濟大論文集(自然). 35. 203-208. 1992
Cheju Univ. Jour. (Natural Sci.).
35. 203-208. 1992

Analysis of Hopfield Neural Network by Using Vector Field

*Min-Je Kang**

벡터장을 이용한 호프필드 신경회로망의 분석

康 珉 齊*

Introduction

Single-layer feedback neural networks, also called Hopfield networks [Hopfield, 1984; Hopfield and Tank, 1985; Tank and Hopfield, 1986], display a number of interesting properties. Once properly trained and initialized, they converge, in time, towards a stable solution which is one of the minima of their computational energy function.

The algorithm for convergence of a network using neurons with discrete activation functions is asynchronous due to the stochastic nature of the convergence process. The minimum energy solution of such a discrete-time operation is random even for deterministic input. For continuous activation functions, however, the solution is deterministic, due to the properties of the system.

This paper presents a general time-domain analysis using the vector field approach of continuous-time single-layer feedback neural networks. The method allows capturing the transients and equilibrium points of such networks.

Vector Field Method

The n neuron system is described by the following state equations (1-3)

$$u_i = \frac{1}{C_i}(i_i + \sum_{j=1}^{n} w_{ij} - G_i u_i), \quad (1)$$
$$i = 1, 2, \ldots, n,$$

where u_i, v_i are state and output variables, i_i, are biasing currents, and G_i, C_i are total conductances and capacitances, respectively, connected directly to the input of the i-th neuron.

* 공과대학 전자공학과(Dept. of Electronic Engineering, Cheju-do, 690-756, Korea)

The description of the system in (1) can be formalized in the output space v as follows

$$\frac{dv_i}{dt} = \Psi_i[\underline{v}(t)], \quad 1 = 1, 2, \ldots, n. \qquad (2)$$

Expression (2) can be used to compute the derivatives $\Psi_1(\underline{v})$, $\Psi_2(\underline{v})$, .., $\Psi_n(\underline{v})$ at each point of space v. Assuming that $dt \cong \Delta t$, the computed entries Ψ_i would correspond to $\Delta v_1/\Delta t$, $\Delta v_2/\Delta t$, .., $\Delta v_n/\Delta t$. This provides components of the vector $\Delta v/\Delta t$ for this nonlinear system. The vector field obtained using this method contains a number of segments of trajectories of evolving system for $[0, \infty)$.

Vector Field Generation for Single-Layer Neural Networks

To apply the vector field method to nonlinear differential equation (1), one needs to assume a specific form of the activation function $v_i = f(u_i)$. Using the following activation function

$$v_i = (1 + e^{-\lambda_i u_i})^{-1} \qquad (3)$$

we obtain

$$dv_i = \lambda_i (v_i - \frac{v_i^2}{a}) du_i, \qquad (4)$$

where the i-th neuron gain is proportional to λ_i and $a = 1$V.

Vector components $\Delta v_i/\Delta t$ approximating the left sides of (2) can be now obtained as

$$\Psi_i(\underline{v}) = \frac{\lambda_i (v_i - \frac{v_i^2}{a})}{C_i} (i_i + \sum_{j=1}^{n} w_{ij} v_j - G_j f^{-1}(v_i)) \qquad (5)$$

$$i = 1, 2, \ldots, n.$$

components Ψ_i determine the emotion of the system output in the direction v_i. Approximated actual displacements of the output are equal to the products $\Psi_i \Delta t$.

The approximation for v_i^{k+1} is

$$v_i^{k+1} = v_i^k + \Delta v_i^k, \quad i = 1, 2, \ldots, n, \qquad (6)$$

where Δv_i is the component of a displacement-step. It is equal to the product of the normalized vector component by a displacement-step

$$\Delta v_i^k = \alpha(\Psi_i(\underline{v}^k))d, \quad i = 1, 2, \ldots, n, \qquad (7)$$

where d is a user-selectable displacement-step, and $\alpha(\Delta_i(\underline{v}^k))$, called normalized vector field component, is defined as follows

$$\alpha(\Psi_i(\underline{v}^k)) = \frac{\Psi_i(\underline{v}^k)}{(\sum_{i=1}^{n} \Psi_i^2(\underline{v}^2))^{\frac{1}{2}}} \qquad (8)$$

Thus, the length of the sum of the vector components of a displacement-step is equal to the displacement-step. The value of the stable displacement-step depends on a system. The reasonable step for the system under study has been chosen as $d = 0.01$. Thus, the system moves by the distance of 0.01 within each iteration.

Case Study

A 2-bit A/D converter representing a class of or gradient-type networks [Zurada and Kang, 1989; Zurada and Kang, 1991] was selected as a case study to test the method. The state space equation (1) describing the converter are

$$C_1 u_1 = x - 0.5 - 2v_2 - (g_1 - 2)u_1$$

$$C_2 u_2 = 2x - 2 - 2v_1 - (g_2 - 2)u_2$$

where g_i are the parasitic input conductances of the i-th neuron, and x is the analog voltage value to be converted to the binary representation $\underline{v} = [v1, v2]^t$.

Equation (9) can be rearranged with form of equation (5) as

$$\begin{aligned} v_1 &= \frac{\lambda_1}{C_1}(v_1 - v_1^2)[x - 0.5 - 2v_2 \\ &\quad -(g_1 - 2)\lambda_1^{-1}\ln(\frac{v_1}{1 - v_1})] \end{aligned}$$

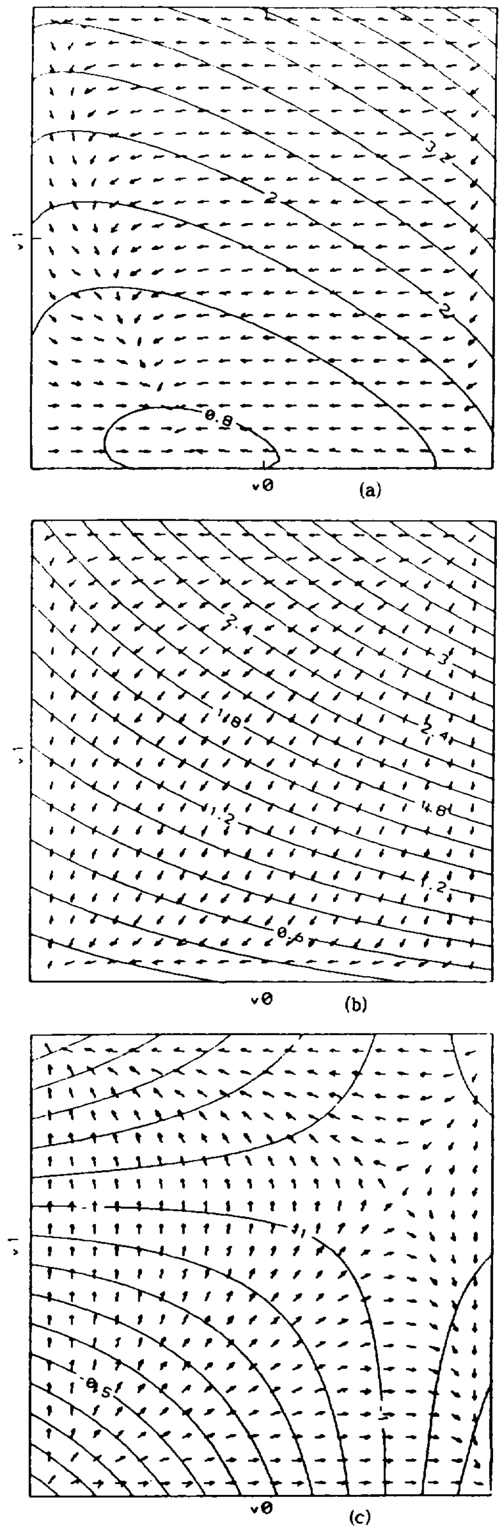
$$\begin{aligned} v_2 &= \frac{\lambda_2}{C_2}(v_2 - v_2^2)[2x - 2 - 2v_1 \\ &\quad -(g_2 - 2)\lambda_2^{-1}\ln(\frac{v_2}{1 - v_2})] \end{aligned}$$

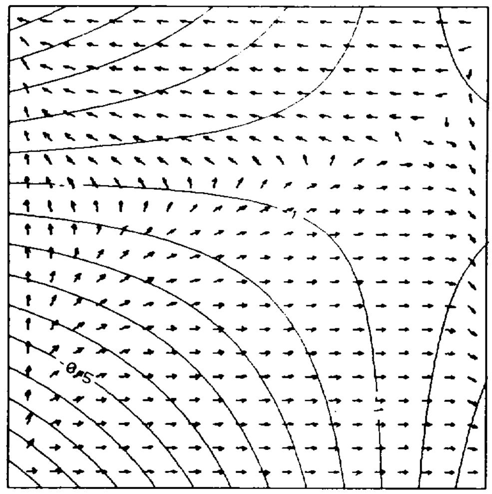
The normalized vector field of this system can now be produced for known values of x, g_i, C_i, and λ $(i = 1, 2)$.

Examples of the normalized vector fields for the example converter are shown in Figure 1. The figure illustrates four different cases of convergence to the equilibrium point for values $\lambda = 10$, 100. The other convergence parameters are conductances g_i, and the capacitance ratio C_2/C_1. The convergence has been evaluated for $x = 0$ (Figure 1 a, b) and $x = 1.8$ (Figure 1 c, d). The energy contours, or equipotential lines, have also been marked.

The trajectories become horizontally skewed for large C_2/C_1. This condition of uneven capacitances promotes movement of the system by slowly varying v_2 with respect to v_1. This is due to the fact that C_2 holds the majority of charge and the voltage u_2 across it changes more slowly than u_1.

The vector field approach provides insight in network behavior with the energy



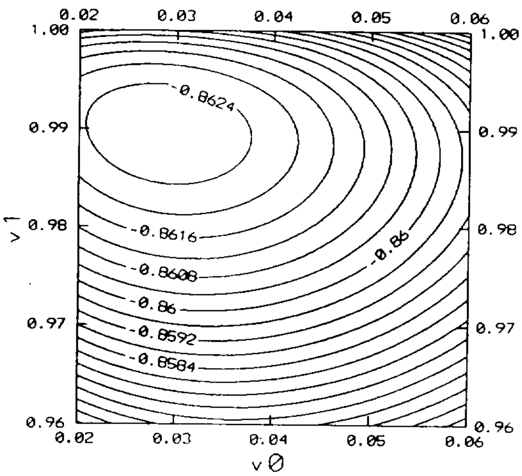


v0 (d)

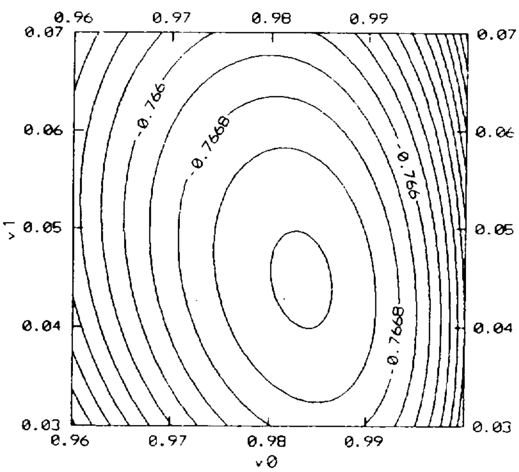
Fig. 1. Selected vector field solutions for 2-bit A/D converter.

(a) $x=0,\ \lambda=100,\ C_2/C_1=1,\ g=1.0$
(b) $x=0,\ \lambda=10,\ C_2/C_1=10,\ g=10$
(c) $x=1.8,\ \lambda=100,\ C_2/C_1=1,\ g=0.1$
(d) $x=1.8,\ \lambda=10,\ C_2/C_1=10,\ g=1.0$

function saddle as shown in Figure 1 c, d. It can also be seen that the trajectories lead to either of the solutions $\underline{v} = [0,1]^t$ or $\underline{v} = [1.0]^t$, dependent upon the initial condition Comparison of convergence with a zero initial condition at $\underline{v} = [0.5, 0.5]^t$ indicates that the correct solution (equal to 2) for representing $x = 1.8$ is reached as shown in Figure 1 c for equal capacitances C_1, C_2. The case shown in Figure 1 d depicts the incorrect solution due to the horizontally biased movement of the system towards the right. To gain more detailed insight into convergence, the energy maps near $v = [1,0]$, and $v = [0,1]$ have been expanded as shown in Fig. 2c,d. It can be seen that the energy minima near the corners are indeed within the square.



a) macro map for upper left corner



b) macro map for lower right corner

Fig. 2. Vector field macro map for $x=1.6$, $\lambda=2$, $C_2/C_1=1$, $g=2.5$.

Conclusion

In conclusion, the vector field approach provides a detailed insight into the transient behavior and stability conditions of the network. This method can be applied to networks using actual neurons and when complete sets of solutions, as opposed to a single time-domain solution, are needed. Although the method can be graphically illustrated only for $n < 3$, it can be applied to networks of any dimensional size.

References

Hopfield, J. J., 1984, Neurons with graded response have collective computational properties like those of two state neurons, *Proc. Natl. Acad. Sci.* 81, 3088 ~ 3092.

Hopfield, J. J. and D. W. Tank, 1985, Neural computation of decisions in optimization problems, *Biolog. Cybern.* 52, 141-154.

Tank, D. W. and J. J. Hopfield, 1986, Simple neural optimization networks: An A/D converter, signal decision circuit and a linear programming circuit, *IEEE Trans. on Circ. and Syst.* CAS-33(5), 533-541.

Zurada, J. M. and M. J. Kang, 1989, Computational circuits using neural optimization concept, *Int. Journal of Electron.*, 67(3), 311-320.

Zurada, J. M. and M. J. Kang, 1991, Dynamic and Static Numerical Modeling of Actual Gradient-type Neural Networks, *Proc. of the 1991 IEEE International Symp. on Circ. and Syst., Singapore*, 11-14.

〈國文抄錄〉

벡터장을 이용한 호프필드 신경회로망의 분석

이 논문에서 호프필드 신경회로망을 분석함에 있어서 E^n-공간에서의 모든 궤적을 생성하는 것을 가능케 하는 벡터장 방법을 제시하였다. 궤적은 실제 전기소자, 즉 캐패시터, 저항, *op-amp*등으로 구성된 신경회로망의 초기상태에서부터 안정상태로의 모든 변화를 알아볼 수 있게 한다.