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# ON CONVERGENCES OF MEASURABLE SETS AND FUNCTIONS

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〈國文抄錄〉

# 可測集合과 函數의 收斂에 관한 硏究


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實數列 < $a_n$ >의 收斂에 관한 여러가지 성질중 잘 알려진 중요한 結果는 다음과 같다.

(1) 실수열 < $a_n$ >이 $a \in R$에 수렴하기 위한 필요충분조건은 $\lim a_n = \lim a_n = a$.

(2) (Cauchy 의 判定法) 실수열 < $a_n$ >이 收斂하기 위한 必要充分條件은 < $a_n$ >이 Cauchy 數列이다.





本 論文에서는 위 결과의 暗示에서 集合으로 구성된 數列의 下限(limit inferior), 上限(limit superior), 極限에 관한 性質, 1904년 Henri Lebesque에 의해 소개된 길이의 개념을 一般化한 測度를 이용하여 정의된 可測集合의 極限과 可測函數列의 여러가지 收斂 개념의 性質등을 조사하였다.

Ⅱ절에서는 集合으로 구성된 單調增加數列, 單調減少數列, 數列의 極限, 下限, 上限과 極限의 性質에 대하여 조사는 물론, 合集合, 共通集合등을 이용하여 두 개의 集合列에서 새롭게 구성된 集合列의 極限에 대하여 밝히었다.

Ⅲ절에서는 可測集合의 同値條件과 單調減少 可測集合列 < $A_n$ >의 極限의 測度

는 $A_n$의 測度로 구성된 실수열 $\langle m(A_n) \rangle$의 極限과 같게되는 일반적인 條件제시뿐만 아니라 單調增加集合列 $\langle A_n \rangle$의 上限의 測度와 $A_n$의 測度로 구성된 실수열 $\langle m(A_n) \rangle$의 上限과의 관계를 조사하였다.

Ⅳ절에서는 고정된 測度空間에서 정의된 可測函數列의 여러가지 수렴개념 (Convergence almost everywhere, Convergence in measure, Convergence almost uniformly 등)의 여러가지 성질 (상수곱, 합, 절대치, max, min 함수에 대한 보존문제), Cauchy立場에서 본 이들의 개념으로 부터 (2)와 같은 결과의 성립여부 및 여러가지 수렴개념사이의 관계를 연구하였다.

# I. INTRODUCTION

Let $A \subset R$ be the union of a finite number of pairwise disjoint intervals. In a sense, the "length" of this set $A$ is the sum of the lengths of its separate intervals. Even if $A$ is the countably infinite union of pairwise disjoint intervals $I_1, I_2, \ldots$, then the series $\sum_{i=1}^{\infty} l(I_i)$ either converges or diverges to $\infty$; in either case the result is intuitively " the lenth of the set ".

Since an open set $G \subset R$ is either an open interval or a countable union of pairwise disjoint open intervals, then $G$ seems to have, theoretically at least, a length. But is there a reasonable (and useful) way of assigning a " length " to a set which is not necessarily a countable union of intervals ?

In 1904 Henri Lebesque introduced a generalization of the notion of length, which is both intuitive and has many applications, extensions, and abstractions.



On the other hand, it is well-known that (a) A sequence $\{a_n\}$ of real numbers converge to $a \in R$ if and only if

$$\underline{\lim}\, a_n = \overline{\lim}\, a_n = a,$$

(b) (Cauchy Convergence Criterion) A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

These facts give us a motivation to investigate several concepts of convergences about sequence of sets, sequence of measurable sets and sequence of measurable functions.

In section II, we will study properties of limit interior and limit superior of sequence $\{A_n\}$ of sets. Also given two sequences $\{A_n\}$ and $\{B_n\}$ of sets, we give the inclusion relations among limits of the new sequences $\{A_n \cap B_n\}$, and $\{A_n \cup B_n\}$.

In section III , for each sequence $\{A_n\}$ of measurable sets, we will investigate the relations between the measure of limit inferior (resp.superior) of $\{A_n\}$ and the limit inferior (resp.superior) of measures of $A_n$.

In section IV , we will introduce several concepts on convergence (convergence almost everywhere, convergence in measure, convergence almost uniformly etc.) of sequence of measurable functions, and study the properties of these convergences, and the relations among these.

## II. SEQUENCE OF SETS

Given any real sequence $\{a_n\}$,

$$\underline{a} = \sup_{n\geq 1}\left(\inf_{k\geq n} a_k\right) \quad \text{and} \quad \overline{a} = \inf_{n\geq 1}\left(\sup_{k\geq n} a_k\right)$$

are the limit inferior and limit superior respectively of the sequence $\{a_n\}$, and denoted by $\underline{a} = \underline{\lim}\, a_n = \liminf a_n$ and $\overline{a} = \overline{\lim}\, a_n = \limsup a_n$.

**Definition 2.1.** A sequence of sets $\{A_n\}$ is said to be monotonically decreasing if $A_1 \supset A_2 \supset A_3 \supset \cdots$, and monotonically increasing if $A_1 \subset A_2 \subset A_3 \subset \cdots$.

For example, the sequence $\{A_n\}$ defined by $A_n = \{x \mid 0 \leq x \leq 1 + \frac{1}{n}\}$ is monotonically decreasing,

$$\cap A_n = \bigcap_{k=m}^{\infty} A_k = \{x \mid 0 \leq x \leq 1\} \quad \text{and} \quad \bigcup_{k=n}^{\infty} A_k = A_n.$$

**Definition 2.2.** Given a sequence of sets $\{A_n\}$, form the sequence $\{\underline{A}_n\}$ and $\{\overline{A}_n\}$ by setting,

$$\underline{A}_n = \bigcap_{k\geq n} A_k \quad \text{and} \quad \overline{A}_n = \bigcup_{k\geq n} A_k \quad \text{for } n = 1, 2, 3, \cdots$$

and then set

$$\underline{A} = \bigcup_{n\geq 1} \underline{A}_n \quad \text{and} \quad \overline{A} = \bigcap_{n\geq 1} \overline{A}_n.$$

$\underline{A}$ is called the lower limit (or limit inferior) and $\overline{A}$ is the upper limit (or limit superior) of $\{A_n\}$, and denoted by $\underline{A} = \underline{\lim}_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$ and $\overline{A} = \overline{\lim}_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$.

If $\overline{A} = \underline{A}$, then this common set is denoted by $A$, the sequence $\{A_n\}$ is said to have limit $A$, and this is denoted by $A = \lim A_n$.

**Example 2.3.** Let $\{A_n\}$ be defined by

$$A_n = \begin{cases} [0, 2, -\frac{1}{n}] & \text{for } n \text{ odd.} \\ [0, 3 + \frac{1}{n}] & \text{for } n \text{ even.} \end{cases}$$



Then

$$\underline{\lim}\, A_n = [0, 2] \quad \text{and} \quad \overline{\lim}\, A_n = [0, 3].$$

**Theorem 2.4.** If $m$ and $n$ are any positive integers, then

$$\underline{A}_n \subset \overline{A}_m.$$

Proof. If $n \leq m$, then

$$\underline{A}_n = \bigcap_{k\geq n} A_k \subset \bigcap_{k\geq m} A_k \subset \bigcup_{k\geq m} A_k = \overline{A}_m,$$

but if $m \leq n$, then

$$\overline{A}_m = \bigcup_{k \geq m} A_k \supset \bigcup_{k \geq n} A_k \supset \bigcap_{k \geq n} A_k = \underline{A}_n.$$

**Theorem 2.5.** $\underline{\lim}\, A_n \subset \overline{\lim}\, A_n$.

Proof. Let $n \in N$, where $N$ is the set of natural numbers. Then from Theorem 2.4,

$$\underline{A}_n \subset \cap \overline{A}_m = \overline{A}.$$

Thus $\underline{A} = \cup \underline{A}_n \subset \cup \overline{A} = \overline{A}$.

**Corollary 2.6.** $\{\underline{A}_n\}$ is monotonically increasing and $\{\overline{A}_n\}$ is monotonically decreasing.



**Theorem 2.7.** Any monotonically decreasing sequence $\{A_n\}$ has

$$\lim A_n = \cap A_n.$$

Proof. By hypothesis,

$$\underline{A}_n = \bigcap_{k \geq n} A_k = \bigcap_{k \geq n-1} A_k = \cdots = \bigcap_{k \geq 1} A_k.$$

and hence

$$\underline{A} = \cup \underline{A}_n = \cap A_k,$$

but also

$$\overline{A}_n = \bigcup_{k \geq n} A_k = A_n \quad \text{so that} \quad \overline{A} = \cap \overline{A}_n = \cap A_n.$$

**Theorem 2.8.** Any monotonically increasing sequence $\{A_n\}$ has

$$\lim A_n = \cup A_n.$$

Proof. By hypothesis,

$$\overline{A}_n = \bigcup_{k \geq n} A_k = \bigcup_{k \geq n-1} = \cdots = \bigcup_{k \geq 1} A_k.$$

and hence

$$\overline{A} = \cap \overline{A}_n = \cap [\bigcup_{k \geq 1} A_k] = \cup A_k.$$



but also

$$\underline{A}_n = \bigcap_{k \geq n} A_k = A_n \quad \text{so that} \quad \underline{A} = \bigcup_{n \geq 1} \underline{A}_n = \cup A_n.$$

**Theorem 2.9.** For every sequence of sets $\{A_n\}$,

$\underline{\lim} A_n = \{x \mid x \in A_n$ for all except finitely many $n\}$; that is, $x \in \underline{A}$ iff $\exists n_0 \in N$ such that $n \geq n_0 \Rightarrow x \in A_n$.

Proof. Let $x \in \underline{A} = \cup \underline{A}_n$. Hence $\exists n_0$ such that $x \in \underline{A}_{n_0} = \bigcap_{n \geq n_0} A_n$.

Therefore $n \geq n_0 \Rightarrow x \in A_n$.

Conversely, let $x$ and $n_0$ be such that $n \geq n_0 \Rightarrow x \in A_n$. Then

$$x \in \bigcap_{n \geq n_0} A_n = \underline{A}_n \subset \bigcup_{n \geq n_0} A_n \subset \bigcup \underline{A}_n = \underline{A}.$$

**Theorem 2.10.** For every sequence of sets $\{A_n\}$,

$\overline{\lim}\, A_n = \{\, x \mid x \in A_n \quad \text{for infinitely many } n \,\}$; that is, $x \in \overline{A}$ iff for each $k \in N$ there is an $n \geq k$ for which $x \in A_n$.

Proof. Let $x \in \overline{A} = \cap \overline{A}_n$. Then for each $k$, $x \in \overline{A}_k$. Therefore for each $k$, $\exists\, n_0$ such that $n_0 \geq k$ and $x \in A_{n_0}$.

Conversely, assume that for each $k, \exists\, n \geq k$ for which $x \in A_n$. Then for each $k, x \in \bigcup_{n \geq k} A_n = \overline{A}_k$. Therefore $x \in \bigcap \overline{A}_k = \overline{A}$.

**Theorem 2.11.** Let $A_n$ be a subset of a universal set $U$. Then



$$\underline{\lim}\, {A_n}^c = (\overline{\lim}\, A_n)^c \quad \text{and} \quad \overline{\lim}\, {A_n}^c = (\underline{\lim}\, A_n)^c.$$

Proof. Let $B_n = {A_n}^c$. Then

$$\underline{B}_n = \bigcap_{k \geq n} B_k = \bigcap_{k \geq n} {A_k}^c = \left( \bigcup_{k \geq n} A_k \right)^c = (\overline{A}_n)^c \quad \text{and}$$

$$\underline{B} = \cup \underline{B}_n = \cup (\overline{A}_n)^c = (\cap \overline{A}_n)^c = (\overline{A})^c.$$

Therfore $\underline{\lim}\, {A_n}^c = (\overline{\lim}\, A_n)^c$. Similarly, $\overline{\lim}\, {A_n}^c = (\underline{\lim}\, A_n)^c$.

**Theorem 2.12.** Given two sequence $\{A_n\}$ and $\{B_n\}$, form the sequence $\{C_n\}$ by setting $C_n = A_n \cup B_n$. Then

$$\underline{A} \cup \underline{B} \subset \underline{C} \subset \left\{ \begin{matrix} \underline{A} \cup \overline{B} \\ \overline{A} \cup \underline{B} \end{matrix} \right\} \subset \overline{C} = \overline{A} \cup \overline{B}.$$

Proof. (1) Let $x \in \underline{A}$. By Theorem 2.9 , choose $n_0$ such that $n \geq n_0 \Rightarrow x \in A_n$. Then $n \geq n_0 \Rightarrow x \in A_n \cup B_n = C_n$. Thus $x \in \underline{C}$. Hence $\underline{A} \subset \underline{C}$ and similarly $\underline{B} \subset \underline{C}$, so that $\underline{A} \cup \underline{B} \subset \underline{C}$.

(2) Let $x \in \underline{C}$. By Theorem 2.9, choose $n_1$ such that $n \geq n_1 \Rightarrow x \in C_n = A_n \cup B_n$. If $x \in B_n$ for only finitely many $n$, $\exists\, n_2$ such that $n \geq n_2 \Rightarrow x \in A_n$ and hence with $n_3 = \max(n_1, n_2), n \geq n_3 \Rightarrow x \in A_n$ so that $x \in \underline{A} \subseteq \underline{A} \cup \overline{B}$. If $x \in B_n$ for infinitely many $n$, then by Theorem 2.10, $x \in \overline{B}$. In either case, $x \in \underline{A} \cup \overline{B}$. Hence $\underline{C} \subset \underline{A} \cup \overline{B}$. Similarly, $\underline{C} \subset \overline{A} \cup \underline{B}$.

(3) Let $x \in \underline{A} \cup \overline{B}$. Then $x \in \underline{A}$ or $x \in \overline{B}$. If $x \in \underline{A}$, then $\exists\, n_1$ such that $n \geq n_1 \Rightarrow x \in A_n \subset C_n$. Hence $x \in \underline{C} \subset \overline{C}$. Thus $\underline{A} \subset \overline{C}$. If $x \in \overline{B}$, then for each $n_2$ $\exists\, n \geq n_2$ such that $x \in B_n \subset C_n$. Hence $x \in \overline{C}$. Thus $\overline{B} \subset \overline{C}$. Therefore $\underline{A} \cup \overline{B} \subset \overline{C}$. Similarly, $\overline{A} \cup \underline{B} \subset \overline{C}$.

(4) Let $x \in \overline{C}$. Then for each $n_0$, $\exists\, n \geq n_0$ for which $x \in C_n$. Hence $x \in A_n$ or $x \in B_n$. Thus $x \in \overline{A}$ or $x \in \overline{B}$. Therefore $\overline{C} \subset \overline{A} \cup \overline{B}$. Let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then for each $n_1$ $\exists\, n \geq n_1$ for which $x \in A_n \subset C_n$. Thus $\overline{A} \subset \overline{C}$. Similarly, $\overline{B} \subset \overline{C}$. Therefore $\overline{A} \cup \overline{B} \subset \overline{C}$.

**Theorem 2.13.** Given two sequence $\{A_n\}$ and $\{B_n\}$, form the sequence $\{C_n\}$ by setting $C_n = A_n \cap B_n$. Then

$$\underline{A} \cap \underline{B} = \underline{C} \subset \left\{ \begin{matrix} \overline{A} \cap \underline{B} \\ \underline{A} \cap \overline{B} \end{matrix} \right\} \subset \overline{C} \subset \overline{A} \cap \overline{B}.$$

Proof. (1) Let $x \in \underline{A} \cap \underline{B}$. Then $x \in \underline{A}$ and $x \in \underline{B}$. Since $x \in \underline{A}$, then there exists $n_1 \in N$ such that $n \geq n_1 \Rightarrow x \in A_n$. and similarly, since $x \in \underline{B}$, there exists $n_2 \in N$ such that $n \geq n_2 \Rightarrow x \in B_n$. Taking $n_3 = \max(n_1, n_2)$, $n \geq n_3 \Rightarrow x \in A_n \cap B_n = C_n$. Hence $x \in \underline{C}$.

Conversely, let $x \in \underline{C}$. Then there exists $n_0$ such that $n \geq n_0 \Rightarrow x \in C_n = A_n \cap B_n$. Hence $n \geq n_0 \Rightarrow x \in A_n$ and $x \in B_n$. Therefore $x \in \underline{A}$ and $x \in \underline{B}$, by Theorem 2.9 and so $x \in \underline{A} \cap \underline{B}$.

(2) By Theorem 2.5, $\underline{C} = \underline{A} \cap \underline{B} \subset \overline{A} \cap \underline{B}$. Similarly, $\underline{C} \subset \underline{A} \cap \overline{B}$.

(3) Let $x \in \overline{A} \cap \underline{B}$. Then $x \in \overline{A}$ and $x \in \underline{B}$. Since $x \in \overline{A}$, for each $n_1 \in N$ there exists $n \geq n_1$ for which $x \in A_n$. Since $x \in \underline{B}$, there exists $n_2 \in N$ such that $n \geq n_2 \Rightarrow x \in B_n$. For each $n_1 \in N$, there exists $n \geq \max(n_1, n_2)$ for which $x \in C_n = A_n \cap B_n$. Hence $x \in \overline{C}$.

(4) Let $x \in \overline{C}$. Then for each $n_0$ there exists $n \geq n_0$ for which $x \in C_n = A_n \cap B_n$. Thus $x \in \overline{A}$ and $x \in \overline{B}$ i.e.,$x \in \overline{A} \cap \overline{B}$.

**Remark 2.14.** Let $A, B$ are any sets on $R$. Define $A \times B = \{(a, b) : a \in A, b \in B\}$. Then

(1) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and $B$ is any set, then put $C_n = A_n \times B$ for $n = 1, 2, 3, \cdots$. Hence $C_n \subseteq C_{n+1}$ and so $\lim_{n\to\infty} C_n = \cup C_n$.

(2) If $A_1 \supseteq A_2 \supseteq A_3 \cdots$ and $B$ is any set, then put $C_n = A_n \times B$ for $n = 1, 2, 3, \cdots$. Thus $C_n \supseteq C_{n+1}$ and so $\lim_{n\to\infty} C_n = \cap C_n$.

(3) Given two sequence $\{A_n\}$ and $\{B_n\}$, Put $C_n = A_n \times B_n$ for $n = 1, 2, 3, \cdots$.

Then $\underline{A} \times \underline{B} \subset \underline{C} \subset \left\{ \begin{matrix} \underline{A} \times \overline{B} \\ \overline{A} \times \underline{B} \end{matrix} \right\} \subset \overline{C} = \overline{A} \times \overline{B}$.

## III. SEQUENCE OF MEASURABLE SETS

**Definition 3.1** The outer measure of a set $A \subset R$ is denoted by $m^*(A)$ and defined by

$$m^*(A) = \inf\{ \sum l(I_n) \mid A \subset \cup I_n \} = \inf_{A \subset \cup I_n} \sum l(I_n),$$

where each $I_n$ is an open interval in $R$ and $l(I_n)$ is its length.

From the definition, it is easy to show that

(a) $m^*(A) \geq 0$.

(b) If $A \subset B$, then $m^*(A) \leq m^*(B)$.

(c) $m^*(x) = 0$ for any $x \in R$.

(d) $m^*(\emptyset) = 0$.

(e) If $A$ is countable, then $m^*(A) = 0$.



**Theorem 3.2.** For any sequence $\{A_n\}$,

$$m^*(\cup A_n) \leq \sum m^*(A_n).$$

Proof. If one of the sets $A_n$ has infinite outer measure, the inequality holds trivially. If $m^*(A_n)$ is finite for every $n$,then given $\epsilon > 0$, there exists a sequence $\{I_{n,i}\}_i$, where $i = 1, 2, 3, \cdots$ such that $A_n \subset \bigcup_i I_{n,i}$ and $\sum_i l(I_{n,i}) \leq$

$m^*(A_n) + 2^{-n}\epsilon$. Now the collection $\{I_{n,i}\}_{n,i} = \bigcup_n \{I_{n,i}\}_i$ is countable being the union of countable number of countable collections, and covers $\cup A_n$. Thus

$$m^*(\cup A_n) \le \sum_{n,i} l(I_{n,i}) = \sum_n \sum_i l(I_{n,i})$$

$$\le \sum_n \left(m^*(A_n) + \epsilon\, 2^{-n}\right)$$

$$= \sum m^*(A_n) + \epsilon.$$

Since $\epsilon$ was arbitrary positive number,

$$m^*(\cup A_n) \le \sum m^*(A_n).$$

**Definition 3.3.** A set $A$ is said to be measureble, if for each set $E$,



(1) $$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c).$$

**Remark 3.4.** (a) A set $A$ is measurable iff for every set $A$,

(2) $$m^*(E) \ge m^*(E \cap A) + m^*(E \cap A^c).$$

(b) A set $A$ is measurable iff $A^c$ is measurable.

**Theorem 3.5.** A set $A$ is measurable iff (2) holds for every set $E$ having $m^*(E) < \infty$.

Proof. If $m^*(E) = \infty$ and $A$ is any set whatever, then

$$m^*(E \cap A) + m^*(E \cap A^c) = \infty.$$

Hence if $m^*(E) = \infty$, then (1) holds as $\infty = \infty$ whether $A$ is measurable or not. Thus a set $A$ is measurable iff (2) holds for all set $E$ with $m^*(E) < \infty$.

**Theorem 3.6.** If $m^*(A) = 0$, then $A$ is measurable.

Proof. Let $E$ be any set. Since $E \cap A \subset A$ and $E \cap A^c \subset E$. Therefore,

$$\begin{aligned} m^*(E \cap A) + m^*(E \cap A^c) &\le m^*(A) + m^*(E) \\ &= m^*(E). \end{aligned}$$

**Theorem 3.7.** If $A_1, A_2, A_3, \cdots, A_n$ with $n \in N$ are each measurable, then



$$\bigcup_{k=1}^{n} A_k \quad \text{and} \quad \bigcap_{k=1}^{n} A_k$$

are measurable.

Proof. First, for $n = 2$, let $E$ be any set. Since $A_2$ is measurable,

$$(*) \qquad m^*(E \cap A_1{}^c) = m^*(E \cap A_1{}^c \cap A_2) + m^*(E \cap A_1{}^c \cap A_2{}^c)$$

and

$$E \cap (A_1 \cup A_2) = (E \cap A_1) \cup (E \cap A_2 \cap A_1{}^c).$$

Therefore

$$m^*(E \cap [A_1 \cup A_2]) + m^*(E \cap [A_1 \cup A_2]^c)$$
$$\leq m^*(E \cap A_1) + m^*(E \cap A_2 \cap A_1{}^c) + m^*(E \cap A_1{}^c \cap A_2{}^c)$$
$$= m^*(E \cap A_1) + m^*(E \cap A_1{}^c) \qquad \text{,by } (*)$$
$$= m^*(E) \quad \text{,by Definition.}$$

Thus $A_1 \cup A_2$ is measurable.

Next suppose,

$$m^*(E) = m^*\left(E \cap [\bigcup_{k=1}^{n-1} A_k]\right) + m^*\left(E \cap [\bigcup_{k=1}^{n-1} A_k]^c\right)$$

Then

$$m^*\left(E \cap [\bigcup_{k=1}^{n} A_k]\right) + m^*\left(E \cap [\bigcup_{k=1}^{n} A_k]^c\right)$$
$$= m^*[E \cap \{A_n \cup \left(\bigcup_{k=1}^{n-1} A_k\right)\}] + m^*[E \cap A_n{}^c \cap \left(\bigcup_{k=1}^{n-1} A_k\right)^c]$$
$$\leq m^*(E \cap A_n) + m^*[(E \cap A_n{}^c) \cap \left(\bigcup_{k=1}^{n-1} A_k\right)]$$
$$+ m^*[(E \cap A_n{}^c) \cap \left(\bigcup_{k=1}^{n-1} A_k\right)^c]$$
$$= m^*(E \cap A_n) + m^*(E \cap A_n{}^c)$$
$$= m^*(E).$$

By induction, $\bigcup_{k=1}^{n} A_k$ is measurable.

Finally by the similar argument, $\bigcap_{k=1}^{n} A_k$ is measurable.

**Theorem 3.8.** Let $A$ be a measurable subset of a set $B$. If $W$ is any set with $m^*(W) < \infty$, then

$$m^*(W \cap B \cap A^c) = m^*(W \cap B) - m^*(W \cap A).$$

Proof. Since $A$ is a measurable set,

$$m^*(U) = m^*(U \cap A) + m^*(U \cap A^c)$$

for any set $U$. Hence with $U = W \cap B$,



$$\begin{aligned} m^*(W \cap B) &= m^*(W \cap B \cap A) + m^*(W \cap B \cap A^c) \\ &= m^*(W \cap A) + m^*(W \cap B \cap A^c), \quad \text{from } A \subset B \end{aligned}$$

Since $m^*(W \cap A) \leq m^*(W) < \infty$, $m^*(W \cap B \cap A^c) = m^*(W \cap B) - m^*(W \cap A)$.

**Lemma 3.9.** A monotonically decreasing sequence

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

of measurable sets has measurable intersection

$$D = \bigcap_{n=1}^{\infty} A_n.$$

Proof. Let $W$ be any set with $m^*(W) < \infty$. The numerical sequence $\{m^*(W \cap A_n)\}$ has finite-valued terms, is convergent, and

$$m^*(W \cap D) \leq \lim_{n\to\infty} m^*(W \cap A_n),$$

since $D \subset A_n$. Also $D^c = A_1{}^c \cup (A_1 \cap A_2{}^c) \cup (A_2 \cap A_3{}^c) \cup \cdots$ and

$$\begin{aligned}
& m^*(W \cap D^c) \\
& \leq m^*(W \cap A_1{}^c) + \sum_{n=1}^{\infty} m^*(W \cap A_n \cap A_{n+1}{}^c) \\
& = m^*(W \cap A_1{}^c) + \lim_{n\to\infty} \sum_{k=1}^{n} [m^*(W \cap A_k) - m^*(W \cap A_{k+1})] \\
& = m^*(W \cap A_1{}^c) + m^*(W \cap A_1) - \lim_{n\to\infty} m^*(W \cap A_{n+1}) \\
& \leq m^*(W) - m^*(W \cap D)
\end{aligned}$$



from the measurability of $A$,and Theorem 3.8. Thus

$$m^*(W) \geq m^*(W \cap D) + m^*(W \cap D^c)$$

for every set $W$ having $m^*(W) < \infty$. Hence $D$ is measurable.

**Theorem 3.10.** A sequence $\{A_n\}$ of measurable sets has measurable intersection, union, limit superior, and limit inferior.

Proof. By Theorem 3.7, each of the sets

$$B_n = \bigcap_{k=1}^{n} A_k, \quad n = 1, 2, 3, \cdots$$

is measurable. Since $B_1 \supset B_2 \supset B_3 \supset \cdots$ and $\cap B_n = \cap A_n$, then $\cap A_n$ is measurable by Lemma 3.9. Hence $\cup A_n = (\cap {A_n}^c)^c$ is measurable.

With $B_n$ redefined as $B_n = \bigcup_{k=n}^{\infty} A_k$, then $B_n$ is measurable and

$$\bigcap_{n=1}^{\infty} B_n = \overline{\lim}\, A_n$$

is measurable. Hence $\underline{\lim}\, A_n = (\overline{\lim}\, {A_n}^c)^c$ is measurable.



**Remark 3.11.** Let $A$ be a measurable set. If $B$ is any set, then it is easy to show that

(1) $$m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B),$$

and if in addition $A \cap B = \emptyset$, then

(2) $$m^*(A \cup B) = m^*(A) + m^*(B).$$

From now on, let $m(A)$ be the outer measure of a measurable set $A$.

**Theorem 3.12.** A sequence $\{A_n\}$ of pairwise disjoint measurable sets has

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n).$$

Proof. If $\{A_n\}$ is a finite sequence of disjoint measurable sets, then its holds from the above fact (2) by induction. Let $\{A_n\}$ be an infinite sequence of pairwise disjoint measurable sets.

Then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \geq m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(E_i).$$

By letting $n \to \infty$, we have $m(\cup A_n) \geq \sum m(A_n)$. But the reverse inequality holds by Theorem 3.2 and thus the equality holds.



**Theorem 3.13.** A monotonically increasing sequence

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

of measurable sets has

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

Proof. If $m(A_k) = \infty$ for some $k \in N$, then

$$m(\cup A_n) \geq m(A_k), \quad m(A_n) = \infty \quad \text{for each } n \geq k.$$

and the equality holds. Hence consider $m(A_n) < \infty, \quad n = 1, 2, 3, \cdots$ . Since

$$\cup A_n = A_1 \cup (A_2 \cap A_1{}^c) \cup (A_3 \cap A_2{}^c) \cup \cdots$$

is the union of pairwise disjoint measurable sets, by **Remark 3.11**,

$$\begin{aligned} m(\cup A_n) &= m(A_1) + m(A_2 \cap A_1{}^c) + m(A_3 \cap A_2{}^c) + \cdots \\ &= m(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n} [m(A_{i+1}) - m(A_i)] \qquad \text{,by Theorem 3.8} \\ &= \lim_{n \to \infty} m(A_{n+1}). \end{aligned}$$

**Theorem 3.14.** A monotonically decreasing sequence

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

of measurable set, with $m(A_n) < \infty$ for at least one $n$ has



$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} m(A_n).$$

Proof. Let $k \in N$ be such that $m(A_k) < \infty$. Then

$$n \geq k \Rightarrow m(A_k) \geq m(A_n) \geq m(A_{n+1}) \geq \cdots \geq 0.$$

Therefore

$$\lim_{n \to \infty} m(A_n) \leq m(A_k) < \infty.$$

Let $D = \bigcap_{n=1}^{\infty} A_n$. Then

$$A_k = D \cup (A_k \cap A_{k+1}{}^c) \cup (A_{k+1} \cap A_{k+2}{}^c) \cup \cdots .$$

express $A_k$ as the union of pairwise disjoint measurable sets. Hence, from Theorem 3.12,

$$m(A_k) = m(D) + \sum_{i=k}^{\infty} m(A_i \cap A_{i+1}{}^c)$$

$$= m(D) + \lim_{n\to\infty} \sum_{i=k}^{k+n} [m(A_i) - m(A_{i+1})] \quad \text{,by Theorem 3.8}$$

$$= m(D) + m(A_k) - \lim_{n\to\infty} m(A_{k+n+1}).$$

Therefore $m(D) = \lim_{n\to\infty} m(A_n)$.



Note that "with $m(A_n) < \infty$ for at least one $n \in N$" in the above Theorem is a necessary condition.

For example, let $A_n = \{ x \in R \mid x \geq n \}$ $(n = 1, 2, 3, \cdots)$. Then

$$A_1 \supset A_2 \supset A_3 \supset \cdots, m(A_n) = \infty \quad \text{and} \lim_{n\to\infty} m(A_n) = \infty.$$

However, $\cap A_n = \emptyset$ and thus $m(\cap A_n) = 0$.

**Theorem 3.15.** A sequence $\{A_n\}$ of measurable sets has

$$m(\underline{\lim}\, A_n) \leq \underline{\lim}\, m(A_n).$$

If in addition $m(A_n \cup A_{n+1} \cup A_{n+2} \cup \cdots) < \infty$ for at least one $n \in N$, then

$$m(\overline{\lim} A_n) \geq \overline{\lim}\, m(A_n).$$

Proof. Let $B_n = \bigcap_{k=n}^{\infty} A_k$. By Definition, $\underline{\lim} A_n = \cup B_n$. Also, $m(\cup B_n) = \lim m(B_n)$ by Theorem 3.13. Thus

$$m(\underline{\lim} A_n) = \lim m(B_n).$$

For each $n \in N$, $B_n \subset A_{n+k}$ and $m(B_n) \leq m(A_{n+k})$ for $k = 1, 2, 3, \cdots$ Thus

$$m(B_n) \leq \underline{\lim}_{k\to\infty} m(A_{n+k}) = \underline{\lim}_{k\to\infty} m(A_k) \quad \text{for each } n \in N.$$

Consequently $\lim_{n\to\infty} m(B_n) \leq \underline{\lim}\, m(A_n)$. Therfore $m(\underline{\lim} A_n) \leq \underline{\lim}\, m(A_n)$. Assume that in condition $m(A_n \cup A_{n+1} \cup \cdots) < \infty$ for at least one $n \in N$. Let $C_n = \bigcup_{k=n}^{\infty} A_k$. By Definition, $\overline{\lim} A_n = \cap C_n$. Also by hypothesis $m(C_n) < \infty$ for at least one $n$. Thus $m(\cap C_n) = \lim m(C_n)$ by Theorem 3.14. Hence



$$m(\overline{\lim} A_n) = \lim m(C_n).$$

For each $n \in N, C_n \supset A_{n+k}$ and so $m(C_n) \geq m(A_{n+k})$ for $k = 1, 2, 3, \cdots$ Thus

$$m(C_n) \geq \overline{\lim}_{k\to\infty} m(A_{n+k}) = \overline{\lim}_{k\to\infty} m(A_k)$$

for each $n \in N$. Consequently

$$\lim_{n\to\infty} m(C_n) \geq \overline{\lim_{n\to\infty}}\, m(A_n)$$

Therfore $m\left(\overline{\lim}\, A_n\right) \geq \overline{\lim}\, m(A_n)$.

Note that "with $m\left(\bigcup_{k\geq m} A_k\right) < \infty$ for at least one $n \in N$" in the above Theorem is necessary condition.

In $R$ let $A_n = \begin{cases} (n,\infty) & \text{for } n \text{ odd} \\ (-\infty,-n) & \text{for } n \text{ even.} \end{cases}$

Then for all $n$, $m(A_n) = \infty$. Hence $m\left(\bigcup_{k\geq n} A_k\right) = \infty$ and so $\overline{\lim}\, m(A_n) = \infty$, However,

$$\overline{\lim}\, A_n = \overline{A}_1 \cap \overline{A}_2 \cap \cdots = \emptyset$$



and thus $m\left(\overline{\lim}\, A_n\right) = 0$.

**Corollary 3.16.** If $\{A_n\}$ is a convergent sequence of measurable sets with each $A_n \subset B$ where $m^*(B) < \infty$, then $\lim m(A_n)$ exists and

$$m(\lim A_n) = \lim m(A_n).$$

Proof. Since $\{A_n\}$ is a convergent sequence,

$$\lim A_n = \underline{\lim}\, A_n = \overline{\lim}\, A_n.$$

Also by the above Theorem,

$$m(\lim A_n) = m(\underline{\lim} A_n) \le \underline{\lim}\, m(A_n).$$

Since $A_n \subset B$ for each $n$ and $m^*(B) < \infty$,

$$m(A_n \cup A_{n+1} \cup \cdots) \le m^*(B) < \infty \quad \text{for each } n.$$

Again applying the above Theorem,

$$m(\lim A_n) = m(\overline{\lim} A_n) \ge \overline{\lim}\, m(A_n).$$

Hence $\overline{\lim}\, m(A_n) \le m(\lim A_n) \le \underline{\lim}\, m(A_n)$.

i.e.

$$m(\lim A_n) = \underline{\lim}\, m(A_n) = \overline{\lim}\, m(A_n).$$

Therefore



$$m(\lim A_n) = \lim m(A_n).$$

## IV. SEQUENCE OF MEASURABLE FUNCTIONS

**Definition 4.1.** Let $f(x)$ be a real-valued function defined on a measurable set $E$. $f$ is called a measurable function, if for each $\alpha \in R$ the set $\{ x \in E \mid f(x) > \alpha \}$ is measurable.

Let $c$ be a constant and $f$ and $g$ be measurable functions defined on the same domain. Then the functions $f + c, cf, f + g, g - f$, and $fg$ are also measurable.

**Definition 4.2.** A measure space is a triple $(X, \tau, \mu)$ consisting of a set $X$,a $\sigma$-algebra $\tau$ of subsets of $X$, and a measure $\mu$ defined on $X$. We will denote by $M$ the class of Lebesque measurable sets. Then the class $M$ is a $\sigma$-algebra, and so $(R,M,m)$ is a measure space. A null set means a measurable set of measure zero, and a property is said to hold almost everywhere (abbreviated a.e.) if the set of points where it fails to holds is a null set.

In this section, we shall consider only real-valued functions defined on a fixed measure space $(X, \tau, \mu)$.

**Definition 4.3** (a) A sequence $f_n$ of functions is said to converge almost everywhere to the real-valued function $f$, defined on X, in case there exists a null set $E$ such that $x \in X - E$ implies $f_n(x) \to f(x)$. Briefly, $f_n \to f$. a.e.

(b) A sequence $f_n$ of functions is said to be fundamental almost everywhere if there exists a null set $E$ such that $x \in X - E$ implies that $f_n(x)$ is a Cauchy sequence ; Briefly, $f_n$ is fundamental a.e.

**Theorem 4.4.** Suppose, $f_n \to f$ a.e. Then

(a) $f_n$ is fundamental a.e.,

(b) $f = g$ a.e.,if $f_n \to g$ a.e.

(c) $f_n \to g$ a.e., if $g$ is the real valued function such that $f = g$ a.e.

Proof. Let $E$ be a null set such that $f_n(x) \to f(x)$ for all $x \in X - E$.

(a) $x \in X - E$ implies $f_n(x)$ is a Cauchy sequence.

(b) If $F$ is a null set such that $f_n(x) \to g(x)$ for all $x \in X - F$, then $E \cup F$ is a null set such that $f(x) = \lim f_n(x) = g(x)$ for all $x \in X - (E \cup F)$.

(c) If $F$ is a null set such that $f(x) = g(x)$ for $x \in X - F$, then $E \cup F$ is a null set such that $f_n(x) \to f(x) = g(x)$ for all $x \in X - (E \cup F)$.



**Theorem 4.5.** If $f_n \to f$ a.e., and $f_n = g_n$ a.e. for each $n$, then

(a) $g_n \to f$ a.e.

(b) $f = g$ a.e. if $g_n \to g$ a.e.

(c) $g_n \to g$ a.e. if $f = g$ a.e.

Proof. Let $E$ be a null set such that $f_n(x) \to f(x)$ for all $x \in X - E$, and $E_n$ a null set such that $f_n(x) = g_n(x)$ for all $x \in X - E_n$. Let $K = E \cup (\cup E_n)$. Then $K$ is a null set;

(a) For all $x \in X - K$, $g_n(x) = f_n(x) \to f(x)$.

(b) Let $F$ be a null set such that $g_n(x) \to g(x)$ for all $x \in X - F$. Then $K \cup F$ is a null set and so

$$f(x) = \lim f_n(x) = \lim g_n(x) = g(x) \quad \text{for all } x \in X - (K \cup F).$$

(c) Let $G$ be a null set such that $f(x) = g(x)$ for all $x \in X - G$. Then $K \cup G$ is a null set, and so

$$g_n(x) = f_n(x) \to f(x) = g(x) \quad \text{for all } x \in X - (K \cup G).$$

**Theorem 4.6.** If $f_n \to f$ a.e., $g_n \to g$ a.e., $c$ is a real number, and $A$ is any subset of $X$, then

(a) $cf_n \to cf$ a.e.,

(b) $f_n + g_n \to f + g$ a.e.,

(c) $|f_n| \to |f|$ a.e.,

(d) $\begin{cases} \max(f_n, g_n) \to \max(f, g) & \text{a.e., and} \\ \min(f_n, g_n) \to \min(f, g) & \text{a.e.,} \end{cases}$

(e) $f_n^+ \to f^+$ a.e., and $f_n^- \to f^-$ a.e.,

(f) $\chi_A f_n \to \chi_A f$ a.e.,

(g) $f_n g_n \to fg$ a.e..

Proof. Let $E$ be a null set such that both $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for all $x \in X - E$.

(a) For all $x \in X - E$, $\lim cf_n(x) = c \lim f_n(x) = cf(x)$.

(b) $\lim(f_n(x)+g_n(x)) = \lim f_n(x) + \lim g_n(x) = f(x)+g(x)$ for all $x \in X - E$.

(c) $\lim|f_n(x)| = |\lim f_n(x)| = |f(x)|$ for all $x \in X - E$.

(d) Since $\max(f,g) = \frac{1}{2}(f+g+|f-g|)$ and $\min(f,g) = \frac{1}{2}(f+g-|f-g|)$, (d) holds by (a),(b),and (c).

(e) These follow from (d) and the fact that $f^+ = \max(f,0)$ and $f^- = -\min(f,0)$

(f) $\lim(\chi_A f_n)(x) = \chi_A(x)\lim f_n(x) = \chi_A(x)f(x)$ for all $x \in X - E$.

(g) $\lim(f_n g_n)(x) = \lim f_n(x)\lim g_n(x) = f(x)g(x)$ for all $x \in X - E$.

**Remark 4.7.** (1) The characteristic function $\chi_E$ of the set $E$, is measurable iff $E$ is measurable.

(2) Let $< f_n >$ be sequence of measurable functions, and $f_n \to f$ pointwise. Then $f$ is a measurable function.



**Theorem 4.8.** If $f_n$ is fundamental a.e., there exists a function $f$ such that $f_n \to f$ a.e. If moreover the $f_n$ are measurable, we may take $f$ to be measurable.

Proof. Let $E$ be a null set such that $f_n(x)$ is a Cauchy sequence for all $x \in X - E$. Define $f(x) = \lim f_n(x)$ for $x \in X - E$, and $f(x) = 0$ for $x \in E$. Clearly $f_n \to f$ a.e.

Suppose,moreover, that $f_n$ are measurable. Define $g_n = \chi_{X-E} f_n$. Then $g_n$ are measurable, and $g_n(x) = (\chi_{X-E} f_n)(x) \to (\chi_{X-E} f)(x) = f(x)$ for all $x \in X - E$. Hence $f$ is measurable.

**Theorem 4.9.** Suppose, $f_n \to f$ a.e. and $f_n, f, g$ are real valued functions. Then

(a) $f \geq 0$ a.e. if $f_n \geq 0$ a.e.,

(b) $f \leq g$ a.e. if $f_n \leq g$ a.e. for each $n$,

(c) $|f| \leq |g|$ a.e. if $|f_n| \leq |g|$ a.e..

Proof. Let $E$ be a null set such that $x \in X - E$ implies $f_n(x) \to f(x)$.

(a) Let $E_n$ be a null set such that $f_n(x) \geq 0$ for $x \in X - E_n$. Then $F = E \cup (\cup E_n)$ is a null set, and $x \in X - F$ implies $f(x) = \lim f_n(x) \geq 0$.

(b) Since $g - f_n \geq 0$ a.e., and $g - f_n \to g - f$ a.e., by (a), thus $g - f \geq 0$ a.e.



(c) Since $|f_n| \to |f|$ a.e.,$|f| \leq |g|$ a.e. by (b).

**Definition 4.10.** (a) A sequence $f_n$ of measurable real-valued functions is said to converge in measure to a measurable real-valued function if for each $\epsilon > 0$, $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$. Briefly, $f_n \to f$ is measurable.

(b) A sequence $f_n$ of measurable function is said to be fundamental in measure if, for each $\epsilon > 0$, $\lim_{m,n\to\infty} \mu(\{x : |f_m(x) - f_n(x)| \geq \epsilon\}) = 0$. Briefly, $f_n$ is fundamental in measure.

**Theorem 4.11.** If $f_n \to f$ in measure and $g$ is a measurable function, then

(a) $f_n$ is fundamental in measure,

(b) $f = g$ a.e. if $f_n \to g$ in measure,

(c) $f_n \to g$ in measure if $f = g$ a.e.

Proof. Let $E_n = \{ x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2} \}$ for any $\varepsilon$.

(a) Define $E_{mn} = \{ x : |f_m(x) - f_n(x)| \geq \varepsilon \}$.

The relation $|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)|$ implies that $E_{mn} \subset E_m \cup E_n$, hence $\lim_{m,n\to\infty} \mu(E_{mn}) \leq \lim_{m,n\to\infty} \mu(E_m) + \lim_{m,n\to\infty} \mu(E_n) = 0$.

(b) Given any $\varepsilon > 0$, we have

$$\{ x : |f(x) - g(x)| \geq \varepsilon \} \subset \{ x : |f(x) - f_n(x)| \geq \frac{\varepsilon}{2} \} \cup \{ x : |f_n(x) - g(x)| \geq \frac{\varepsilon}{2} \},$$

hence it is clear that $\mu(\{ x : |f(x) - g(x)| \geq \varepsilon \}) = 0$. Our assertion then follows from the relation



$$\{ x \mid (f - g)(x) \neq 0 \} = \bigcup_{m=1}^{\infty} \{ x : |f(x) - g(x)| \geq \frac{1}{m} \}$$

and the fact that a countable union of null sets is a null set.

(c) Let $E$ be a null set such that $f(x) \neq g(x)$ implies $x \in E$. Then for any $\varepsilon > 0$,

$$\{ x : |f_n(x) - g(x)| \geq \varepsilon \} \subset \{ x : |f(x) - g(x)| \geq \frac{\varepsilon}{2} \} \cup \{ x : |f_n(x) - f(x)|$$

$$\geq \frac{\varepsilon}{2} \} \subset E \cup \{ x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2} \}$$ and so (c) holds.

**Theorem 4.12.** If $f_n \to f$ in measure and $f_n = g_n$ a.e., then

(a) $g_n \to f$ in measure if $g_n$ is a sequence of measurable functions.

(b) $f = g$ a.e. if $g_n \to g$ in measure.

Proof. Let $E_n$ be a null set such that $f_n(x) = g_n(x)$ for all $x \in X - E_n$.

(a) Put $E = \cup E_n$. Then $E$ is a null set.

For any $\epsilon > 0$, $\{ x : |g_n(x) - f(x)| \geq \epsilon \} \subset E \cup \{ x : |f_n(x) - f(x)| \geq \epsilon \}$ for all $n$. Therefore $g_n \to f$ in measure.

(b) For any $\epsilon > 0$,

$$\{ x : |f(x) - g(x)| \geq \epsilon \} \subset \{ x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2} \} \cup \{ x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2} \} \cup E_n.$$

for all $n$. By hypothesis, $\{ x : |f(x) - g(x)| \geq \epsilon \}$ is a null set for any $\epsilon > 0$. Therefore $f = g$ a.e.



**Theorem 4.13.** If $f_n \to f$ in measure, $g_n \to g$ in measure, $c$ is a real number, and $A$ is a locally measurable set, then

(a) $cf_n \to cf$ in measure.

(b) $f_n + g_n \to f + g$ in measure.

(c) $|f_n| \to |f|$ in measure.

(d) $\begin{cases} \max(f_n, g_n) \to \max(f, g) & \text{in measure, and} \\ \min(f_n, g_n) \to \min(f, g) & \text{in measure.} \end{cases}$

(e) $\chi_A f_n \to \chi_A f$ in measure.

Proof. (a) This is clear if $c = 0$ ; if $c \neq 0$, then

$$\{ x : |cf_n(x) - cf(x)| \geq \epsilon \} = \{ x : |f_n(x) - f(x)| \geq \frac{\epsilon}{|c|} \} \quad \text{for any } \epsilon > 0.$$

(b) Let $h_n = f_n + g_n, h = f + g$. The relation $|h_n(x) - h(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$ implies

$$\{ x : |h_n(x) - h(x)| \geq \epsilon \} \subset \{ x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2} \} \cup \{ x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2} \}.$$

(c) Since $||f_n(x)| - |f(x)|| \leq |f_n(x) - f(x)|, |f_n| \to |f|$ in measure,

(d) Since $\max(f, g) = \frac{1}{2}(f + g + |f - g|)$, (d) holds by (a),(b) and (c).

(e) Since $|\chi_A f_n - \chi_A f| = \chi_A |f_n - f| \leq |f_n - f|$, it holds.

**Theorem 4.14.** If $f_n, f, g$ are measurable, $f_n \to f$ in measure, then

(a) $f \geq 0$ a.e. if $f_n \geq 0$ a.e.

(b) $f \leq g$ a.e. if $f_n \leq g$ a.e. for all $n$.

(c) $|f| \leq |g|$ a.e. if $|f_n| \leq |g|$ a.e. for all $n$.

Proof. (a) Modifying on a null set, we may assume that $f_n \geq 0$ everywhere for all $n$. Given $\epsilon > 0$, define $E = \{ x \mid f(x) \leq -\epsilon \}$ and $E_n = \{ x : |f(x) - f_n(x)| \geq \epsilon \}$ By hypothesis, $\lim \mu(E_n) = 0$. If $x \in E$, then $f(x) \leq -\epsilon$; Since $f(x) = [f(x) - f_n(x)] + f_n(x) \geq f(x) - f_n(x), f(x) - f_n(x) \leq -\epsilon$, and hence $|f(x) - f_n(x)| \geq \epsilon$. i.e., $E \subset E_n$ for all $n$, hence $m(E) = 0$. Thus

$$\{ x : f(x) < 0 \} = \bigcup_{m=1}^{\infty} \{ x : f(x) \leq -\frac{1}{m} \}$$ is a null set, and so $f \geq 0$ a.e.

(b) We have $g - f_n \geq 0$ a.e., and $g - f_n \to g - f$ in measure by Theorem 4-13(b), hence $g - f \geq 0$ a.e. by (a)

(c) Since $|f_n| \to |f|$ in measure by Theorem 4.15(c), we have $|f| \leq |g|$ a.e. by (b).

**Remark 4.15.** (a) Convergence a.e. does not always imply convergence in measure. For example, let $\mu$ be discrete measure on the class $\tau$ of all the set $X = \{1, 2, 3, \cdots\}$. If $f_n$ is the characteristic function of the singleton $\{n\}$, then $f_n(x) = \chi_{\{n\}}(x) \to 0$ pointwise on $X$. Taking $\epsilon = 1, \mu(\{\, x : |f_n(x)| \geq 1 \,\}) = \mu\{n\} = 1$, for all $n$, i.e. $f_n \not\to 0$ in measure.

(b) Convergence in measure does not imply convergence a.e. For example, let $m$ be a Lebesque measure, and consider the sequence of intervals $[0,1), [0, \frac{1}{2}), [\frac{1}{2}, 1), [0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), \cdots$. If $f_n$ is the characteristic function of the nth term of this sequence. i.e. $f_1 = \chi_{[0,1)}, f_2 = \chi_{[0,\frac{1}{2})}, f_3 = \chi_{[\frac{1}{2},1)}, \cdots$. Since $m(\{\, x : |f_n(x)| \geq \epsilon \,\})$ = length of the $n$th term of this sequence of intervals for any $0 < \epsilon \leq 1, f_n \to 0$ in measure. But $f_n(x)$ does not converge, for all $x \in E = [0,1)$, and $f_n(x) \to 0$ for all $x \in X - E$. Therefore $f_n$ does not converge to 0 a.e.



**Theorem 4.16.** If a sequence $f_n$ is measurable functions which is fundamental in measure, then there exists a measurable function $f$ such that $f_n \to f$ in measure.

Proof. For every positive integer $k$ we can find a positive integer $n_k$ such that for $n, m \geq n_k$,

$$\mu\left(\{x : |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\}\right) < \frac{1}{2^k},$$

and we may assume that for each $k, n_{k+1} > n_k$. Let $E_k = \{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\}$. Then, if $x \notin \bigcup_{k=m}^{\infty} E_k$, we have for $r > s \geq m$

$$(*) \qquad |f_{n_r}(x) - f_{n_s}(x)| \leq \sum_{i=s+1}^{r} |f_{n_i}(x) - f_{n_{i-1}}(x)| < \sum_{i=s+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^s}.$$

So $\{f_{n_k}(x)\}$ is a Cauchy sequence for each $x \notin \limsup E_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$,



But, for all $m, \mu(\limsup E_k) \leq \mu\left(\bigcup_{k=m}^{\infty} E_k\right) \leq \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}$. So $\{f_{n_k}\}$ converges a.e. to some measurable function $f$. Also from $(*)$ $\{f_{n_k}\}$ is uniformly fundamental in $X - [\bigcup_{k=m}^{\infty} E_k]$, for each $m$.

So $f_{n_k} \to f$ uniformly on $X - [\bigcup_{k=m}^{\infty} E_k]$, and hence for every positive $\varepsilon$,

$$(**) \qquad \mu\left(\{x : |f_{n_k}(x) - f(x)| \geq \frac{\varepsilon}{2}\}\right) \to 0 \quad \text{as } k \to \infty.$$

But $\{ x : |f_n(x) - f(x)| \geq \varepsilon \} \subseteq \{ x : |f_n(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2} \} \cup \{ x : |f(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2} \}$. If $n$ and $m$ are sufficiently large, then $\mu(\{ x : |f_n(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2} \}) \to 0$ from (**) and that $f_n$ is a fundamental in measure. Therefore $\mu(\{ x : |f_n(x) - f(x)| \geq \varepsilon \}) \to 0$ as $n \to \infty$.

**Definition 4.17.** (a) A sequence $f_n$ of functions is said to converge almost uniformly to the function $f$, if given $\delta > 0$, there exists a measurable set $F$ such that $\mu(F) \leq \delta$ and $f_n(x) \to f(x)$ uniformly on $X - F$. Briefly, $f_n \to f$ a.u.

(b) A sequence $f_n$ of functions is said to be almost uniformly fundamental, if given any $\delta$, there exists a measurable set $F$ such that $\mu(F) \leq \delta$, and such that $f_n(x)$ is uniformly fundamental on $X - F$. i.e. For each $\varepsilon > 0$, there exists a positive integer $n_0$ such that $m, n \geq n_0$ imply $|f_m(x) - f_n(x)| \leq \varepsilon$ for all $x \in X - F$. Briefly, $f_n$ is fundamental a.u.



If $f_n \to f$ a.u., then $f_n$ is fundamental a.u.; this is immediate from the relation $|f_m - f_n| \leq |f_m - f| + |f - f_n|$.

**Theorem 4.18.** If the sequence $f_n$ is fundamental a.u., there exists a function $f$ such that $f_n \to f$ a.u. If moreover the $f_n$ are measurable, one can take $f$ to be measurable.

Proof. For each positive integer $m$, let $F_m$ be a measurable set such that $\mu(F_m) \leq \frac{1}{m}$ and $f_n(x)$ is uniformly fundamental on $X - F_m$. Then the set $F = \bigcap_{m=1}^{\infty} F_m$ is measurable, and $\mu(F) \leq \mu(F_m) \leq \frac{1}{m}$ for all $m$, hence $F$ is null set. Moreover, for each $x$ in the set $X - F = \bigcup_{m=1}^{\infty} (X - F_m)$, the sequence $f_n(x)$ is Cauchy. For each $x \in X - F$, define $f(x) = \lim f_n(x)$, and $f(x) = 0$ for $x \in F$. For each $m, f_n(x) \to f(x)$ on $X - F_m$, and $f_n(x)$ is uniformly fundamental on $X - F_m$, hence $f_n(x) \to f(x)$ uniformly on $X - F_m$. It follows at once that $f_n \to f$ a.u. If the $f_n$ are also measurable, then so are the function $g_n = \chi_{X-F} f_n$. Since $g_n(x) \to f(x)$ for all $x$, $f$ is measurable by Remark 4.7.

**Theorem 4.19.** If $f_n \to f$ a.u., then $f_n \to f$ a.e. If moreover the $f_n$ and $f$ are measurable, then $f_n \to f$ in measure.

Proof. For each positive integer $m$, let $F_m$ be a measurable set such that $\mu(F_m) \leq \frac{1}{m}$ and $f_n(x) \to f(x)$ uniformly on $X - F_m$. Then $F = \bigcap_{m=1}^{\infty} F_m$ is a null set, and $f_n(x) \to f(x)$ for each $x \in X - F$, thus $f_n \to f$ a.e. Suppose, in addition, that the $f_n$ and $f$ are measurable. Given any $\epsilon > 0$, define $E_n = \{ x : |f_n(x) - f(x)| \geq \epsilon \}$ We must show that $\mu(E_n) \to 0$ as $n \to \infty$. Given $\delta > 0$, choose a positive integer $m$ so that $\frac{1}{m} < \delta$, and consider the set $F_m$ defined earlier. Since $f_n(x) \to f(x)$ uniformly on $X - F_m$, there exists an positive integer $n_0$, such that $n \geq n_0$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in X - F_m$ and hence $(X - F_m) \subset (X - E_n)$ whenever $n \geq n_0$. Then $n \geq n_0$ implies $E_n \subset F_m$, and hence $\mu(E_n) \leq \mu(F_m) \leq \frac{1}{m} < \delta$.

**Remark 4.20.** Convergence a.e. does not always imply convergence almost uniformly. For example, let $X = [0, \infty)$, and let

$$f_n(x) = \begin{cases} 1 - n(x-k) & , \text{ if } \quad k \leq x \leq \frac{1}{n} + k, \\ 0 & , \text{ if } \quad \frac{1}{n} + k \leq x < k+1, \text{ for } k = 0, 1, \cdots . \end{cases}$$

Then $\lim_{n\to\infty} f_n(x) = 0$ except for $x = 0, 1, 2, \cdots$, but $m(\{x : f_n(x) \geq \epsilon\}) = \infty$ for each $\epsilon$ with $0 < \epsilon < 1$ and for each $n$. So $f_n \to 0$ a.e. but not in measure. Therefore $f_n \to 0$ a.e. but not a.u.



**Theorem 4.21.** Let $f_n \to f$ a.e. and $f_n, f$ are measurable functions. If $\mu(X) < \infty$, then $f_n \to f$ a.u.

Proof. We suppose without loss of generality that $f_n$ converges at every point of $X$ to $f$. If $m, n \in N$, let $E_n(m) = \bigcup_{k=n}^{\infty} \{x \in X : |f_k(x) - f(x)| \geq \frac{1}{m}\}$, so that $E_n(m) \in \tau$ and $E_{n+1}(m) \subseteq E_n(m)$. Since $f_n(x) \to f(x)$ for all $x \in X$, it follows that $\bigcap_{m=1}^{\infty} E_n(m) = \emptyset$. Since $\mu(X) < \infty$, we infer that $\mu(E_n(m)) = 0$ as $n \to \infty$. If $\delta > 0$, let $k_m$ be such that $\mu(E_{k_m}(m)) <$

$\frac{\delta}{2^m}$ and let $E_\delta = \bigcup_{m=1}^{\infty} E_{k_m}(m)$, so that $E_\delta \in \tau$ and $\mu(E_\delta) < \delta$. If $x \in X - E_\delta$,

then $x \in X - E_{k_m}(m)$, so that $|f_k(x) - f(x)| < \frac{1}{m}$ for all $k \geq k_m$. Therefore $f_k \to f$ uniformly converge on $X - E_\delta$. i.e. $f_k \to f$ a.u.

**Theorem 4.22.** Suppose $f_n \to f$ in measure. Then

(a) $f_n \to f$ a.u. if $f_n$ is fundamental a.u.

(b) $f_n \to f$ a.e. if $f_n$ is fundamental a.e.

Proof. Since $f_n \to f$ in measure, for any $\epsilon > 0$ $\mu(E_n) = \mu(\{x : |f_n - f| \geq \epsilon\}) \to 0$ as $n \to \infty$. For every $\delta > 0$, choose $n_0 \in N$ such that $\frac{1}{n_0} < \delta$ (i.e. $n_0 > \frac{1}{\delta}$) and $\mu(E_n) < \delta$ for $n \geq n_0$.

(a) Since $X - E_n = \{x : |f_n(x) - f(x)| < \epsilon\}$, $|f_n(x) - f(x)| < \epsilon$ for each $x \in X - E_n$. Hence $n \geq n_0$ implies $f_n(x) \to f(x)$ for $x \in X - E_n$. Therefore $f_n \to f$ a.u.

(b) Let $E = \bigcap_{n=1}^{\infty} E_n$. Then $\mu(E) \leq \mu(E_n)$ for all $n$, hence $E$ is a null set and $f_n(x) \to f(x)$ for $x \in X - E$. Therefore $f_n \to f$ a.e.

**Theorem 4.23.** Suppose $f_n \to f$ a.u.

(a) $f = g$ a.e. if $f_n \to g$ a.u.,

(b) $f_n \to g$ a.u. if $f = g$ a.e.,

(c) $g_n \to g$ a.u. if $g_n = f_n$ a.e.

Proof. For every $k \in N$, let $F_k$ be a measurable set such that $\mu(F_k) < \frac{1}{k}$ and $f_n(x) \to f(x)$ for all $x \in X - F_k$.

(a) Let $G_k$ be a measurable set such that $m(G_k) < \frac{1}{k}$ and $f_n(x) \to g(x)$ for all $x \in X - G_k$. Put $H = \bigcap_{k=1}^{\infty} (F_k \cup G_k)$. Then $\mu(H) \leq \mu(F_k \cup G_k) \leq \frac{2}{k}$ for all $k$ and so $H$ is a null set. Hence $f(x) = g(x)$ for all $x \in X - H$. i.e. $f = g$ a.e.

(b) Since $f = g$ a.e., there exists a null set $E$ such that $f(x) = g(x)$ on $X - E$. Put $H_k = F_k \cup E$ $(k = 1, 2, \cdots)$. Then $\mu(H_k) \leq \mu(F_k) + \mu(E) = \mu(F_k) < \frac{1}{k}$ and $f_n(x) \to f(x) = g(x)$ for all $x \in X - H_k$. i.e. $f_n \to g$ a.u.

(c) Since $f_n = g_n$ a.e. for all $n$, for each $n \in N$, there exists null set $E_n$ such that $f_n(x) = g_n(x)$ for each $x \in X - E_n$. Put $E = \bigcup_{n=1}^{\infty} E_n$. Then $E$ is a null set and $f_n(x) = g_n(x)$ for any $x \in X - E$. Put $H = F_k \cup E$ $(k = 1, 2, \cdots)$. Then $\mu(H) \leq \mu(F_k) + \mu(E) = \mu(F_k) < \frac{1}{k}$ and $f_n(x) = g_n(x) \to f(x)$ for each $x \in X - H$. Therefore $g_n \to f$ a.u.

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