is complete.

PROOF. Let (x_n) be any Cauchy sequence in the space ℓ^2 where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \cdots)$. Then for every $\varepsilon > 0$ there is a positive integer N such that for all m, n> N $|x_m - x_n| = (\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}|^2)^{\frac{1}{2}} < \varepsilon$. (1)

It follows that for every j=1, 2,……, we have $|\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon$ (m, n>N) (2). We choose a fixed j. Form (2) we see that $(\xi_j^{(1)}, \xi_j^{(2)}, \cdots\cdots)$ is a Cauchy sequence of numbers. It converges since R and $\mathbb{C}$ are complete, say $\xi_j^{(m)} \to \xi_j$ as $m \to \infty$. Using these limits, we define $x = (\xi_1, \xi_2, \cdots)$ and show that $x \in \ell^2(\mathbb{C})$ and $x_m \to x$. From (1) we have for all m, n>N

$$\sum_{j=1}^{k} |\xi_j^{(m)} - \xi_j^{(n)}|^2 < \varepsilon^2 (k = 1, 2, \cdots).$$

Letting $n \to \infty$, we obtain for m>N

$$\sum_{j=1}^{k} |\xi_j^{(m)} - \xi_j|^2 \leq \varepsilon^2 (k = 1, 2, \cdots).$$

Letting $k \to \infty$, then we have for m>N

$$\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j|^2 \leq \varepsilon^2. \quad (3)$$

This shows that $x_m - x = (\xi_j^{(m)} - \xi_j) \in \ell^2(\mathbb{C})$. Since $x_m \in \ell^2(\mathbb{C})$, it follows by means of the Minkowski inequality that $x = x_m + (x - x_m) \in \ell^2(\mathbb{C})$. Furthermore (3) implies that $x_m \to x$. Since (x_m) was an arbitrary Cauchy sequence in $\ell^2(\mathbb{C})$, this proves completeness of $\ell^2(\mathbb{C})$.

From this theorem, $\ell^2(\mathbb{C})$ is a Hilbert space.

DEFINITION 2.5. [7]. A space X is separable if there exists a countable dense subset in X.

THEOREM 2.6. [2, 7]. $\ell^2(\mathbb{C})$ is a separable Hilbet space.

PROOF. From Lemma 2.4, it suffices to show that $\ell^2(\mathbb{C})$ is separable, Let M be the set of all sequence y of the form $y=(\eta_1, \eta_2, \cdots, \eta_n, 0, 0, \cdots)$ where n is any positive integer and the η_j's are complex rationals. Then M is countable. We show that M is dense in $\ell^2(\mathbb{C})$. Let $x=(\xi_n)\in \ell^2(\mathbb{C})$ be arbitrary. Then for every $\varepsilon>0$ there is an n (depending on ε) such that $\sum_{j=n+1}^{\infty} |\xi_j|^2 < \frac{\varepsilon^2}{2}$.

Since the complex rationals are dense in $\mathbb{C}$, for each ξ_j there is a complex rational η_j close to it. Hence we can find a $y\in$M satisfying $\sum_{j=1}^{n} |\xi_j - \eta_j|^2 < \frac{\varepsilon^2}{2}$.

It follows that $\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2 = \sum_{j=1}^{n} |\xi_j - \eta_j|^2 + \sum_{j=n+1}^{\infty} |\xi_j|^2 < \varepsilon^2$.

Hence M is dense in $\ell^2(\mathbb{C})$.

We note that the standard basis $e_1=(1, 0, 0, \cdots)$, $e_2=(0, 1, 0, \cdots), \cdots$ is an infinite orthonormal system in $\ell^2(\mathbb{C})$.

We shall denote by $\ell^2_+(\mathbb{C})$ the set of all sequences $x=(\xi_n)=(\xi_0, \xi_1, \xi_2, \cdots)$ of complex numbers for which $\sum_{n=0}^{\infty} |\xi_n|^2 < \infty$. Also we denote by $\ell^2_\pm(\mathbb{C})$ the set of all sequence $x=(\xi_n)=(\cdots \xi_{-1}, \xi_0, \xi_1, \cdots)$ of complex numbers for which $\sum_{n=-\infty}^{\infty} |\xi_n|^2 < \infty$.

Similarly we can show that $\ell^2_+(\mathbb{C})$ and $\ell^2_\pm(\mathbb{C})$ are Hilbert spaces.

Let H be a Hilbert space with inner product $\langle\, , \rangle$. Let ℓ^2(H) denote the set of all sequences (x_n) of elements in H for which $\sum_{n=1}^{\infty} \| x_n \|^2 < \infty$. Define addition and scalar multiplication by $x+y=(x_n+y_n)$, $\alpha x=(\alpha x_n)$ where $x=(x_n)$, $y=(y_n)\in$

ℓ^2(H). Then ℓ^2(H) is clearly a vector space by Schwarz inequality and triangle inequality. Also we define the inner product (,) on ℓ^2(H) by $(x, y) = \sum_{n=1}^{\infty} \langle x_n, y_n \rangle$, and the morn $\| x \|$ on ℓ^2(H) is defined by $\| x \| = (x, x)^{\frac{1}{2}} = (\sum_{n=1}^{\infty} \langle x_n, x_n \rangle)^{\frac{1}{2}}$. Hence ℓ^2(H) is a normed space, and also ℓ^2(H) is clearly an inner product space.

THEOREM 2.7. ℓ^2(H) is a Hilbert space.

PROOF. From the above remark it suffices to show that ℓ^2(H) is complete. Let (a_n) be a Cauchy sequence in the space ℓ^2(H) where $a_m = (x_1^{(m)}, x_2^{(m)}, \cdots)$. Then for every $\varepsilon > 0$ there exists an N such that for all m, n>N

$$\| a_m - a_n \| = (\sum_{k=1}^{\infty} \| x_k^{(m)} - x_k^{(n)} \|^2)^{\frac{1}{2}} < \varepsilon . \qquad (1)$$

It follows that for every j=1, 2,⋯ we have

$$\| x_k^{(m)} - x_k^{(n)} \| < \varepsilon \ (m, n > N). \qquad (2)$$

We choose a fixed k. From (2) we see that $(x_k^{(1)}, x_k^{(2)}, \cdots)$ is a Cauchy sequence of elements in H. It converges since H are complete, say $x_k^{(m)} \to x_k$ as m → ∞. Using these limits, we define $a = (x_1, x_2, \cdots)$ and show that $a \in \ell^2$(H) and $a_n \to a$. From (1) we have for all m, n>N

$$\sum_{k=1}^{j} \| x_k^{(m)} - x_k^{(n)} \|^2 < \varepsilon^2 \ (j = 1, 2, \cdots) .$$

Letting n → ∞, we obtain for m>N

$$\sum_{k=1}^{j} \| x_k^{(m)} - x_k \|^2 < \varepsilon^2 \ (j = 1, 2, \cdots) .$$

Letting $j \to \infty$, we have for $n > N$

$$\sum_{k=1}^{\infty} \| x_k^{(m)} - x_k \|^2 \leq \varepsilon^2 . \qquad (3)$$

This shows that $a_m - a = (x_k^{(m)} - x_k) \in \ell^2(H)$. Since $a_m \in \ell^2(\mathbb{C})$, it follows from Lemma 2.1.(b) that $a = a_m + (a - a_m) \in \ell^2(H)$. Since (a_m) was an arbitrary Cauchy sequence in $\ell^2(H)$, this proves completeness of $\ell^2(H)$.

Let $\omega = (\omega_1, \omega_2, \cdots)$ be a sequence of strictly positive numbers and let $\ell^2_\omega(\mathbb{C})$ to be the set of complex sequences $x = (\xi_1, \xi_2, \cdots)$ with $\sum_{j=1}^{\infty} \omega_j |\xi_j|^2 < \infty$. With respect to the coordinate linear operations and the inner product defined by $(x, y)_\omega = \sum_{j=1}^{\infty} \omega_j \xi_j \bar{\eta}_j$. We can similarly show that the set $\ell^2_\omega(\mathbb{C})$ is a Hilbert space. In particular, if $\omega = (1, \alpha, \alpha^2, \cdots)$ for $\alpha > 1$ then $x = (1, \frac{1}{\alpha}, \frac{1}{\alpha^2}, \cdots) \in \ell^2_\omega(\mathbb{C})$.

On the other hand, let $\ell^2(N \times N)$ be the set of all double sequences (ξ_{mn}) with $\sum_{m,n=1}^{\infty} |\xi_{mn}|^2 < \infty$. With respect to the coordinate linear operations and the inner product defined by $(\xi, \eta) = ((\xi_{mn}), (\eta_{mn})) = \sum_{m,n=1}^{\infty} \xi_{mn} \bar{\eta}_{mn}$, we see that $\ell^2(N \times N)$ is a Hilbert space.

THEOREM 2.8. Let $\ell^2_F(\mathbb{C})$ be the subspace of $\ell^2(\mathbb{C})$ consisting of all sequence (x_n) where $x_n = 0$ for all but at most finite number of n. With the inner product inherited from $\ell^2(\mathbb{C})$. $\ell^2_F(\mathbb{C})$ is an inner product space, but $\ell^2_F(\mathbb{C})$ is not a Hilbert space.

PROOF. Clearly $\ell^2_F(\mathbb{C})$ is an inner product space. It suffices to show that $\ell^2_F(\mathbb{C})$ is not complete.

Let $a_n = (\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\cdots, \frac{1}{2^n}, 0, \cdots\cdots, 0) \in \ell^2_F(\mathbb{C})$.

Then for n>m

$$\|a_n - a_m\|^2 = \|(\frac{1}{2}, \frac{1}{2^2}, \cdots\cdots, \frac{1}{2^n}, 0, \cdots\cdots, 0) -$$

$$(\frac{1}{2}, \frac{1}{2^2}, \cdots\cdots, \frac{1}{2^m}, 0, \cdots\cdots 0)\|^2$$

$$= \|(0, \cdots\cdots, 0, \frac{1}{2^{m+1}}, \cdots\cdots, \frac{1}{2^n}, 0, \cdots\cdots, 0\|^2$$

$$= \sqrt{0 + \cdots\cdots + 0 + (\frac{1}{2^{m+1}})^2 + \cdots\cdots + (\frac{1}{2^n})^2 + 0 + \cdots\cdots}$$

$$< \frac{1}{2^{m+1}} + \cdots\cdots + \frac{1}{2^n} < \sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^m}$$

Since $\| a_n - a_m \| \to 0$ as $n \to \infty$, $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, which converges in $\ell^2(\mathbb{C})$ to $a = (\frac{1}{2}, \frac{1}{2^2}, \cdots\cdots) \notin \ell^2_F(\mathbb{C})$. Consequently (x_n) can not converge to a vector in $\ell^2_F(\mathbb{C})$ since limits of sequence in $\ell^2(\mathbb{C})$ are unique.

THEOREM 2.9. Let $\ell^2_E(\mathbb{C})$ be the subspace of $\ell^2(\mathbb{C})$ consisting of all sequences (ξ_n) where $\xi_{2n}=0$ for all n. With the inner product inherited from $\ell^2(\mathbb{C})$, $\ell^2_E(\mathbb{C})$ is a Hilbert space.

PROOF. Clearly $\ell^2_E(\mathbb{C})$ is an inner product space. It suffices to show that $\ell^2_E(\mathbb{C})$ is complete. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in the space $\ell^2_E(\mathbb{C})$, where $x_n=(\xi_1^{(n)}, \xi_2^{(n)}, \cdots)$. Then $\xi_{2k}^{(n)}=0$ for every $k=1, 2, \cdots\cdots$, and $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in the space $\ell^2(\mathbb{C})$. By Theorem 2. 4, $(x_n)_{n=1}^{\infty}$ converges to ξ_k for every $k=1, 2, \cdots\cdots$, and $\xi_{2k}=0$ for $k=1,2,\cdots\cdots$. Hence $x \in \ell^2_E(\mathbb{C})$.

CHAPTER III
SPECTRA OF SHIFT OPERATORS

DEFINITION 3.1. [1,2]. Let H_1 and H_2 be Hilbert spaces over the complex number $\mathbb{C}$. A linear operator $T: H_1 \to H_2$ is called bounded if $\sup_{\|x\| \leq 1} \|Tx\| < \infty$. The norm of T, written $\|T\|$ is the nonnegative number $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$.

Let $B(H_1, H_2)$ be the set of all bounded linear operators from H_1 into H_2, and put $B(H,H) \equiv B(H)$. Note that $B(H, \mathbb{C})$ is the set of all bounded linear functional on H, and that for a bounded linear operator $T \in B(H_1, H_2)$,

(a) $\|T\| = \sup\{\|Tx\| / \|x\| : x \neq 0\} = \sup\{\|Tx\| : \|x\| = 1\}$

$= \inf\{c > 0 : \|Tx\| \leq c\|x\|, x \in H_1\}$.

(b) $\|T\| = \sup\{\langle Tx, y\rangle : \|x\| = \|y\| = 1, x \in H_1, y \in H_2\}$

$= \sup\{\langle Tx, y\rangle : \|x\| \leq 1, \|y\| \leq 1, x \in H_1, y \in H_2\}$.

EXAMPLE 3.2. (a) Let $S_r : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ be defined by $S_r(\xi_1, \xi_2, \cdots) = (0, \xi_1, \xi_2, \cdots)$. The operator S_r is called right shift operator (or unilateral shift) [2,3]. Obviously Sr is linear and $\|S_r x\| = \|x\|$ $(x \in \ell^2(\mathbb{C}))$. Thus S_r is an isometry of $\ell^2(\mathbb{C})$ into $\ell^2(\mathbb{C})$ and $\|S_r\| = 1$. In fact, S_r maps $\ell^2(\mathbb{C})$ onto a proper subspace, namely the set of all absolutely square sequence having first term zero. i. e., S_r is not surjective. Therefore S_r is not invertible.

(b) Let $S_\ell : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ be defined by $S_\ell(\xi_1, \xi_2, \cdots) = (\xi_2, \xi_3, \cdots)$. The operator S_ℓ is called left shift operator (or backward shift) [2,3]. Obviously S_ℓ is linear, and $\| S_\ell \| = 1$, but S_ℓ is not one-one. Hence S_ℓ is not invertible.

(c) Let $\omega = (\omega_n)$ be a sequence of complex numbers and let $T : \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C})$ be defined by $T(x_1, x_2, \cdots) = (\omega_1 x_1, \omega_2 x_2, \cdots)$. The operator T is called the diagonal operator [2,3]. It is easy to show that (1) T is bounded if and only if $\omega = (\omega_n)$ is bounded, in this case $\| T \| = \sup |\omega_n|$, (2) $\inf |\omega_n| \, \| x \| \leqq \| Tx \|$ for any $x \in \ell^2(\mathbb{C})$, and (3) T is invertible if and only if $\inf |\omega_n| > 0$.

(d) Certain infinite matrices give rise to bounded linear operators on $\ell^2(\mathbb{C})$ as follows: Given an infinite matrix $(a_{ij})_{i,j=1}^{\infty}$ where

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty, \quad (*) \qquad \text{define } T: \ \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C}) \ \text{ by}$$

$$T(\xi_1, \xi_2, \cdots\cdots) = (\eta_1, \eta_2, \cdots\cdots) \qquad \text{where} \quad y_i = \sum_{j=1}^{\infty} a_{ij} \xi_j .$$

The operator T is a bounded linear operator on ℓ^2 and

$$\| T \|^2 \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 ,$$

since $|\eta_i| \leq \sum_{j=1}^{\infty} |a_{ij} \xi_j| \leq (\sum_{j=1}^{\infty} |a_{ij}|^2)^{\frac{1}{2}} (\sum_{j=1}^{\infty} |\xi_j|^2)^{\frac{1}{2}}$ (by Cauchy-Schwarz inequality) implies that for $x = (\xi_1, \xi_2, \cdots\cdots)$

$$\| Tx \|^2 = \sum_{i=1}^{\infty} |\eta_i|^2 \leq \| x \|^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 .$$

Condition (*) is not a necessary condition for T to be bounded.

Since the identity matrix $(a_{ij}) = (\delta_{ij})$ does not satisfy (*), yet T=I. Here δ_{ij} denotes the Kronecker delta.

By Riesz Representation theorem [1], we see that a functional f on $\ell^2(\mathbb{C})$ is bounded and linear if and only if there exists a $y=(\eta_1, \eta_2, \cdots) \in \ell^2(\mathbb{C})$ such that for all $x = (\xi_1, \xi_2, \cdots\cdots) \in \ell^2(\mathbb{C})$, $f(x) = \sum_{n=1}^{\infty} \xi_n \bar{\eta}_n$.

Similarly we obtain the following theorem.

THEOREM 3.3. Let $\omega = \{\omega_n\}$ be a bounded sequence of positive numbers. A functional f is bounded linear on the Hilbert space ($\ell^2_\omega(\mathbb{C})$, $\langle\, ,\rangle_\omega$) if and only if there exists a $y=(\eta_1, \eta_2, \cdots) \in \ell^2_\omega(\mathbb{C})$ such that

$$f(x) = \langle x, y \rangle_\omega = \sum_{n=1}^{\infty} \omega_n \xi_n \bar{\eta}_n \quad \text{for all } x=(\xi_1, \xi_2, \cdots) \in \ell^2_\omega(\mathbb{C}).$$

PROOF. Suppose f is a bounded linear functional on $\ell^2_\omega(\mathbb{C})$. Then by Riesz Representation theorem, there exists a unique $y=(\eta_1, \eta_2, \cdots) \in \ell^2_\omega(\mathbb{C})$ such that for all $x=(\xi_1, \xi_2, \cdots) \in \ell^2_\omega(\mathbb{C})$, $f(x)=\langle x, y \rangle_\omega$. Hence $f(x) = \langle x, y \rangle_\omega = \sum_{n=1}^{\infty} \omega_n \xi_n \bar{\eta}_n$.

Conversely suppose that there exists a $y=(\eta_1, \eta_2, \cdots) \in \ell^2_\omega(\mathbb{C})$ such that $f(x) = \langle x, y \rangle_\omega = \sum_{n=1}^{\infty} \omega_n \xi_n \bar{\eta}_n$. Then f is clearly a linear functional and $\| f \| = \sum_{k=1}^{\infty} \omega_k | \eta_k |^2 < \infty$ so f is a bounded linear functional .

Also we apply the Riesz Representation theorem to $\ell^2(H)$, and see that a functional f on $\ell^2(H)$ is bounded and linear if and only if there exists a $y=(y_1, y_2, \cdots) \in \ell^2(H)$ such that for all $x=(x_1, x_2, \cdots) \in \ell^2(H)$, $f(x) = \sum_{n=1}^{\infty} \langle x_n, y_n \rangle$.

DEFINITION 3.4. [1,2]. If $T \in B(H_1, H_2)$, then the unique operator S in $B(H_2, H_1)$ satisfying $\langle Tx, y \rangle = \langle x, Sy \rangle$ for all x in H_1 and y in H_2 is called the adjoint of T and is denoted by $S = T^*$

We note that if an operator on $\mathbb{C}^n$ is represented by a matrix, then its adjoint is represented by the conjugate transpose of the matrix. Indeed, let H be the n-dimensional unitary space $\mathbb{C}^n$ with basis $\{e_k\}_{k=1}^n$ and let the linear operator T be given by a matrix $\alpha = (\alpha_{k,\ell})_{1 \leq k, \ell \leq n}$. If $x = \sum_{k=1}^n \xi_k e_k$ and $C = \sum_{k=1}^n \eta_k e_k$ are any vectors in $H = \mathbb{C}^n$, Then we have

$$\langle Tx, y \rangle = \langle \sum_{k=1}^n \left(\sum_{\ell=1}^n \alpha_{k,\ell} \xi_\ell \right) e_k , \sum_{k=1}^n \eta_k e_k \rangle = \sum_{k=1}^n \sum_{\ell=1}^n \bar{\eta}_k \alpha_{k,\ell} \xi_\ell .$$

Let $\alpha^* = (\alpha^*_{k,\ell})_{1 \leq k, \ell \leq n}$ be the matrix with the elements $\alpha^*_{k,\ell} = \overline{\alpha_{\ell,k}}$.

(α^* is called the adjoint matrix of α) Then we have

$$\langle x, T^* y \rangle = \langle \sum_{k=1}^n \xi_k e_k , \sum_{k=1}^n \left(\sum_{\ell=1}^n \alpha^*_{k,\ell} \eta_\ell \right) e_k \rangle$$

$$= \sum_{k=1}^n \sum_{\ell=1}^n \xi_k \overline{\alpha^*_{k,\ell} \eta_\ell} = \sum_{k=1}^n \sum_{\ell=1}^n \bar{\eta}_\ell \alpha_{\ell,k} \xi_k = \langle Tx, y \rangle .$$

The operator T^* corresponding to the matrix $\alpha^* = (\alpha^*_{k,\ell})_{1 \leq k, \ell \leq n}$ is therefore the adjoint of T, and adjoint operators correspond to adjoint matrices.

DEFINITION 3.5. [1,3]. Given $T \in B(H)$, (a) a point $\lambda \in \mathbb{C}$ is called a regular point of T if $\lambda I - T$ is invertible. The set $\rho(T)$ of regular points is called the resolvent set of T. The spectrum $\sigma(T)$ of T is the complement of $\rho(T)$. (b) a point $\lambda \in \mathbb{C}$ is called an eigenvalue of T if $\ker(\lambda I - T) \neq \{0\}$. The set $\sigma_p(T)$ of eigenvalues of T is called the point spectrum of T. (c) The approximate point spectrum $\sigma_{ap}(T)$ of T is defined by $\sigma_{ap}(T)=\{\lambda \in \mathbb{C} \mid$ there is a sequence $\{x_n\}$ in H such that $\|x_n\|=1$ for all n and $\|(T-\lambda I)x_n\| \to 0\}$.

Note that $\sigma_p(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T)$ and $\partial\sigma(T) \subseteq \sigma_{ap}(T)$, where $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$. [1, 3]. Also we recall that for every T in B(H), $\sigma(T)$ is a nonempty compact subset of $\mathbb{C}$, $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leqq \|T\|\}$, and $\sigma(T^*) = \sigma(T)^*$, where for any subset $\triangle$ of $\mathbb{C}$, $\triangle^* \equiv \{\bar{z} : z \in \triangle\}$.

LEMMA 3.6. Let $\omega=(\omega_n)$ be a sequence of strictly positive numbers such that $\sup_j \omega_j < \infty$ and $\inf_j \omega_j > 0$. Define $S_r^\omega : \ell_\omega^2(\mathbb{C}) \to \ell_\omega^2(\mathbb{C})$ by

$$S_r^\omega(\xi_1, \xi_2, \cdots\cdots) = (0, \xi_1, \xi_2, \cdots\cdots).$$

Then $(S_r^\omega)^*$ is given by $(S_r^\omega)^*(\xi_1, \xi_2, \cdots\cdots) = (\frac{\omega_2}{\omega_1}\xi_2, \frac{\omega_3}{\omega_2}\xi_3, \cdots\cdots)$.

In particular, $S_r^* = S_\ell$, where S_r denotes the right shift operator on $\ell^2(\mathbb{C})$.

PROOF. For any $x=(\xi_n)$ and $y=(\eta_n)$ in $\ell_\omega^2(\mathbb{C})$,

$$((S_r^\omega)^*x, y) = (x, S_r^\omega y) = ((\xi_1, \xi_2, \xi_3, \cdots), (0, \eta_1, \eta_2, \cdots)) =$$

$$\omega_2 \xi_2 \bar{\eta}_1 + \omega_3 \xi_3 \bar{\eta}_2 + \cdots\cdots = \omega_1 (\frac{\omega_2}{\omega_1}\xi_2)\bar{\eta}_1 + \omega_2(\frac{\omega_3}{\omega_2}\xi_3)\bar{\eta}_2 + \cdots\cdots$$

$$= ((\frac{\omega_2}{\omega_1}\xi_2, \frac{\omega_3}{\omega_2}\xi_3, \cdots\cdots), (\eta_1, \eta_2, \cdots\cdots)).$$

Since this holds for every $y=(y_n)$, the result is proved. In particular, if $\omega=(1, 1, \cdots)$, then $S_r^* = S_\ell$ since $S_r^\omega = S_r$.

COROLLARY 3.7. Let $\omega=(\omega_n)$ be a sequence of strictly positive numbers such that $\sup_j \omega_j<\infty$ and $\inf_j \omega_j>0$. Define $S_l^\omega : \ell_\omega^2(\mathbb{C})\rightarrow \ell_\omega^2(\mathbb{C})$ by $S_l^\omega(\xi_1, \xi_2,\cdots)=(\xi_2, \xi_3,\cdots)$. Then $(S_l^\omega)^*$ is given by $(S_l^\omega)^*(\xi_1, \xi_2, \cdots\cdots) = (0, \frac{\omega_1}{\omega_2}\xi_1, \frac{\omega_2}{\omega_3}\xi_2, \cdots\cdots)$. In particular, $S_l^* = S_r$.

THEOREM 3.8. Let S_r be the right shift operator on $\ell^2(\mathbb{C})$. Then $\sigma(S_r)=\{\lambda \in \mathbb{C} : |\lambda| \leqq 1\}$, $\sigma_p(S_r)=\phi$, and $\sigma_{ap}(S_r)=\{\lambda \in \mathbb{C} : |\lambda|=1\}$.

PROOF. Since $\|S_r\|=1$, $\sigma(S_r)\subseteq\{\lambda \in \mathbb{C} : |\lambda| \leqq 1\}$. Suppose $x=(\xi_1, \xi_2,\cdots)\in \ell^2(\mathbb{C})$ and $\lambda \neq 0$. If $S_r x=\lambda x$, then $0=\lambda\xi_1$, $x_1=\lambda\xi_2,\cdots$. Hence $0=\xi_1=\xi_2=\cdots$. Since $\lambda\neq 0$, $x=0$, and so $\lambda \notin \sigma_p(S_r)$. Also since $\|S_r x\| = \|x\|$ for each x in $\ell^2(\mathbb{C})$. i. e., S_r is an isometry, $\ker S_r=\{0\}$, and so $\lambda=0\notin \sigma_p(S_r)$. Hence $\sigma_p(S_r)=\phi$. Let $|\lambda|<1$ and put $x=(1, \lambda, \lambda^2,\cdots)$. Then $\|x\|^2 = \sum_{n=0}^{\infty} |\lambda^2|^n < \infty$. Also $S_l x= (\lambda, \lambda^2,\cdots)=\lambda x$. Hence $\lambda \in \sigma_p(S_l)$ and $x\in \ker(S_l-\lambda I)$. Since $S_r^*= S_l$, $\{\lambda \in \mathbb{C} : |\lambda|<1\}\subseteq\sigma(S_l)=\sigma(S_r)\subseteq\{\lambda \in \mathbb{C} : |\lambda| \leqq 1\}$. If $|\lambda|<1$ and $x=(\xi_n)\in \ell^2(\mathbb{C})$, then $\|(S_r-\lambda I)x\| = \|S_r x-\lambda x\| \geq |\,\|S_r x\| - |\lambda|\,\|x\|\,| = |\,\|x\| - |\lambda|\,\|x\|\,| =(1-|\lambda|)\|x\|$. By the equivalent condition of $\sigma_{ap}(T)$ [1], $\lambda \notin \sigma_{ap}(S_r)$. Hence $\sigma_{ap}(S_r)\subseteqq\{\lambda \in \mathbb{C} : |\lambda|=1\}$. Also by the fact that $\partial\sigma(T)\subseteqq\sigma_{ap}(T)$, $\sigma_{ap}(S_r)=\{\lambda \in \mathbb{C} : |\lambda|=1\}$.

COROLLARY 3.9. Let S_l be the left shift operator on $\ell^2(\mathbb{C})$. Then $\sigma(S_l)=\{\lambda \in \mathbb{C} : |\lambda| \leqq 1\}$, and $\sigma_{ap}(T)=\{\lambda \in \mathbb{C} : |\lambda|=1\}$.

PROOF. Our results follow from the theorem and the fact that $\sigma(S_r^*) = \sigma(S_l)$.

REMARK 3.10. Let S_r be the right shift operator on $\ell^2(\mathbb{C})$. Then $\sigma(S_r S_r^*) \neq \sigma(S_r^* S_r)$. Indeed since $S_r^* S_r = S_l S_r = I$, $\sigma(S_r^* S_r) = \{1\}$. On the other hand since $S_r S_r^*(\xi_1, \xi_2, \cdots) = S_r S_l(\xi_1, \xi_2, \cdots) = S_r(\xi_2, \xi_3, \cdots) = (0, \xi_2, \xi_3, \cdots)$, $\sigma(S_r S_r^*) = \{\lambda \in \mathbb{C} : S_r S_r^* - \lambda I \text{ is not invertible}\} = \{0, 1\}$. Thus $\sigma(S_r^* S_r) \neq \sigma(S_r S_r^*)$.

THEOREM 3.11. Suppose $0 < \omega_1 \leqq \omega_2 \leqq \cdots$ such that $\lim_{n\to\infty} \omega_n = r < \infty$. Define $T : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ by $T(\xi_1, \xi_2, \cdots) = (0, \omega_1\xi_1, \omega_2\xi_2, \cdots)$.
Then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leqq r\}$, $\sigma_p(T) = \phi$ and $\sigma_{ap}(T) = \partial\sigma(T)$.

PROOF. Since $\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup \omega_n = r$, $\sigma(T) \subseteqq \{\lambda \in \mathbb{C} : |\lambda| \leqq r\}$. Suppose $x = (\xi_1, \xi_2, \cdots) \in \ell^2(\mathbb{C})$ and $\lambda \neq 0$. If $Tx = \lambda x$, then $0 = \lambda\xi_1$, $\omega_1\xi_1 = \lambda\xi_2, \cdots$. Hence $0 = \xi_1 = \xi_2 = \cdots$ i.e., $x = 0$. Since $\lambda \neq 0$, $x = 0$, and so $\lambda \notin \sigma_p(T)$. Since $\|Tx\| = \|(0, \omega_1\xi_1, \omega_2\xi_2, \cdots)\| \geqq \omega_1\|x\|$ for $x = (\xi_n)$ in $\ell^2(\mathbb{C})$, ker $T = \{0\}$, and so $\lambda = 0 \notin \sigma_p(T)$. Hence $\sigma_p(T) = \phi$. Let $|\lambda| < r$ and put $x = \left(1, \frac{\lambda}{\omega_1}, \frac{\lambda^2}{\omega_1\omega_2}, \cdots\cdots\right)$. Then $\|x\|^2 = 1 + \left|\frac{\lambda}{\omega_1}\right|^2 + \left|\frac{\lambda^2}{\omega_1\omega_2}\right|^2 + \cdots = \sum_{n=0}^{\infty} a_n < \infty$,

since
$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|\frac{\lambda^{n+1}}{\omega_1\cdots\omega_n\omega_{n+1}}\right|^2}{\left|\frac{\lambda^n}{\omega_1\cdots\cdots\omega_n}\right|^2} = \lim_{n\to\infty}\frac{|\lambda|^2}{\omega_{n+1}^2}$$

$$= |\lambda|^2 \lim_{n\to\infty}\frac{1}{\omega_{n+1}^2} = \frac{|\lambda|^2}{r^2} < 1$$

Also since $T^*(\xi_n) = (\omega_1 \xi_2, \ \omega_2 \xi_3, \ \cdots\cdots)$, $T^* x = (\lambda, \frac{\lambda^2}{\omega_1}, \frac{\lambda^3}{\omega_1 \omega_2}, \cdots\cdots)$ $= \lambda(1, \frac{\lambda}{\omega_1}, \frac{\lambda^2}{\omega_1 \omega_2}, \cdots\cdots) = \lambda x$. Thus $\lambda \in \sigma_p(T^*)$, and $x \in \ker(T^* - \lambda I)$. Hence $\{\lambda \in \mathbb{C} : |\lambda| < r\} \subseteqq \sigma(T^*) = \sigma(T)^* \subseteqq \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

Since $\sigma(T)$ is necessarily closed, $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

Finally since $\partial\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$ and $\partial\sigma(T) \subseteqq \sigma_{ap}(T)$, $\sigma_{ap}(T) = \{\lambda : |\lambda| = r\}$.

CHAPTER IV
COMPACT OPERATORS ON ℓ^2 SPACE

A Hilbert space has two useful topologies(weak and strong),but the space of operators on a Hilbert space has several.

DEFINITION 4.1. [3, 5]. Let H be a Hilbert space with inner product $\langle\,,\,\rangle$. (a) The norm operator topology or the uniform operator topology on B(H) is the metric topology induced by the norm. (b) The weak operator topology on B(H) is the weak topology defined by the collection of functions $T \rightarrow \langle Tx, y \rangle$ from B(H) to $\mathbb{C}$ for x and y in H. (c) The strong operator topology on B(H) is the strong topology defined by the collection of functions $T \rightarrow Tx$ from B(H) to H for x in H.

A sequence of operator $\{T_n\}$ converges to T in the weak [strong, uniform respectively] operator topology if $\lim \langle T_n x, y \rangle = \langle Tx, y \rangle$ [$\lim T_n x = Tx$, $\lim T_n = T$ respectively] for every x and y in H.

We will investigate the relation between these topological concepts and shift operators.

(a) Let S_r be the unilateral shift operator on $\ell^2(\mathbb{C})$, and put $T_k = S_r^{*k} = S_\ell^k$ $(k=1, 2, \cdots)$. Then (1) $T_k \rightarrow 0$ strongly, but the sequence $\{T_k^*\}$ is not strongly convergent to anything, i. e., the adjoint is discontinuous with respect to the strong operator topology. For $\| T_k(\xi_1, \xi_2, \cdots\cdots)\|^2 = \|(\xi_{k+1}, \xi_{k+2}, \cdots\cdots)\|^2 = \sum_{n=k+1}^{\infty} |\xi_n|^2$ so that $\| T_k x \|^2$ is, for each $x = (\xi_n)$, the tail of a convergent series and therefore $T_k x \to 0$. But if $x=(\xi_n)\neq 0$, then $\| T_{m+n}^* x - T_n^* x \|^2 = \| S_r^{m+n} x - S_r^n x \|^2 = \| S_r^m x - x \|^2 = \| S_r^m x \|^2 - 2\,\mathrm{Re}(S_r^m x, x) + \| x \|^2 = 2(\| x \|^2 - \mathrm{R_e}(x, S_r^{*m} x))$. Since $\| S_r^{*m} x \| \to 0$ as $m \to \infty$, $\| T_{m+n}^* x - T_n^* x \|$ is nearly equal to $\sqrt{2}\,\| x \|$, i. e., $\{T_k^* x\}$ is not a Cauchy sequence.

(b) Let S_r be the right shift on $\ell^2(\mathbb{C})$ and put $T_n = S_r^{*n}$ and $S_n = S_r^n$, $n=1,2,\cdots$. Since $T_n \to 0$ strongly by (a), it follows that $T_n \to 0$ weakly, and hence that $S_n \to 0$ weakly since $|\langle S_n x, y \rangle| = |\langle x, S_n^* y \rangle| = |\langle x, T_n y \rangle| = |\langle T_n y, x \rangle|$ for any $x, y \in \ell^2(\mathbb{C})$. Since $T_n S_n = I$ for all n, it is not true that $T_n S_n \to 0$ weakly. Therefore multiplication is not weakly sequentially continuous.

The image or range of an linear operator $T: H_1 \to H_2$, written Im T, is the subspace $TH_1 = \{Tx : x \in H_1\}$. Also the kernel of T, written ker T, is the closed subspace $\ker T = \{x \in H_1 : Tx = 0\}$.

DEFINITION 4.2. [2, 5, 8]. A linear operator $T: H_1 \to H_2$ is called an operator of finite rank if Im T is finite dimensional. T is called a compact operator if for each sequence $\{x_n\}$ in H_1, $\| x \| = 1$, the sequence $\{Tx_n\}$ has a subsequence which converges in H_2 .

EXAMPLE 4.3. (a) Every operator $T: H_1 \to H_2$ of finite rank is compact. For, suppose $\{x_n\} \subset H_1$, $\| x \| = 1$. Then $\{Tx_n\}$ is a bounded sequence in the finite dimensional space Im T. Since Im T is linearly isometric to $\mathbb{C}^k$ for some k, It follows that $\{Tx_n\}$ has a convergent subsequence.

(b) The identity operator on an infinite dimensional Hilbert space is not compact. For if $e_1, e_2, \cdots$ is an infinite orthonormal set in H, then $\| Ie_n - Ie_m \| = \sqrt{2}$ which implies that $\{Ie_n\}$ has no convergent subsequence even though $\| e_n \| = 1$.

LEMMA 4.4. [2]. Suppose $\{k_n\}$ is a sequence of compact operators in $L(H_1, H_2)$ and $\| k_n - k \| \to 0$ as $n \to \infty$ where k is in $L(H_1, H_2)$. Then k is compact.

PROOF. Let $\{x_n\}$ be a sequence in H_1, $\| x_n \| = 1$, since k_1 is compact. There exists a subsequence $\{x_{1n}\}$ of $\{x_n\}$ such that $\{k_1 x_{1n}\}$ converges. Since k_2 is compact, there exists a subsequence $\{x_{2n}\}$ of $\{x_{1n}\}$ such that $\{k_2 x_{2n}\}$ converges. Continuing in this manner, we obtain for each integer $j \geq 2$, a subsequence $\{x_{jn}\}_{n=1}^{\infty}$ of $\{x_{(j-1)n}\}_{n=1}^{\infty}$ such that $\{k_j x_{jn}\}_{n=1}^{\infty}$ converges. We now show that the "diagonal" sequence $\{k x_{nn}\}$ converges, which proves that k is compact. Given $\varepsilon > 0$, there exists by hypothesis an integer p such that $\| k - k_p \| < \varepsilon / 2$. (1)

Now $\{k_p x_{nn}\}$ converges since $n \geq p$ implies that $\{k_p x_{nn}\}$ is a subsequence of the convergent sequence $\{k_p x_{pn}\}$. By $\| k x_{nn} - k x_{mm} \| \leq \| k x_{nn} - k_p x_{nn} \| + \| k_p x_{nn} - k_p x_{mm} \| + \| k_p x_{mm} - k x_{mm} \| \leq 2 \| k_p - k \| + \| k_p x_{nn} - k_p x_{mm} \| < \varepsilon + \| k_p x_{nn} - k_p x_{mm} \| \to \varepsilon$ as $n, m \to \infty$. Thus it follows that $\{k x_{nn}\}$ is a Cauchy sequence which must converge since H_2 is complete.

THEOREM 4.5. A diagonal operator on $\ell^2(\mathbb{C})$ with diagonal $\{\omega_n\}$ is compact if and only if $\omega_n \to 0$ as $n \to \infty$.

PROOF. Let T be a diagonal operator on $\ell^2(\mathbb{C})$ with diagonal $\{\omega_n\}$, and for each positive integer n, consider the diagonal operator T_n with diagonal $\{\omega_1, \cdots, \omega_n, 0, 0, \cdots\}$.. Since $T-T_n$ is a diagonal operator with diagonal $\{0, \cdots, 0, \omega_{n+1}, \omega_{n+2}, \cdots\}$, so that $\| T-T_n \| = \sup_k | \omega_{n+k} |$, it is clear that the assumption $\omega_n \to 0$ implies the condition $\| T-T_n \| \to 0$. By Lemma 4.4, if $\omega_n \to 0$, then T is compact.

To prove the converse, consider the standard orthonormal basis $\{e_n\}$ that makes T diagonal. If T is compact, then $Te_n \to 0$ strongly because $e_n \to 0$ weakly. Since $\| \omega_n e_n \| = | \omega_n | \; \| e_n \| \to 0$, $w_n \to 0$.

THEOREM 4.6. Let $(a_{ij})_{i,j=1}^{\infty}$ be an infinte matrix where $\sum_{i,j=1}^{\infty} | a_{ij} |^2 < \infty$. Define $T : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ by $T(\xi_1, \xi_2, \cdots\cdots) = (\eta_1, \eta_2, \cdots\cdots)$ where $\eta_i = \sum_{j=1}^{\infty} a_{ij} \xi_j$. Then Tis compact.

PROOF. Let $T_n \in \ell^2(\mathbb{C})$, n=1,2,$\cdots$ be the operator corresponding to the matrix $(a_{ij}^{(n)})_{i,j=1}^{\infty}$, where $a_{ij}^{(n)} = a_{ij}$, $1 \leqq i, j \leqq n$ and $a_{ij}^{(n)} = 0$ otherwise. Since T_n is of finite rank, it is compact. Moreover $\| T - T_n \| \leq \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} | a_{ij} |^2 \to 0$ as $n \to \infty$. Hence By Lemma 4.4, T is compact.

REFERENCES

1. J. B. Conway, A course in functional analysis, Springer-Verlag, 1985.

2. I. Gohberg & S. Goldberg, Basic Operator Theory, Birkhäuser Boston, 1980.

3. P. R. Halmos, A Hilbert space problem book, Springer Verlag, 1982.

4. ______, Ten problems in Hilbert space, Bull. Amer. Math. Soc, 76 (1970), 887 ~933.

5. G. Helmberg, Introduction to spectral theory in Hilbert space, North-Holl and Publ. Co. 1969.

6. S. Karanasios, On certain commuting families of rank one operators, Proc. Edinburgh Math. Soc. 27 (1984), 115~129.

7. E. Kreyszig, Introductory Functional Analysis with applications, John Wiley- & Sons, Inc, 1978.

8. J. Weidmann, Linear Operators in Hilbert spaces, Springer-Verlag New York, 1980.

9. B. S. Yadaw & S. Chatterjee, On a characterization of invariant subspace lattices of weighted shifts, Proc. Amer. Math. Soc. 84 (1982), 492~496.

< 國文抄錄 >

힐버트空間 ℓ^2 上의 線型作用素에 관한 硏究

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本 論文에서는 첫째로 여러 종류의 ℓ^2空間의 基本性質과 이들 空間 및 部分空間이 Hilbert 空間이 되는지 여부를 조사한다. 둘째로 Riesz 表現定理를 이용하여 여러 ℓ^2 空間上의 汎函數의 特性을 조사하고 이 空間上에서 左(右)推移 作用素와 加重推移 作用素의 基本性質과 點 스펙트럼, approximate 點 스펙트럼 및 스펙트럼을 구한다. 세째로 ℓ^2 空間上에서 推移 作用素에 의한 여러 位相의 性質과 이 空間上에서 compact 作用素의 同値條件을 밝힌다.