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Classification of Joint Spectra on a Hilbert Space

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CONTENTS

I. INTRODUCTION

Let H be a complex Hilbert space with the saclar product $\langle , \rangle$ and the norm $\| \cdot \|$, and let T be a bounded liner operator. It is wellknown that (a) the approximate spectrum $\sigma_{ap}(T)$ and the spectrum $\sigma(T)$ of T are nonempty compact subsets of the complex number field C, (b) $\sigma_{ap}(T) \subseteq \sigma(T)$. and (c) $\partial\sigma(T) \subseteq \sigma_{ap}(T)$ where $\partial\Delta$ denote the boundary set Δ. ([2]. [6])

A 2-tuple of bounded linear operator A_1, A_2 will be denoted by $A=(A_1, A_2)$. In recent years, there have been many definitions of joint spectrum of a 2-tuple of commuting operators on a Hilbert space. ([1], [3], [4], [5]) In this paper, we give equivalent conditions of some definitions of joint spectrum, compare these and study properties, in particular, compactness of these in detail.

Throughout this paper, all operators on H will be assumed to be a bounded linear transformations of H into it self. The real number field and the complex number field are denoted by R and C respectively.

The organization of this paper is as follows. In section Ⅱ, we introduce some notations and give various spectral results needed in the sequel.

In section Ⅲ, we give equivalent conditions and properties, in particular, compactness of joint point and joint approximate spectra.

In section Ⅳ, we study properties, in particular, compactness of several joint spectra in the sense of Taylor, Dash, Harte etc. Also we give an equivalent condition for a nonsingularity of $A=(A_1, A_2)$, and show in detail that for a 2-tuple $A=(A_1, A_2)$ of commuting normal operators, the joint spectra of Dash, Harte, Taylor respectively are equal to the joint approximate point spectrum of $A=(A_1, A_2)$.

Ⅱ. PROPERTIES OF SPECTRA OF OPERATOR

Definition 2. 1. [6] Let X be a vector space over C. An inner product on X is a complex valued function $\langle , \rangle$, defined on X with the following properties.

(a) $\langle x+y,\ z\rangle = \langle x,\ z\rangle + \langle y,\ z\rangle$.

(b) $\langle \alpha x,\ y\rangle = \alpha\langle x,\ y\rangle$.

(c) $\overline{\langle x,\ y\rangle} = \langle y,\ x\rangle$.

(d) $\langle x, x\rangle \geqq 0$ and $\langle x,\ x\rangle = 0$ *iff* $x=0$.

X, together with an inner product is called an inner product space.

The norm, $\| x \|$, on X is given by $\| x \| = \langle x, x\rangle^{+}$.

Examples 2. 2. (a) Let $\ell^2(C)$ be the set of those sequence$(\xi_1, \xi_2, \cdots)$ of complex numbers for which $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$, together with operations of addition and scalar multiplication defined as follows.

For $x=(\xi_1, \xi_2, \cdots)$ and $y=(\eta_1, \eta_2, \cdots)$ in $\ell^2(C)$,

$x+y=(\xi_1+\eta_1, \xi_2+\eta_2, \cdots)$ and $\alpha x=(\alpha\xi_1, \alpha\xi_2, \cdots)$, $\alpha\in C$.

The complex valued function $\langle , \rangle$, defined on $\ell^2(C)\times\ell^2(C)$ by $\langle x, y\rangle = \sum_{i=1}^{\infty} \xi_i\overline{\eta_i}$, where $x=(\xi_1, \xi_2, \cdots)$, $y=(\eta_1, \eta_2, \cdots)$ difines an inner product on $\ell^2(C)$. That is, $\ell^2(C)$ is an inner product space.

(b) Let $L^2([a, b])$ be the vector space of all complex valued Lebesque measurable functions f defined on the interval $a\leqq x\leqq b$ with the property that $|f|^2$ is Lebesque integrable. If f and $g\in L^2([a, b])$, then Hölders inequality implies $f\cdot\overline{g}\in L^2([a, b])$. By indentifying functions which are equal almost everywhere, $\langle f, g\rangle = \int_a^b f(x)\,\overline{g(x)}\,dx$ defines an inner product on $L^2([a, b])$.

Theorem 2.3. ([2], [3]) Let X be an inner product space. For x and y in X.

(a) $|\langle x, y\rangle| \leqq \| x \| \ \| y \|$ (Cauchy-Bunyakowsky-Schwarz inequality.)

(b) $\| x+y \| \leqq \| x \| + \| y \|$ (Triangle inequality)

(c) $\| \alpha x \| = |\alpha| \, \| x \|$ for all α in C.

Definition 2.4. ([2],[6]) a sequence $\{x_n\}$ in an inner product space H is said to be converge to $x \in$ H, written $x_n \to x$, if $\| x_n - x \| \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ in H is called a Cauchy sequence if $\| x_m - x_n \| \to 0$ as $m, n \to \infty$.

If every Cauchy sequence in H converges to a vector in H, then H is complete. A complete inner product space is called a Hilbert space.

Enample 25. (a) C^n is an n-dimensional Hilbert space.

(b) $\ell^2(C)$ is an infinite dimensional Hilbert space.

(c) $L^2([a, b])$ is an infinite dimensional Hilbert space, since the function $1, x, x^2, \cdots$ are linearly independent. For, if $\sum_{k=0}^{n} a_k x^k$ is the zero function, the $a_k = 0 (0 \leqq k \leqq n)$, since any polynomial of degree n has at most n zeros.

Definition 2.6. H_1 and H_2 Hilbert spaces over the complex number C. A linear operator $T : H_1 \to H_2$ is called bounded if $\sup_{\|x\| \leq 1} \| Tx \| < \infty$. The norm of T, written $\| T \|$, is the nonnegative number $\| T \| = \sup_{\|x\|=1} \| Tx \|$.

Let $B(H_1, H_2)$ be the set of all bounded linear operator from H_1 into H_2, and put $B(H, H) = B(H)$. Note that for a bounded linear operator T, $S \in B(H)$,

(a) $\| T \| = \sup \{ \| Tx \| / \| x \| : x \neq 0 \}$

$= \sup \{ \| Tx \| : \| x \| = 1 \}$

$= \inf \{ c > 0 : \| Tx \| \leqq c \| x \|, \ x \in H \}$.

(b) $\| TS \| \leqq \| T \| \| S \|$. ([2], [7], [8])

Example 2. 7. (a) Let $S_r : \ell^2(C) \rightarrow \ell^2(C)$ be difined by $S_r(\xi_1, \xi_2, \cdots) = (0, \xi_1, \xi_2, \cdots)$. The operator S_r is called right shift operator(or unilateral shift). Obviously S_r is linear and $\| S_r x \| = \| x \|$. Thus S_r is an isometry of $\ell^2(C)$ into $\ell^2(C)$ and $\| S_r \| = 1$. In fact S_r maps $\ell^2(C)$ onto a proper subspace namely the set of all absolutely square sequence having first term zero, i. e. S_r is not surjective. Therefore S_r is not invertible.

(b) Let $S_\ell : \ell^2(C) \rightarrow \ell^2(C)$ be defined by $S_\ell(\xi_1, \xi_2, \cdots) = (\xi_2, \xi_3, \cdots)$.

The operator S_ℓ is called left shift operator. Obviously S_ℓ is linear, and $\| S_\ell \| = 1$, but S_ℓ is not one-one. Hence S_ℓ is not invertible.

Lemma 2. 8. [2] Let $T : H_1 \rightarrow H_2$ be a linear operator. Then, the following conditions are equivalent;

(a) T is continuous at a point.

(b) T is continuous on H_1.

(c) T is bounded.

Definition 2. 9. [2] If $T \in B(H_1, H_2)$, then the unique operator S in $B(H_2, H_1)$ satisfying $\langle Tx, y \rangle = \langle x, Sy \rangle$ for all x in H_1 and y in H_2 is called the adjoint of T denoted by $S = T^*$.

We note the following;

(a) $0^* = 0^*$ and $I^* = I$.

(b) For each S, $T \in B(H)$, and $\lambda \in C$,

$(\lambda S + T)^* = \lambda^* S^* + T^*$, $(ST)^* = T^* S^*$ and $T^{**} = T$.

(c) $\| T^* \| = \| T \|$ for each $T \in B(H)$.

(d) If an operator on C^n is represented by a matrix, then its adjoint is represented by the conjugate transpose of the matrix.

Example 2.10. Let S_r be the right shift operator on $\ell^2(C)$. Given $y=(\zeta_1, \zeta_2, \cdots)$ and $x=(\xi_1, \xi_2, \cdots)$ in $\ell^2(C)$,

$$\langle S_r x, y\rangle = \langle(0, \xi_1, \xi_2, \cdots), (\zeta_1, \zeta_2, \cdots)\rangle$$

$$= \xi_1 \cdot \overline{\zeta}_2 + \xi_2 \cdot \overline{\zeta}_3 + \cdots = \langle x, z\rangle,$$

where $z=(\zeta_2, \zeta_3, \cdots)$. Thus $S_r^*(\zeta_1, \zeta_2, \cdots)=(\zeta_2, \zeta_3, \cdots)$, i.e. S_r^* is the left shift operator.

Theorem 2.11. ([2], [9]) If $T \in B(H)$ such that $\|I-T\|<1$, then T is invertible, T^{-1} exists as a bounded linear operator on H and $T^{-1}=\sum_{n=0}^{\infty}(I-T)^n = I+(I-T)+(I-T)^2+\cdots\cdots$

proof. Let $S=I-T$. Then $r=\|S\|<1$. Since $\|S^n\| \leqq \|S\|^n = r^n$, the series $\sum_{n=0}^{\infty}\|S^n\|$ converges.

Hence $U=\sum_{n=0}^{\infty} S^n$ converges in $B(H)$, since H is complete. If $U_n = I+S+S^2+\cdots+S^n$,

Then $U_n(I-S)=(I+S+S+S^2+\cdots+S^n)-(S+S^2+S^3+\cdots+S^{n+1})$

$=I-S^{n+1}$

But $\|S^{n+1}\| \leqq r^{n+1}$, so $S^{n+1} \to 0$ as $n \to \infty$.

Hence $U(I-S)=\lim_{n\to\infty} U_n(I-S)=I$.

Similarly, $(I-S)U=I$. So $I-S=T$ is invertible and $T^{-1}=(I-S)^{-1}=U=\sum_{n=0}^{\infty} S^n = \sum_{n=0}^{\infty}(I-T)^n$.

Corollary 2. 12. If T∈B(H) such that ‖T‖<1, then $(I-T)^{-1}$ exists $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$, where the series on the right is convergent in the norm on B(H)

Definition 2. 13. ([2], [7], [8]) Given T∈B(H), (a) the spectrum of T, denoted by $\sigma(T)$, is defined by

$$\sigma(T) = \{\lambda \in C : T-\lambda \text{ is not invertible}\}.$$

The resolvent set of T is defined by $\rho(T) = C - \sigma(T)$.

(b) A point $\lambda \in C$ is called an eigenvalue of T if $\ker(T-\lambda) = \{0\}$.

The set $\sigma(T)$ of all eigenvalues of T is called the point spectrum of T.

(c) The approximate point spectrum $\sigma_{ap}(T)$ is defined by $\sigma_{ap}(T) = \{\lambda \in C \mid$ There is a sequence $\{x_n\}$ of unit vectors in H such that $\|(T-\lambda)x_n\| \to 0$ as $n \to \infty\}$.

Note that $\sigma_p(T) \subseteq \sigma_{ap}(T)$ and $\sigma(T^*) = \sigma(T)^*$, where for any subset Δ of C $\Delta^* = \{\bar{z} : z \in \Delta\}$.

Lemma 2. 14. [2] If T∈B(H) and $\alpha \in C$, the follwing statements are equivalent;

(a) $\lambda \not\in \sigma_{ap}(T)$.

(b) $\text{Ker}(T-\lambda) = \{0\}$ and $\text{ran}(T-\lambda)$ is chosed.

(c) There is a constant c>o such that $\|(T-\lambda)x\| \geqq c\|x\|$ for all x.

proop. Clearly it may be assumed that $\lambda = 0$, since $\lambda \in \sigma_{ap}(T)$ if and only if $0 \in \sigma_{ap}(T-\lambda)$

(a) ⇨ (c) ; Suppose (c) fails to hold. Then for every n, there is a nonzero vector x_n with $\|Tx_n\| \leqq \|x_n\|/n$.

If $y_n = x_n / \|x_n\|$, then $\|y\| = 1$ and $\|Ty_n\| \to 0$. Hence $o \in \sigma_{ap}(T)$.

(c)⇨(b) : Suppose there is a constant $c > o$ such that $\|Tx\| \geqq c\|x\|$ for all x. Clearly $\mathrm{Ker}\,T = \{0\}$. If $Tx_n \to y$, then $\|x_n - x_m\| \leqq \frac{1}{c}\|Tx_n - Tx_m\|$. So $\{x_n\}$ is a Cauchy sequence. Let $x = \lim x_n$. Therefore $Tx = y$ and ran T is closed.

(b) ⇨ (a) : Let M=ran T. Then T : H→M is a contiuous bijection. By the Inverse Mapping Theorem, there is a bounded operator S : M → H such that S $Tx = x$ for all $x \in H$. Thus if $\|x\| = 1$, then $| = \|STx\| \leqq \|S\|\,\|Tx\|$. That is, $\|Tx\| \geqq 1/\|S\|$, whenever $\|x\| = 1$. Hence $o \notin \sigma_{ap}(T)$.

Lemma 2.15. [2] if $T \in B(H)$, then $\sigma_{ap}(T) \subset \sigma(T)$.

proof. If $\lambda \notin \sigma(T)$, then $T - \lambda$ is invertible and hence $\|x\| = \|(T-\lambda)^{-1}(T-\lambda)x\| \leqq \|(T-\lambda)^{-1}\|\,\|(T-\lambda)x\|$ for every vector x. That is, $\|(T-\lambda)x\| \geqq \varepsilon\|x\|$, with $\varepsilon = 1/\|(T-\lambda)^{-1}\|$, for every x. By the **Lemma 2.14**, $\lambda \in \sigma_{ap}(T)$.

Theorem 2.16. ([2], [6], [8]) For each $T \in B(H)$, (a) the resolvent set $\rho(T)$ of T is an open set. (b) closed $\sigma(T)$ is contained in $\{\lambda : |\lambda| \leqq \|T\|\}$. (c) $\sigma(T)$ is compact.

proof. (a) Suppose $\mu \in \rho(T)$. Then $T - \lambda$ is invertible. So there exists $\varepsilon > 0$ such that if $|\lambda - \mu| < \varepsilon$, then $T - \lambda$ is invertible, because $T - \lambda = (T-\mu)[I - (\lambda - \mu)(T - \mu^{-1})]$ and $\|(\lambda-\mu)(T-\lambda)^{-1}\| = |\lambda - \mu|\,\|(T-\lambda)^{-1}\| < 1$ implies $|\lambda - \lambda| < \varepsilon = 1/\|(T-\mu)^{-1}\|$. Hence $\rho(T)$ is open.

(b) $\sigma(T) = C \setminus \rho(T)$ is closed. If $|\lambda| \geqq \|T\|$, then $\|\frac{1}{\lambda}T\| < 1$ and so $I - \frac{T}{\lambda}$ is invertible. Therefore $T - \lambda = (-\lambda)(I - \frac{T}{\lambda})$ is also invertible.

(c) From (a) and (b), $\sigma(T)$ is a compact set.

THeorem 2.17. For each $T \in B(H)$, $\sigma_{ap}(T)$ is a compact set.

proof. If $\mu \in \sigma_{ap}(T)$, then by **Lemma 2.14** there exists $\varepsilon > 0$ such that $\| (T-\lambda) x \| \geqq \varepsilon$ for all unit vector x. Hence if x is a unit vector and if $|\lambda - \mu| < \frac{\varepsilon}{2}$, then

$$\| (T-\lambda)x \| = \| (T-\mu)x + (\mu-\lambda)x \| \geqq \| (T-\mu)x \| - \| (\mu-\lambda)x \|$$
$$= \| (T-\mu)x \| - | \mu-\lambda | \geqq \frac{\varepsilon}{2}.$$

Hence $\lambda \notin \sigma_{ap}(T)$, and therefore $\sigma_{ap}(T)^c$ is open, that is, $\sigma_{ap}(T)$ is closed. Since $\sigma_{ap}(T) \subseteq \sigma(T)$ and for each $\lambda \in \sigma(T)$, $|\lambda| \leqq \| T \|$, $\sigma_{ap}(T)$ is bounded.

Theorem 2.18. [2] For each $T \in B(H)$, $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.

proof. Let $\lambda \in \partial\sigma(T)$ and let $\{\lambda_n\}$ be a sequence in $C \setminus \sigma(T)$ such that $\lambda_n \to \lambda$. Claim $\| (T-\lambda_n)^{-1} \| \to \infty$ as $n \to \infty$. If the claim were false, then by passing to a subsequence if necessary, there is a constant M such that $\| (T-\lambda_n)^{-1} \| \leqq M$ for all n. Choose n sufficiently large that $|\lambda_n - \lambda| < \frac{1}{M}$.
Then $\| (T-\lambda)-(T-\lambda_n) \| = (\lambda_n - \lambda) < \frac{1}{M} \leqq 1 / \| (T-\lambda_n)^{-1} \|$, and so $\| I-(T-\lambda_n)^{-1}(T-\lambda) \| = \| (T-\lambda_n)^{-1}[(T-\lambda_n)-(T-\lambda)] \| = \| (\lambda-\lambda_n)(T-\lambda_n)^{-1} \| \leqq |\lambda_n - \lambda| \|(T-\lambda_n)^{-1}\| < 1$. Hence $T-\lambda$ is invertible, a contradiction. Let $\| x_n \| = 1$ such that $\alpha_n = \|(T-\lambda_n)^{-1} x_n \| > \| (T-\lambda_n)^{-1} \| - \frac{1}{n}$. Then $x_n \to \infty$. Put $y_n = \frac{1}{\alpha_n}(T-\lambda_n)^{-1} x_n$. Then $\| y_n \| = 1$, and $(T-\lambda)y_n = (T-\lambda_n)y_n + (\lambda_n - \lambda)y_n = \alpha_n^{-1} x_n + (\lambda_n - \lambda)y_n$. Thus $\| (T-\lambda)y_n \| \leqq \alpha_n^{-1} + |\lambda - \lambda_n|$, so that $\| (T-\lambda)y_n \| \to 0$ as $n \to \infty$, i.e, $\lambda \in \sigma_{ap}(T)$.

Examples 2.19. (a) If $T = \mu I$, then the only eigenvalue is μ, i.e. $\sigma_p(T) = \{\mu\}$.

(b) Let S_r be the right shift operator on $\ell^2(C)$. Then $\sigma_p(S_r) = \phi$, $\sigma(S_r) = \{\lambda \in$

$C : |\lambda| \leq 1\}$, and $\sigma_{ap}(S_r) = \{\lambda \in C : |\lambda| = 1\}$

proof. $\sigma(S_r) \subseteq \{\lambda \in C : |\lambda| \leq 1\}$ since $\| Sr \| = 1$. Suppose $x = (\xi_1, \xi_2, \cdots) \in \ell^2(C)$ and $\lambda \neq 0$. If $S_r x = \lambda x$, then $0 = \lambda \xi_1, \xi_1 = \lambda \xi_2, \cdots$. Hence $0 = \xi_1 = \xi_2 = \cdots$. Since $\lambda \neq 0$, $x = 0$ and so $\lambda \notin \sigma_p(S_r)$. Also since $\|S_r x\| = \|x\|$ for each x in $\ell^2(C)$, $\ker S_r = \{0\}$, and so $0 \notin \sigma_p(S_r)$. Hence $\sigma_p(S_r) = \phi$. Let $|\lambda| < 1$ and put $x = (1, \lambda, \lambda^2, \cdots)$. Then $\|x\|^2 = \sum_{n=0}^{\infty} |\lambda^2|^n < \infty$. Also $S_l x = (\lambda, \lambda^2, \lambda^3, \cdots) = \lambda x$, since $S_l = S_r^*$. Hence $\lambda \in \sigma_p(S_l)$ and $x \in \ker(S_l - \lambda)$, and therefore $\{\lambda \in C : |\lambda| < 1\} \subseteq \sigma(S_l) = \sigma(S_r) \subseteq \{\lambda \in C : |\lambda| \leq 1\}$. Since $\sigma(S_r)$ is compact, $\sigma(S_r) = \{\lambda \in C : |\lambda| \leq 1\}$. If $|\lambda| < 1$ and $x = (\xi_1, \xi_2, \cdots) \in \ell^2(C)$, then $\|(S_r - \lambda)x\| = \|S_r x - \lambda x\| \geq | \|S_r x\| - |\lambda| \|x\| | = | \|x\| - |\lambda| \|x\| | = (1 - |\lambda|) \|x\|$. By the equivalent condition of $\sigma_{ap}(T)$, $\lambda \notin \sigma_{ap}(S_r)$. Hence $\sigma_{ap}(S_r) \subseteq \{\lambda \in C : |\lambda| = 1\}$. Since $\partial\sigma(S_r) \subseteq \sigma_{ap}(S_r)$, $\partial\sigma_{ap}(S_r) = \{\lambda \in C : |\lambda| = 1\}$.

Definition 2.20. [2] If $T \in B(H)$, then (a) T is hermitian or selfadjoint if $T = T^*$; (b) T is normal if $TT^* = T^*T$; (c) T is positive if $\langle Tx, x \rangle \geq 0$ for all $x \in H$; (d) T is unitary if $TT^* = T^*T = I$.

We have the following properties ; (I) The following conditions are equivalent ; (a) T is selfadjoint. (b) $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$. (c) $\langle Tx, x \rangle = \langle x, Tx \rangle$ for all $x \in H$. $\langle Tx, x \rangle$ is real for all $x \in H$.

(II) If S, T are selfadjoint and $\lambda \in R$, then $S + T$, λT are selfadjoint, and ST is selfadjoint if and only if $ST = TS$.

(III) Any eigenvalue of a selfadjoint operators T is real, for if $Tx = \lambda x$ and $x \neq 0$, then $\lambda \|x\|^2 = \langle Tx, x \rangle = \langle x, Tx \rangle = \bar{\lambda} \|x\|^2$, i.e. $\lambda = \bar{\lambda}$.

(Ⅳ) The following conditions are equivalent ; (a) T is normal. (b) T^* is normal. (c) $\| T^*x \| = \| Tx \|$ for all $x \in H$.

(Ⅴ) For any operator T, T^*T is positive. ([2], [6], [7], [8])

Ⅲ. JOINT POINT SPECTRA AND JOINT APPROXIMATE SPECTRA

For a 2-tuple of operators A_1 and A_2 on H, we write $A=(A_1, A_2)$ and $A^*=(A_1^*, A_2^*)$

Definition 3.1. [4] Let $A=(A_1, A_2)$ be a 2-tuple of operators on H. We shall say that.

(a) $\lambda=(\lambda_1, \lambda_2)\in C^2$ is in the joint point spectrum $\sigma_p(A)$ of A if there exists a nonzero vector x in H such that $(A_i-\lambda_i)x=0$ for each $i=1, 2$.

(b) $\lambda=(\lambda_1, \lambda_2)$ is in the joint residual spectrum $\sigma_r(A)$ of A if $\lambda^*=(\lambda_1^*, \lambda_2^*)$ is in $\sigma_p(A^*)$, where λ_i^* denotes the complex conjugate of λ_i

(c) $\lambda=(\lambda_1, \lambda_2)$ is in the joint point-residual spectrum $\sigma_{pr}(A)$ of A if there exists a nonzero vector x in H such that $(A_1-\lambda_1)x=0=(A_2-\lambda_2)^*x$.

(d) $\lambda=(\lambda_1, \lambda_2)$ is in the joint residual-point spectrum $\sigma_{rp}(A)$ of A if there exists a nonzero vector x in H such that $(A_1-\lambda_1)^*x=0=(A_2-\lambda_2)x$.

(e) $\lambda=(\lambda_1, \lambda_2)$ is in the joint approximate point spectrum $\sigma_\pi(A)$ of A if there exists a sequence $\{x_k\}$ of unit vectors in H such that.

$\|(A_i-\lambda_i)x_k\| \to 0$ as $k \to \infty$. $(i=1, 2)$.

(f) $\lambda=(\lambda_1, \lambda_2)$ is in the joint approximate compression spectrum σ_δ (A) of A if λ^* is in $\sigma_\pi(A^*)$.

(g) $\lambda=(\lambda_1, \lambda_2)$ is in the joint approximate point — approximate compression spectrum $\sigma_{\pi\delta}(A)$ of A if there exists a sequence $\{x_k\}$ of unit vectors in H such that $\|(A_1-\lambda_1)x_k\|\to 0$ and $\|(A_2-\lambda_2)^*x_k\|\to 0$ as $k\to\infty$.

(h) $\lambda=(\lambda_1, \lambda_2)$ is in the joint approximate compression–approximate point spectrum $\sigma_{\delta\pi}(A)$ of A if there exists a sequence $\{x_k\}$ of unit vectors in H such that $\|(A_1-\lambda_1)^*x_k\|\to 0$ and $\|(A_2-\lambda_2)x_k\|\to 0$ as $k\to\infty$.

Lemma 3.2. For each 2-tuple $A=(A_1, A_2)$ of operators on H.

(a) $\sigma_p(A) \subseteq \sigma_\pi(A)$

(b) $\sigma_r(A) \subseteq \sigma_\delta(A)$

(c) $\sigma_{pr}(A) \subseteq \sigma_{\pi\delta}(A)$

(d) $\sigma_{rp}(A) \subseteq \sigma_{\delta\pi}(A)$

proof. (a) If $\lambda=(\lambda_1, \lambda_2)\in\sigma_p(A)$, than there exists a nonzero vector y in H such that $(A_i-\lambda_i)y=0\,(i=1,2)$. Put $x_k=y/\|y\|$ for all $k=1,2,3,\cdots$. Then $\{x_n\}$ is a sequence of unit vectors in H and $(A_i-\lambda_i)x_k\to 0$ as $k\to\infty\,(i=1,2)$. Hence $\lambda\in\sigma_\pi(A)$.

(b) if $\lambda\in\sigma_r(A)$, then $\lambda^*=(\lambda_1^*, \lambda_2^*)\in\sigma_p(A^*)$. By (a) $\lambda^*\in\sigma_\pi(A^*)$. Hence $\lambda\in\sigma_\delta(A)$.

(c), (d) : The proofs of (c) and (d) are similar to that of (a).

Lemma 3.3. For each 2-tuple $A=(A_1, A_2)$ of operators on H, $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma_\pi(A)$, if and only if,

$B_1(A_1-\lambda_1)+B_2(A_2-A_2)\neq I$ for all $B_1, B_2\in B(H)$, i.e.

$B(H)(A_1-\lambda_1)+B(H)(A_2-\lambda_2)\neq B(H)$.

proof. If $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma_\pi(A)$, then there exists a sequence $\{x_n\}$ of unit vectors in H such that $\|(A_i-\lambda_i)x_n\|\to 0$ as $n\to\infty\,(i=1,2)$

We claim that the expression $B(H)(A_1-\lambda_1)+B(H)(A_2-\lambda_2)=B(H)$ does not hold. Assume contray. Then there are operators B_1, B_2 in $B(H)$ such that $B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2)=I$. This implies that.

$$\begin{aligned} 1 &= \| x_n \| = \| Ix_n \| = \| [B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2)]x_n \| \\ &= \| [B_1(A_1-\lambda_1)]x_n + [B_2(A_2-\lambda_2)]x_n \| \\ &\leqq \| B_1 \| \, \| (A_1-\lambda_1)x_n \| + \| B_2 \| \, \| (A_2-\lambda_2)x_n \| \to 0 \text{ as } n\to\infty, \end{aligned}$$

which is impossible. Thus for all B_1, B_2 in $B(H)$,

$$B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2) \neq I.$$

Conversely, suppose that $B(H)(A_1-\lambda_1)+B(H)(A_2-\lambda_2) \neq B(H)$. Then this implies in particular that $(A_1-\lambda_1)^*(A_1-\lambda_1)+(A_2-\lambda_2)^*(A_2-\lambda_2)$ is not invertible in $B(H)$. Hence 0 is in the spectrum of the positive bounded operator

$$(A_1-\lambda_1)^*(A_1-\lambda_1)+(A_2-\lambda_2)^*(A_2-\lambda_2).$$

Since the boundary of the spectrum of a single operator is a subset of the approximate point spectrum, there exists a sequence $\{x_n\}$ of unit vectors in H such that $\| [(A_1-\lambda_1)^*(A_1-\lambda_1)+(A_2-\lambda_2)^*(A_2-\lambda_2)]x_n \| \to 0$ as $n\to\infty$. But

$$\begin{aligned} &\| (A_1-\lambda_1)x_n \|^2 + \| (A_2-\lambda_2)x_n \|^2 \\ &= \langle (A_1-\lambda_1)^*(A_1-\lambda_1)x_n, x_n \rangle + \langle (A_2-\lambda_2)^*(A_2-\lambda_2)x_n, x_n \rangle \\ &= \langle [(A_1-\lambda_1)^*(A_1-\lambda_1)+(A_2-\lambda_2)^*(A_2-\lambda_2)]x_n, x_n \rangle \\ &\leqq \| [(A_1-\lambda_1)^*(A_1-\lambda_1)+(A_2-\lambda_2)^*(A_2-\lambda_2)]x_n \|. \end{aligned}$$

There fore $\| (A_i-\lambda_i)x_n \| \to 0$ as $n\to\infty$ for all $i=1, 2$, i. e. λ is in $\sigma_\pi(A)$.

Corollary 3. 4. For each 2-tuple $A=(A_1, A_2)$ of operators on H, $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma_\delta(A)$ if and only if

$$(A_1-\lambda_1)B(H)+(A_2-\lambda_2)B(H) \neq B(H).$$

proof. From the Lemma 3. 3. and taking adjoint of operators, λ is in σ_δ (A) if and only if $\lambda^*=(\lambda_1^*, \lambda_2^*)$ is in $\sigma_\pi(A^*)$ if and only if $B(H)(A_1^*-\lambda_1^*)+B(H)(A_2^*-\lambda_2^*)\neq B(H)$, if and only if $(A_1-\lambda_1)B(H)+(A_2-\lambda_2)B(H)\neq B(H)$

Lemma 3. 5. For each 2-tuple $A=(A_1, A_2)$ of operators on H, $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma_{\pi\delta}(A)$ if and only if $B(H)(A_1-\lambda_1)+B(H)(A_2-\lambda_2)^*\neq B(H)$.

proof. By definition and **Lemma 3. 3**, λ is in $\sigma_{\pi\delta}(A)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in H such that.
$\|(A_1-\lambda_1)x_n\|\to 0$, and $\|(A_2-\lambda_2)^*x_n\|\to 0$ as $n\to\infty$, if and only if $B(H)(A_1-\lambda_1)+B(H)(A_2^*-\lambda_2^*)\neq B(H)$.

Lemma 3. 6. For each 2-tuple $A=(A_1, A_2)$ of operators on H, $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma_{\delta\pi}(A)$ if and only if $B(H)(A_1-\lambda_1)^*+B(H)(A_1-\lambda_2)\neq B(H)$.

proof. The proof is similar to that of **Lemma 3. 5.**

In each case, it is implied that a point $\lambda=(\lambda_1, \lambda_2)$ of C^2 is in a set $\sigma\cdot(A)$ if and only if $0=(0, 0)$ in $\sigma\cdot(A_1-\lambda_1, A_2-\lambda_2)$, so in proofs, we will often confine attention, without loss of generality to the question of whether 0 is in $\sigma\cdot(A)$. From the above lemmas,we have the following : For a 2-tuple of operators on H,

(a) 0 is in $\sigma_\pi(A)$ if and only if for all B_1, B_2 in B(H), $B_1A_1+B_2A_2\neq I$.
(b) 0 is in $\sigma_\delta(A)$ if and only if for all B_1, B_2 in B(H), $A_1B_1+A_2B_2\neq I$.
(c) 0 is in $\sigma_{\pi\delta}(A)$ if and only if for all B_1, B_2 in B(H), $B_1A_1+B_2A_2^*\neq I$.
(d) 0 is in $\sigma_{\delta\pi}(A)$ if and only if for all B_1, B_2 in B(H), $B_1A_1^*+B_2A_2\neq I$.

(e) 0 is in $\sigma_{\pi\delta}(A_1, A_2)$ if and only if 0 is in $\sigma_{\pi}(A_1, A_2^*)$.

(f) 0 is in $\sigma_{\delta\pi}(A_1, A_2)$ if and only if 0 is in $\sigma_{\pi}(A_1^*, A_2)$.

Lemma 3.7. [1] Let $A_1, A_2, \cdots, A_n$ be operators in B(H) such that $A_iA_j = A_jA_i$ for all i and j. If $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ are scalars such that $B(H)(A_1-\lambda_1) + \cdots + B(H)(A_{n-1}-\lambda_{n-1}) \neq B(H)$, then there is a scalar λ_n such that

$$B(H)(A_1-\lambda_1)+\cdots\ B(H)(A_n-\lambda_n) \neq B(H).$$

Theorem 3.8. Let $A=(A_1, A_2)$ be a 2-tuple of commuting operators on H. Then $\sigma_\pi(A)$ is a nonempty compact set.

proof. Since $\sigma_\pi(A_1)$ is nonempty and $\sigma_\pi(A_1) = \{\lambda \in C : B(H)(A_1-\lambda_1) \neq B(H)\}$, by **Lemma 3.7**, $\sigma_\pi(A) = \sigma_\pi(A_1, A_2)$ is nonempty. Cleary $\sigma_\pi(A) \subseteq \sigma_\pi(A_1) \times \sigma_\pi(A_2)$. So it is bounded. Thus we need to show that $\sigma_\pi(A)$ is closed. Suppose $\lambda^k = (\lambda_1^k, \lambda_2^k)$ is in $\sigma_\pi(A)$ for $k=1, 2, \cdots$ and $\lim x_i^k = \lambda_i$ for $i=1, 2$. Suppose, for contradiction, that $\lambda = (\lambda_1, \lambda_2)$ is not in $\sigma_\pi(A)$. Then by **Lemma 3.3**, $B(H) = B(H)(A_1-\lambda_1) + B(H)(A_2-\lambda_2)$, so there exists operators B_1, B_2 in B(H) such that $I = B_1(A_1-\lambda_1) + B_2(A_2-\lambda_2)$. Then

$$\| I - [B_1(A_1-\lambda_1^k) + B_2(A_2-\lambda_2^k)] \|$$
$$= \| (\lambda_1^k-\lambda_1)B_1 + (\lambda_2^k-\lambda_2)B_2 \|$$
$$\leqq |\lambda_1^k-\lambda_1| \, \| B_1 \| + |\lambda_2^k-\lambda_2| \, \| B_2 \|.$$

Hence for large enough k, we have

$$\| I - [B_1(A_1-\lambda_1^k) + B_2(A_2-\lambda_2^k)] \| < 1.$$

But then $B_1(A_1-\lambda_1^k) + B_2(A_2-\lambda_2^k)$ is an invertible operator which contradicts the fact that $\lambda^{*}=(\lambda_1^{*}, \lambda_2^{*})$ is in $\sigma_{\pi}(A)$. Hence $\sigma_{\pi}(A)$ is closed, and therefore $\sigma_{\pi}(A)$ is compact.

Corolary 3.9. Let $A=(A_1, A_2)$ be a 2-tuple of commuting operators on H. Then $\sigma_{\delta}(A)$ is a nonempty compact subset of C^2.

proof. $\lambda\in\sigma_{\delta}(A)$ if and only if $\lambda^{*}\in\sigma_{\pi}(A^{*})$

if and only if $\lambda\in\sigma_{\pi}(A^{*})^{*}$,

i. e. $\sigma_{\delta}(A)=\sigma_{\pi}(A^{*})^{*}$.

Since $A_1A_2=A_2A_1$, $A_2^{*}A_1^{*}=A_1^{*}A_2^{*}$ by taking adjoints. Therefore $A^{*}=(A_1^{*}, A_2^{*})$ is a 2-tuple of commuting operators. By **Theorem 3.8**, $\sigma_{\pi}(A^{*})$ is a nonempty compact subset of C^2, and hence $\sigma_{\delta}(A)=\sigma_{\pi}(A)^{*}$ is a nonempty compact subset of C^2.

Ⅳ. DASH, TAYLOR, HARTE AND COMMUTANT JOINT SPECTRA.

Throughout this section, F will denote the real field R, or the spectrum field C.

An algebra over F is a vector space $\mathscr{A}$ on F that also has a multiplication on it that makes $\mathscr{A}$ into a ring such that if $\alpha \in$ C and $a, b \in \mathscr{A}$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Definition 4.1. [2] A Banach algebra is an algebra $\mathscr{A}$ over F that has a norm $\| \cdot \|$ relative to which $\mathscr{A}$ is a Banach space and such that for all a, b in $\mathscr{A}$, $\| ab \| \leqq \| a \| \ \| b \|$.

If $\mathscr{A}$ has an identity e, then it is assumed that $\| e \| = 1$.

Examples 4.2. (a) Let C([0, 1]) be the set of all continous functions f: [0, 1]→F. For f, g in C([0, 1]), define $f+g$: [0, 1]→F by $(f+g)(x) = f(x) + g(x)$, for α in F, define $(\alpha f)(x) = \alpha[f(x)]$, and define $\| f \| = \sup\{|f(x)| : x \in [0, 1]\}$. Then C([0, 1]) is a Banch space. Furthermore $\mathscr{A}$=C([0, 1]) is a Banach algebra if $(fg)(x) = f(x)g(x)$ whenever $f, g \in$ A and $x \in [0, 1]$. (b) Let H be Hilbert space and $\mathscr{A}$=B(H). If multiplication is defined by composition, then $\mathscr{A}$ is a Banach algebra with identity I.

If $\mathscr{A}$ is an algebra, a left ideal of $\mathscr{A}$ is a subalgebra M of such that $am \in$ M whenever $a \in \mathscr{A}$, $m \in$ M. A right ideal of a is a subalgebra M such that $ma \in$ M whenever $a \in \mathscr{A}$, $m \in$ M. A(bilateral or two-sided) ideal is an subalgebra of $\mathscr{A}$ that is both a left ideal and a right ideal. ([2], [7])

If $\mathscr{A}$ is a Banach algebra, an involution is a map $a \to a^*$ of $\mathscr{A}$ into $\mathscr{A}$ such that the following properties hold for a and b in $\mathscr{A}$ and α in C;

(i) $(a^*)^* = a$ (ii) $(ab)^* = b^* a^*$ (iii) $(\alpha a + b) = \alpha^* a^* + b^*$. ([2], [7])

Definition 4.3. [2] A C^*-algebra is a Banach algebra $\mathscr{A}$ with an involution such that for every a in $\mathscr{A}$, $\| a^* a \| = \| a \|^2$.

For example, (a) C([0, 1]) is a C^*-algebra where $f^*(x) = \overline{f(x)}$ for f in C([0, 1]) and x in X, and (b) B(H) is a C^*-algebra where for each A in B(H), A^* is the adjoint of A.

A sequence(T_n) from B(H) is said to converge to T$\in$B(H) weakly, or $T_n \to T$ in the weak operator topology (WOT) if for all $x \in H$, the sequence$(T_n x)$ weakly converges to Tx in H, i.e. $\langle T_n x, y \rangle \to \langle Tx, y \rangle$ for all x, y in H. An subalgebra of the algebra B(H) is called a weakly closed algebra if it is closed in the weak operator topology in B(H).

Throughout this section, let $A = (A_1, A_2)$ be a 2-tuple of commuting operators on a complex Hilbert space H. Then the commutant $(A)'$ of the set $\mathscr{J} = \{A_1, A_2\}$ is a weakly closed subalgebra of the algebra B(H) containing $\mathscr{J}$ and the identity I, where $(A)'$ is the set of all operators in B(H) that commute with A_1, A_2, i.e. $(A)' = \{B \in B(H) : BA_j = A_j B \text{ for } j = 1, 2\}$. The double commutant $(A)''$ of the set of $\mathscr{J} = \{A_1, A_2\}$ is a weakly closed abelian subalgebra containing $\mathscr{J}$ and the identity I, where $(A)''$ is the set of all operators in B(H) which commute with the operators of B(H) that commute with A_1, A_2, i.e. $(A)'' = ((A)')'$. We denote by(A) the closed subalgegra of B(H) generated by A_1, A_2. Note that $(A) \subseteq (A)'' \subseteq (A)' \subseteq B(H)$ and that (A) and $(A)''$ are commutative Banach algebras. [4]

Definition 4.4. ([3], [4]) Let $A=(A_1, A_2)$ be a 2-tuple of commuting operators on **H**. (a) The commutant joint spectrum $\sigma'(A)$ of A is defined as the set of all points $\lambda=(\lambda_1, \lambda_2)$ in $\mathbf{C}^2$ such that the set $\{A_1-\lambda_1, A_2-\lambda_2\}$ is contained in a proper(two-sided) ideal of $(A)'$. Equivalently, $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma'(A)$ if and only if for all B_1, B_2 in $(A)'$, $B_1(A_1-\lambda_1)+B_2(A_2+\lambda_2) \neq I$.
(b) The double(Dash) commutant joint spectrum $\sigma''(A)$ of A is defined as ths set of all $\lambda=(\lambda_1, \lambda_2)$ in $\mathbf{C}^2$ such that the closed ideal generated by the set $\{A_1-\lambda_1, A_2-\lambda_2\}$ is a proper ideal in $(A)''$. Equivalently, λ is in $\sigma''(A)$ if and only if for all B_1, B_2 in $(A)''$, $B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2) \neq I$. (c) The Harte joint spectrum $\sigma^H(A)$ is defined as $\sigma^H(A)=\sigma^l(A) \cup \sigma^r(A)$, where the left(right) joint spectrum $\sigma^l(A)$ ($\sigma^r(A)$) is the set of all points $\lambda=(\lambda_1, \lambda_2)$ in $\mathbf{C}^2$ such that $\{A_1-\lambda_1, A_2-\lambda_2\}$ generates a proper left(right) ideal in B(H). Equivalently, $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma^H(A)$ if and only if for all B_1, $B_2 \in B(H)$,

$$B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2) \neq I \text{ or } (A_1-\lambda_1)B_1+(A_2-\lambda_2)B_2 \neq I.$$

Consider the sequence

$$E(H, A) : 0 \to H \xrightarrow{\delta_1} H \oplus H \xrightarrow{\delta_0} H \to 0. \qquad (1)$$

where $\delta_1(x)=(-A_2x) \oplus (A_1x)$ and $\delta_0(x_1 \oplus x_2)=A_1x_1+A_2x_2$, for x_1, $x_2 \in H$. Then it is evident that $\delta_0 \circ \delta_1=0$. We say that a 2-tuple $A=(A_1, A_2)$ of commuting operators on H is nonsingular if the sequence E(H, A) is exact, i.e. Ker $\delta_1=0$, Im $\delta_1=$Ker δ_0 and Im $\delta_0=$H.

Definition 4.5. ([3], [9]) (a) A point $\lambda=(\lambda_1, \lambda_2)\in C$ is in the Taylor joint spectrum $\sigma^T(A)$ of A if $A-\lambda=(A_1-\lambda_1, A_2-\lambda_2)$ is singular.

(b) The joint spectrum $\sigma^3(A)$ of $A=(A_1, A_2)$ is defined as the set of all $\lambda=(\lambda_1, \lambda_2)$ such that the closed ideal generated by $\{A_1-\lambda_1, A_2-\lambda_2\}$ is a proper ideal in (A). Equivalently, $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma^3(A)$ if and only if for all B_1, B_2 in (A),

$B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2)\neq I$ or $(A_1-\lambda_1)B_1+(A_2-\lambda_2)B_2\neq I$.

Recall that all these notions of joint spectra coincide in case of a single operator, as well as in the case of operators on finite dimensional spaces.

Lemma 4.6. For each 2-tuple $A=(A_1, A_2)$ of commuting operators on H, $\sigma^H(A)\subseteq\sigma^T(A)\subseteq\sigma'(A)\subseteq\sigma''(A)\subseteq\sigma^3(A)$.

proof. Since $(A)\subseteq(A)''\subseteq(A)'\subseteq B(H)$, $\sigma'(A)\subseteq\sigma''(A)\subseteq\sigma^3(A)$. The first inclusion is a direct consequence of the proposition [10]. The fact that $\sigma^T(A)\subseteq\sigma'(A)$ is discussed in [9].

Even in the very simplest situations, the Dash joint spectrum is empty ; in the algebra $M_{2\times2}(C)$ of complex 2×2 matrices, take $A=(A_1, A_2)$ with $A_1=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $A_2=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Then the only possible point of the spectrum is (0, 0) by the fact that $\sigma^H(A)\subseteq\sigma(A_1)\times\sigma(A_2)=\{0\}\times\{0\}$, but $A_2A_1+A_1A_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is the identity matrix, i.e. $0=(0,0)\not\in\sigma^H(A)$.

Lemma 4.7. Let $A=(A_1, A_2)$ be a 2-tuple of operators on H. Then (a) $\sigma_\pi(A)=\sigma^l(A)$. (b) $\sigma_\delta(A)=\sigma^r(A)$.

(c) $\lambda=(\lambda_1, \lambda_2)\in\sigma_p(A)$ if and only if $0\in\sigma_p[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]$.

(d) $\lambda=(\lambda_1, \lambda_2)\in\sigma^l(A)$ if and only if $0\in\sigma_\pi[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]$.

(e) $\lambda=(\lambda_1, \lambda_2)\in\sigma^r(A)$ if and only if $0\in\sigma_\pi[\sum_{i=1}^{2}(A_i-\lambda_i)(A_i-\lambda_i)^*]$.

proof. (a) $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma_\pi(A)$ if and only if for all B_1, B_2 in B(H), $B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2)\neq I$ if and only if $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma^l(A)$.

(b) The proof is similar to that of (a).

(c) $\lambda=(\lambda_1, \lambda_2)\in\sigma_p(A)$ if and only if there exists a nonzero vector x in H such that $(A_i-\lambda_i)x=0$, for each $i=1, 2$. If $\lambda\in\sigma(A)$, then there exists $x\neq0$ in H such that $(A_i-\lambda_i)x=0$ for each $i=1, 2$. Thus $[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]x=\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)x=0$ and so $0\in\sigma_p[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]$.

Conversely if $0\in\sigma_p[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]$, then there exiets a nonzero vector x in H such that $[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]x=0$. Thus

$$\begin{aligned}0&=\langle[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]x,\ x\rangle\\&=\langle\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)x,\ x\rangle\\&=\sum_{i=1}^{2}\langle(A_i-\lambda_i)^*(A_i-\lambda_i)x,\ x\rangle\\&=\sum_{i=1}^{2}\langle(A_i-\lambda_i)x,\ (A_i-\lambda_i)x\rangle\\&=\sum_{i=1}^{2}\|(A_i-\lambda_i)x\|^2,\ \text{and so}\ \|A_i-\lambda_i\|=0,\end{aligned}$$

for each $i=1, 2$. Hence $A_ix=\lambda_ix$ for $i=1, 2$, i.e. $\lambda=(\lambda_1, \lambda_2)\in\sigma_p(A)$.

(d) if $\lambda=(\lambda_1, \lambda_2)$ is in $\sigma^l(A)$, then for all B_1, B_2 in B(H),

$B_1(A_1-\lambda_1)+B(A_2-\lambda_2)\neq I$.

So this implies in particular that $(A_1-\lambda_1)^*(A_1-\lambda_1)+(A_2-\lambda_2)^*(A_2-\lambda_2)\neq I$.

Thus $0\in\sigma[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]$. Since $\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)$ is a positive algebras and $\partial\sigma(T)\subseteq\sigma_\pi(T)$, for any operaotr T, $0\in\sigma_\pi[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]$.

Conversely, suppose that $0\in\sigma_\pi[\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)]$.

Then there exists a sequence $\{x_k\}$ in H with $\|x_k\|=1$ such that $\|\sum_{i=1}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)x_k\|\to 0$ as $k\to\infty$.

Therofore we have

$$\begin{aligned}\|(A_i-\lambda_i)\|^2 &= \langle (A_i-\lambda_i)x_k,\ (A_i-\lambda_i)x_k\rangle\\ &= \langle (A_i-\lambda_i)^*(A_i-\lambda_i)x_k,\ x_k\rangle\\ &\leqq \langle \sum_{i=0}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)x_k,\ x_k\rangle\\ &\leqq \|\sum_{i=0}^{2}(A_i-\lambda_i)^*(A_i-\lambda_i)x_k\|\to 0 \quad \text{as } k\to\infty,\end{aligned}$$

for each $i=1,2$. Thus $\lambda=(\lambda_1,\ \lambda_2)\in\sigma_\pi(A)$.

Part(e) follows by taking adjoints.

For each 2-tuple $A=(A_1,\ A_2)$ of commuting operators on H, $\sigma_\pi(A)$ is nonempty by **Theorem** 3.8 and so $\sigma^H(A)$, $\sigma^T(A)$, $\sigma'(A)$, $\sigma''(A)$ and $\sigma^3(A)$ are nonempty.

Theorem 4.8. For each 2-tuple $A=(A_1, A_2)$ of commuting operators on H, $\sigma^H(A)$ is a nonempty compact subset of C^2.

proof. $\sigma^H(A) \subseteq \sigma(A_1) \times \sigma(A_2)$. This is because, if for any $i=1,2$, $\lambda \notin \sigma(A_i)$, then we obtain $B=(B_1, B_2)$ such that $B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2)=I$, by setting,

$B_i=(A_i-\lambda_i)^{-1}$, $B_j=0 (j \neq i)$.

Since each $\sigma(A_i)$ is a bounded subset of $\mathbf{C}$, $\sigma(A)$ is a bounded subset of $\mathbf{C}^2$. To show that $\sigma(A)$ is also closed, suppose that $\lambda=(\lambda_1, \lambda_2) \in \mathbf{C}^2$ is not in $\sigma(A)=\sigma^l(A) \cup \sigma^r(A)$. Then there are B_1, B_2 and C_1, C_2 in $\mathbf{B}(\mathbf{H})$ such that $\sum_{i=1}^{2} B_i(A_i-\lambda_i)=\sum_{i=1}^{2}(A_i-\lambda_i)C_i=I$.

Now if $\mu=(\mu_1, \mu_2) \in \mathbf{C}^2$ is so close to $\lambda=(\lambda_1, \lambda_2)$ in $\mathbf{C}^2$ that $\sum_{i=1}^{2} \| B_i \| \, |\mu_i - \lambda_i| < 1$ and $\sum_{i=1}^{2} \| C_i \| \, |\mu_i - \lambda_i| < 1$, then each of the elements $\sum_{i=1}^{2} B_i(A_i-\mu_i)$ and $\sum_{i=1}^{2}(A_i-\mu_i)C_i$ is invertible in $\mathbf{B}(\mathbf{H})$, since $\| I-\sum_{i=1}^{2} B_i(A_i-\mu_i) \| = \| \sum_{i=1}^{2} B_i(A_i-\lambda_i) - \sum_{i=1}^{2} B_i(A_i-\mu_i) \| = \| \sum_{i=1}^{2} B_i(\mu_i-\lambda_i) \| \leqq \sum_{i=1}^{2} \| B_i \| \, |\mu_i-\lambda_i| < 1$ and $\| I-\sum_{i=1}^{2}(A_i-\mu_i)C_i \| < 1$.
Hence $\mu=(\mu_1, \mu_2) \notin \sigma^H(A)$, and so $\sigma^H(A)$ is closed.

From the proof of the above theorem, we see that $\sigma^l(A)$ and $\sigma^r(A)$ are nonempty compact subsets of $\mathbf{C}^2$ for each 2-tuple $A=(A_1, A_2)$ of commuting operators on $\mathbf{H}$.

Theorem 4.9. [9] For each 2-tuple $A=(A_1, A_2)$ of commuting operators, $\sigma^T(A)$ is a nonempty compact subset of $\mathbf{C}^2$.

Lemma 4.10. Let $A=(A_1, A_2)$ be a 2-tuple of commuting operators on $\mathbf{H}$. If $A=(A_1, A_2)$ is non-singular, then $A_1^*A_1+A_2^*A_2$ and $A_1A_1^*+A_2A_2^*$ are invertible.

proof. If $y=x_1\oplus x_2$, then $\delta_0(y)=A_1x_1+A_2x_2$. And the dual map δ_0^* of δ_0 is the map from H to H⊕H such that $\delta_0^*(x)=A_1^*x\oplus A_2^*x$ for $x\in$H. since

$\langle(x_1\oplus x_2,\ \delta_0^*y\rangle=\langle\delta_0(x_1\oplus x_2),\ y\rangle=\langle A_1x_1+A_2x_2,\ y\rangle$

$=\langle x_1, A_1^*y\rangle+\langle x_2, A_2^*y\rangle=\langle x_1\oplus x_2,\ A_1^*y\oplus A_2^*y\rangle$, i. e. $\delta_0(y)=A_1^*y\oplus A_2^*y$.

Also $\delta_1(x)=(-A_2x)\oplus(A_1x)$ for $x\in$H, so the dual map δ_1^* of δ_1 is the map from H ⊕ H to H such that $\delta_1^*(x_1\oplus x_2)=-A_2^*x_1+A_1^*x_2$.

So we obtain the dual sequence of (1), namely

$$0\rightarrow H\xrightarrow{\delta_0^*}H\oplus H\xrightarrow{\delta_1^*}H\rightarrow 0 \qquad (2)$$

Now let us show that $A_1A_1^*+A_2A_2^*$ is injective and surjective on H. If $(A_1A_1^*+A_2A_2^*)x=0$ for a certain $x\in$H, then $A_1^*x\oplus A_2^*x\in\mathrm{Ker}\,\delta_0$. Since $\mathrm{Ker}\,\delta_0=(\mathrm{Im}\,\delta_0^*)^{\perp}$, $A_1^*x\oplus A_2^*x\in(\mathrm{Im}\,\delta_0^*)^{\perp}$. But $A_1^*x\oplus A_2^*x\in\mathrm{Im}\,\delta_0^*$ by difinition of δ_0^*.

Since $(\mathrm{Im}\,\delta_0^*)\cap(\mathrm{Im}\,\delta_0^*)^{\perp}=\{0\}$, $A_1^*x=A_2^*x=0$. Since $\mathrm{Ker}\,\delta_0^*=\{0\}$, we have $x=0$. Hence $A_1A_1^*+A_2A_2^*$ is injective.

Take an arbitrary $y\in$H and let us find an $x\in$H such that $y=(A_1A_1^*+A_2A_2^*)x$. We infer that $\delta_0:(\mathrm{Ker}\,\delta_0)^{\perp}\rightarrow$H is an isomorphism since $\mathrm{Im}\,\delta_0=$H, and therefore $y=\delta_0(y_1\oplus y_2)$ with $y_1\oplus y_2\in(\mathrm{Ker}\,\delta_0)^{\perp}=\mathrm{Im}\,\delta_0^*$. Hence $y_1\oplus y_2=\delta_0^*(x)=A_1^*x\oplus A_2^*x$ for some $x\in$H. Therefore

$y=\delta_0(y_1\oplus y_2)=\delta_0(A_1^*x\oplus A_2^*x)=A_1(A_1^*x)+A_2(A_2^*x)=(A_1A_1^*+A_2A_2^*)x$,

i. e. $A_1A_1^*+A_2A_2^*$ is surjective.

Analogously, the operator $A_1^*A_1+A_2A_2^*$ is invertible and this completes the proof.

Theorem 4. 11. Let $A=(A_1, A_2)$ be a 2-tuple of commuting operators on H. Then A is non-singular on H if and only if the operator

$$\alpha(A)=\begin{bmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{bmatrix}$$

is invertible on $H \oplus H$.

proof. If $A=(A_1, A_2)$ is non-singular on H, then both $A_1A_1^*+A_2A_2^*$ and $A_1^*A_1+A_2^*A_2$ are invertible on H by the above lemma. Clearly the operator

$$\begin{bmatrix} A_1^*(A_1A_1^*+A_2A_2^*)^{-1} & -A_2(A_1^*A_1+A_2^*A_2)^{-1} \\ A_2^*(A_1A_1^*+A_2A_2^*)^{-1} & A_1(A_1^*A_1+A_2^*A_2)^{-1} \end{bmatrix}$$

is a left inverse for the operator $\alpha(A)$, since

$$\begin{bmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{bmatrix}\begin{bmatrix} A_1^*(A_1A_1^*+A_2A_2^*)^{-1} & -A_2(A_1^*A_1+A_2^*A_2)^{-1} \\ A_2^*(A_1A_1^*+A_2A_2^*)^{-1} & A_1(A_2^*A_1+A_2^*A_2)^{-1} \end{bmatrix}$$

$$=\begin{bmatrix} (A_1A_1^*+A_2A_2^*)(A_1A_1^*+A_2A_2^*)^{-1} & (-A_1A_2+A_2A_1)(A_1^*A_1+A_2^*A_2)^{-1} \\ (-A_2^*A_1^*+A_1^*A_2^*)(A_1A_1^*+A_2A_2^*)^{-1} & (A_2^*A_2+A_1^*A_1)(A_1^*A_1+A_2^*A_2)^{-1} \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Hence $\alpha(A)$ is surjective on $H\oplus H$. Let us also notice that $\alpha(A)$ is injective too. Indeed, if $\alpha(A)(x_1\oplus x_2)=0$, then

$$\alpha(A)(x_1\oplus x_2)=\begin{bmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{bmatrix}\begin{bmatrix} x_1 \\ \oplus \\ x_2 \end{bmatrix}=(A_1x_1+A_2x_2)\oplus(-A_2^*x_1+A_1^*x_2)=0.$$

Thus $x_1\oplus x_2 \in \text{Ker}\,\delta_0\cap\text{Ker}\,\delta_0^*=\text{Ker}\,\delta_0\cap(\text{Im}\,\delta_1)^{\perp}=\text{Ker}\,\delta_0\cap(\text{Ker}\,\delta_0)^{\perp}=\{0\}$, and hence $x_1=x_2=0$.

Conversely, suppose that $\alpha(A)$ is invertible on H⊕H. Then $\alpha(A)^*$ is invertible. Therefore

$$\alpha(A)\cdot\alpha(A)^* = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{bmatrix}\begin{bmatrix} A_1^* & -A_2 \\ A_2^* & A_1 \end{bmatrix} = \begin{bmatrix} A_1A_1^*+A_2A_2^* & -A_1A_2+A_2A_1 \\ -A_2^*A_1^*+A_1^*A_2^* & A_2^*A_2+A_1^*A_1 \end{bmatrix}$$

$$= \begin{bmatrix} A_1A_1^*+A_2A_2^* & 0 \\ 0 & A_1^*A_1+A_2^*A_2 \end{bmatrix}$$

is invertible and hence $(A_1A_1^*+A_2A_2^*)^{-1}$ and $(A_1^*A_1+A_2^*A_2)^{-1}$ are operators in B(H).

Let us prove that the sequence $0 \xrightarrow{\delta_1} H \rightarrow H\oplus H \xrightarrow{\delta_0} H \rightarrow 0$ is exact. Indeed, if $\delta_1(x)=(-A_2x)\oplus(A_1x)=0$, then $(A_1^*A_1+A_2^*A_2)x=0$, because $0=\langle(-A_2x)\oplus(A_1x), (-A_2x)\oplus(A_1x)\rangle=\langle(-A_2x), (-A_2x)\rangle+\langle -A_1x, A_1x\rangle$

$=\langle A_2^*A_2x, x\rangle+\langle A_1^*A_1x\rangle=\langle(A_1^*A_1+A_2^*A_2)x, x\rangle$.

Hence $x=0$, and so Ker $\delta_1=\{0\}$. Assume now that $\delta_0(x_1\oplus x_2)=A_1x_1+A_2x_2=0$ (i. e. $x_1\oplus x_2\in$ Ker δ_0). If $y=-A_2^*x_1+A_1^*x_2$, then

$$\alpha(A)(x_1\oplus x_2)=\begin{bmatrix} A_1 & A_2 \\ & \\ -A_2^* & A_1^* \end{bmatrix}\begin{bmatrix} x_1 \\ \oplus \\ x_2 \end{bmatrix}=\begin{bmatrix} A_1x_1+A_2x_2 \\ \oplus \\ -A_2^*x_1+A_1^*x_2 \end{bmatrix} = 0\oplus y.$$

Hence $x_1\oplus x_2=\alpha(A)^{-1}(0\oplus y)$, and Thus by(*),

$$x_1\oplus x_2=\begin{bmatrix} A_1^*(A_1A_1^*+A_2A_2^*)^{-1} & -A_2(A_1^*A_1+A_2^*A_2)^{-1} \\ A_2^*(A_1A_1^*+A_2A_2^*)^{-1} & A_1(A_1^*A_1+A_2^*A_2)^{-1} \end{bmatrix}(0\oplus y)$$

$$\text{and so}\begin{cases} x_1 = -A_2(A_1{}^*A_1 + A_2{}^*A_2)^{-1}\, y \\ x_2 = A_1(A_1{}^*A_1 + A_2{}^*A_2)^{-1}\, y \end{cases}$$

i. e. $x_1 \oplus x_2 = [-A_2 By \oplus A_1 By] = \delta_1(By)$, where $B = (A_1{}^*A_1 + A_2{}^*A_2)^{-1}$, i. e. $x_1 \oplus x_2 \in \mathrm{Im}\,\delta_1$. Hence $\mathrm{Ker}\,\delta_0 \subseteq \mathrm{Im}\,\delta_1$. Since $\delta_0 \circ \delta_1 = 0$, $\mathrm{Im}\,\delta_1 \subseteq \mathrm{Ker}\,\delta_0$. Therfore $\mathrm{Ker}\,\delta_0 = \mathrm{Im}\,\delta_1$.

Finally, if $y \in H$ is arbitrary, the $x_i = A_i{}^*(A_1A_1{}^* + A_2A_2{}^*)^{-1}y$ $(i=1,2)$ satisfy the equation

$$\begin{aligned} A_1x_1 + A_2x_2 &= A_1A_1{}^*(A_1A_1{}^* + A_2A_2{}^*)^{-1}y + A_2A_2{}^*(A_1A_1{}^* + A_2A_2{}^*)^{-1}y \\ &= (A_1A_1{}^* + A_2A_2{}^*)(A_1A_1{}^* + A_2A_2{}^*)^{-1}y = y, \end{aligned}$$

i. e. $y = \delta_0(x_1 \oplus x_2)$ and so $y \in \mathrm{Im}\,\delta_0$. Hence $H = \mathrm{Im}\,\delta_0$.

Therefore $0 \to H \xrightarrow{\delta_1} H \oplus H \xrightarrow{\delta_0} H \to 0$ is exact and so $A = (A_1, A_2)$ is non-singular on H.

Theorem 4. 12. Let $A = (A_1, A_2)$ be an n-tuple of commuting normal operators. Then $\sigma_\pi(A) = \sigma''(A) = \sigma^H(A) = \sigma^T(A)$.

proof. Since $\sigma_r(A) \cup \sigma_l(A) = \sigma^H(A) \subseteq \sigma^T(A) \subseteq \sigma'(A) \subseteq \sigma''(A)$, and $\sigma^r(A) = \sigma_\pi(A)$, it suffices to show that if $0 \in \sigma''(A)$, then $0 \in \sigma_\pi(A)$. By defintion, 0 is in $\sigma''(A)$ if and only if $\sum_{i=1}^{2} B_iA_i \neq I$ for all B_1, B_2 in $(A)''$. Since $(A)''$ is a selfadjoint algebra [4], this means in particular that $A_1{}^*A_1 + A_2{}^*A_2$ is not invertible. Thus there exists a sequence $\{x_k\}$ of unit vectors in H such that $\|(\sum_{i=1}^{2} A_i{}^*A_i)x_k\| \to 0$ as $k \to \infty$. Hence $\sum_{i=1}^{2} \|A_ix_k\|^2 \to 0$ as $k \to \infty$, and therefore $\|A_ix_k\| \to 0$ as $k \to \infty$ for $i = 1, 2$, i. e. $0 = (0, 0) \in \sigma_\pi(A)$

Corollary 4. 13. Let $A(A_1, A_2)$ be a 2-tuple of commuting normal operators on H. Then $A=(A_1, A_2)$ is non-singular if and only if $A_1^*A_1+A_2^*A_2$ is invertible on H.

proof. If $A=(A_1, A_2)$ is non-singular, $A_1^*A_1+A_2^*A_2$ is invertible by Lemma 4.10.

We must show that if $A_1^*A_1+A_2^*A_2$ is invertible, the $A=(A_1, A_2)$ is non-singular. If $A=(A_1, A_2)$ is singular, then $0\in\sigma^T(A)=\sigma_\pi(A)=\sigma^l(A)$ by the above theorem. By Lemma 4.7 (d),

$$0\in\sigma_\pi(\sum_{i=1}^{2} A_i^*A_i)\subseteq\sigma(\sum_{i=1}^{2} A_i^*A_i).$$

Hence $A_1^*A_1+A_2^*A_2$ is not invertible.

From the above facts, we have the following questions ;

(a) Is it true that $\sigma^H(A)=\sigma^T(A)=\sigma'(A)=\sigma''(A)$ for a 2-tuple $A=(A_1, A_2)$ of commuting non-normal operators on H?

(b) Is it true that $\sigma_\pi(A)=\sigma''(A)$ for each 2-tuple $A=(A_1, A_2)$ of commuting non-normal operators on H?

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〈국문초록〉

Hilbert 空間上의 接合 스펙트럼의 분류

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複素 Hilbert 空間上의 有界線型의 作用素 T의 點스펙트럼(point spectrum) $\sigma_p(T)$, 點近似 스펙트럼(approximate point spectrum) $\sigma_{ap}(T)$과 스펙트럼 $\sigma(T)$에 관한 여러가지 성질중 잘 알려진 중요한 結果는 다음과 같다.

(1) $\sigma_p(T) \subset \sigma_{ap}(T) \subset \sigma(T)$.

(2) σ_{ap} (T)와 $\sigma(T)$는 空集合의 아닌 C의 compact 部分集合이고,

(3) $\sigma(T)$의 境界(boundary) $\partial\sigma(T)$는 $\sigma(T)$의 部分集合이다. 本 論文에서는 스펙트럼의 체계적 研究의 시도로서 위 結果들을 배경으로 하여, 첫째, 복소 Hilbert 공간상에서 作用素 A_1, A_2의 쌍 $A=(A_1, A_2)$의 여러종류의 接合點 스펙트럼(joint point spectrum)의 同値條件, 포함관계를 상세하게 조사는 물론 A_1과 A_2가 交換일 때 접합 點近似 스펙트럼(joint approximate point spectrum) $\sigma_{ap}(A)$과 접합 approximate compression 스펙트럼 $\sigma_\delta(A)$는 空集合이 아니고 이차원 복소 유클리드 공간 C^2의 compact 部分集合임을 밝히고 있다.

둘째로, 복소 Hilbert 공간상에서 交換하는 作用素 A_1, A_2의 쌍 $A=(A_1, A_2)$에 대

하여 Dash, Harte, Taylor 등의 立場에서 본 여러 종류의 接合 스펙트럼($\sigma^H(A)$, σ^T $\sigma'(A)$, $\sigma''(A)$)의 同値條件과 포함관계 등의 硏究는 물론 이들 接合 스펙트럼이 공집합이 아니고 C^2의 compact 部分集合 여부를 조사하였다. 한편 交換하는 정규 作用素 A_1, A_2의 여러 접합 스펙트럼과 접합 점근사 스펙트럼이 같음을 밝힘과 동시에 이들에 관한 문제를 제시하였다.

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