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The Instability Theorems for Finite Delay Functional Differential Equations

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Summary

We consider a system of nonautonomous finite delay functional differential equation $x'(t)=F(t, x_t)$ and obtain conditions on a Liapunov functional to insure the instability of the zero solution.

Introduction

It is well-known that Liapunov's direct method sometimes provides a useful tool in the study of stability and instability of functional differential equations. See, for example, Burton(1985), Hale (1965, 1977). The purpose of this paper is to provide two new instability theorems for the finite delay functional differential equations by Liapunov's direct method.

For the remainder of this section, we present the fundamental notation and definitions to which we will refer throughout this paper. Section 2 is devoted to obtaining two new theorems involving Liapunov functionals for instability for finte delay functional differential equations.

For $x \in R^n$ with $x=(x_1, x_2, \cdots, x_n)$, $|x|$ denotes a usual norm in R^n, and, for fixed $h>0$, C denotes the space of ocntinuous functions mapping $[-h, 0]$ into R^n, and for $\phi \in C$,

$$\|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|.$$

Also, C_H denotes the set of $\phi \in C$ with $\|\phi\| < H$. If x is a continuous function of u defined for $-h \le u < A$, with $A>0$, and if t is a fixed number satisfying $0 \le t < A$, then x_t denotes the restriction of x to $[t-h, t]$ so that x_t is an element of C defined by

$$x_t(\theta) = x(t+\theta) \quad \text{for} -h \le \theta \le 0.$$

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We consider the system

$$x'(t) = F(t, x_t), \qquad (1)$$

where $F : R_+ \times C_H \rightarrow R^n$ is continuous and takes closed bounded sets into bounded sets; $0 < H \leq \infty$. We denote by $x(t_0, \phi)$ a solution of (1) with initial condition $\phi \in C$ where $x_{t_0}(t_0, \phi) = \phi$ and we denote by $x(t, t_0, \phi)$ the value of $x(t_0, \phi)$ at t. x' denotes the right-hand derivative. It is well known (Burton (1985), Burton (1989)) that for each $t_0 \in R_+ = [0, \infty)$ and each $\phi \in C_H$, there is at least one solution $x(t_0, \phi)$ defined on an interval $[t_0, t+\alpha)$ and, if there is an $H_1 < H$ with $|x(t, t_0, \phi)| \leq H_1$ for all t for which $x(t, t_0, \phi)$ is defined, then $\alpha = \infty$.

A Liapunov functional is a continuous function $V : R_+ \times C_H \rightarrow R_+$ which is locally Lipschitz with respect to ϕ. The derivative of a Liapunov functional $V(t, \phi)$ along a solution $x(t)$ of (1) may be defined in several equivalent ways. If V is differentiable, the natural derivative is obtained using the chain rule. Then $V'_{(1)}(t, \phi)$ denotes the derivative of functional V with respect to (1) defined by

$$V'_{(1)}(t, \phi) = \limsup_{\delta \rightarrow 0+} \{V(t+\delta, x_{t+\delta}(t, \phi)) - V(t, \phi)\} / \delta.$$

Definition 1.1. A continous function $W : R_+ \rightarrow R_+$ is called a wedge if $W(0) = 0$ and W is strictly increasing on R_+.

Definition 1.2. Let $F(t, 0) = 0$ for all $t \geq 0$.

(a) The zero solution of (1) is said to be stable if for each $\epsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\phi \in C_\delta, t \geq t_0]$ imply $|x(t, t_0, \phi)| < \epsilon$.

(b) The zero solution of (1) is said to be unstable if there exist $\epsilon > 0$ and $t_0 \geq 0$ such that for any $\delta > 0$ there is an ϕ with $\|\phi\| < \delta$ and a $t_1 > t_0$ such that $|x(t_1, t_0, \phi)| \geq \epsilon$.

Notice that stability requires all solutions starting near zero to stay near zero, but instbility calls for the existence of some solutions starting near zero to move well away from zero.

Definition 1.3. A measurable function $\eta : R_+ \rightarrow R_+$ is said to be positive in measure if for every $\epsilon > 0$ there are $T \in R_+$, $\delta > 0$ such that $[t \geq T$, $Q \subset [t-h, t]$ is open, $\mu(Q) \geq \epsilon]$ imply that $\int_Q \eta(t)\,dt \geq \delta$. (Here, $\mu(Q)$ denotes the Lebesgue measure of Q.)

Lemma 1.1. Let $K > 0$ be given and suppose that η is positive in measure. Then for each wedge W and $\alpha > 0$ there are $\beta > 0$ and $T \in R_+$ such that if $f : R_+ \rightarrow R$ is measurable, $f^2(s) \leq K$ for $s \in R_+$, $t \geq T$, $\int_{t-h}^{t} f^2(s)\,ds \geq \alpha$, then $\int_{t-h}^{t} \eta(s) W(|f(s)|)\,ds \geq \beta$.

Proof. The proof follows from Lemma 2 in Burton and Hatvani (1989).

Main Theorems and Examples

Theorem 2.1. Let $H, K > 0$ and $V : R_+ \times C_H \rightarrow R_+$ be continuous with V locally Lipschitz in ϕ, and let $\eta : R_+ \rightarrow R_+$ be a nonnegative function such that $\int_0^\infty \eta(s)\,ds = \infty$. Suppose that there are wedges W_1, W_2 and W_3 such that, for all $t \geq 0$ and $\phi \in C_H$,

(i) $V(t, \phi) \leq W_1(\|\phi\|)$ and

(ii) $V'_{(1)}(t, x_t) \geq K W_2(|x'(t)|) + \eta(t) W_3(|x(t)|)$,

and there are $\alpha > 0$ and r_0 such that $r > r_0$ implies $W_2(r) \geq \alpha r$. Futhermore, if we can choose a continuous initial function ϕ such that $V(t_0, \phi) > 0$ with $\|\phi\| < \delta$ for any $t_0 \geq 0$ and $\delta > 0$. Then the zero solution of (1) is unstable.

Proof. Suppose that $x=0$ is stable. Then for each $\epsilon > 0$ and $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that $\|\phi\| < \delta$ implies $\|x_t(\phi)\| < \epsilon$ for any $t \geq t_0$. Now we may choose the initial function $\phi : [t_0-h, t_0] \rightarrow R^n$ such that $V(t_0, \phi) >$ and $\|\phi\| < \delta$. Then we have

$$\| x_t \| \geq W_1^{-1}(V(t_0, \phi)) \equiv \theta \text{ for any } t \geq t_0.$$

Thus there exists an r_i in each interval

$$I_i = [t_0 + ih, t_0 + (i+1)h] \text{ with } |x(r_i)| > \theta \text{ for } i = 0, 1, 2, 3, \cdots$$

On each I_i either $|x(t)| \geq \theta/2$ for every $t \in I_i$ or there is an s_i with $|x(s_i)| < \theta/2$. In the first case we have

$$\int_{I_i} \eta(s) W_3(|x(s)|) ds \geq W_3(\theta/2) \int_{I_i} \eta(s) ds$$

In the latter case we have

$$\int_{I_i} |x'(s)| ds \geq \left| \int_{r_i}^{s_i} |x'(s)| ds \right| \geq \theta - \theta/2 = \theta/2.$$

Define

$$p_1(t) = \begin{cases} |x'(t)| & \text{if } |x'(t)| \geq r_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$p_2(t) = |x'(t)| - P_1(t).$$

If

$$\int_{I_i} p_1(t) dt \geq \theta/4$$

then by Lemma 1.1 there are $\beta_1 = \beta_1(\theta)$ and $N_1 = N_1(\theta)$ with

$$\int_{I_i} KW_2(P_1(s)) ds \geq \beta_1 \text{ for } i \geq N_1.$$

If

$$\int_{I_i} p_2(t) dt \geq \theta/4$$

then,

$$\int_{I_i} KW_2(p_2(s)) ds \geq K\alpha\theta/4 \equiv \beta_2 > 0.$$

In any case we have

$$\int_{I_j} K\ W_2(|x'(t)|) dt \geq \min\{\beta_1, \beta_2\} \equiv \beta > 0 \text{ for } j \geq N_1.$$

Thus we have

$$V(t, x_t) \geq \sum_{i=0}^{n} \int_{I_i} K\ W_2(|x'(s)|) ds + \sum_{i=0}^{n} \int_{I_i} \eta(s) W_3(|x(s)|) ds \rightarrow \infty$$

as $t \to \infty$ and $n \to \infty$. Hence the proof is complete.

Example 2.1. Consider a scalar equation

$$x'(t) = x(t) + b(t)x(t-h), \tag{2}$$

where $b: R_+ \to R$ is a continuous function with $0 \leq |b(t)| < 1/2$. Then the zero solution of (2) is unstable.

Proof. Consider the Liapunov functional

$$V(t, x_t) = x^2(t) - \int_{t-h}^{t} K(u) x^2(u) du \text{ with } K(u) = |b(t+h)|.$$

Then we have

$$\begin{aligned} V'(t, x_t) &= 2x(t)x'(t) - K(t)x^2(t) + K(t-h)x^2(t-h) \\ &= 2x(t)\{x(t) + b(t)x(t-h)\} - |b(t+h)|x^2(t) + |b(t)|x^2(t-h) \\ &\geq 2x^2(t) + 2b(t)x(t)x(t-h) - |b(t+h)|x^2(t) + b^2(t)x^2(t-h) \\ &= 2x^2(t) + \{x'(t)\}^2 - x^2(t) - b^2(t)x^2(t-h) - |b(t+h)|x^2(t) \\ &\quad + b^2(t)x^2(t-h) = \{1 - |b(t+h)|\}x^2(t) + \{x'(t)\}^2 \\ &\quad + b^2(t)x^2(t-h) = \{1 - |b(t+h)|\}x^2(t) + \{x'(t)\}^2, \end{aligned}$$

which satisfies the conditions in the above theorem. Hence the proof is complete.

In fact the following theorem is the generalization of Theorem 2. 1. Because the condition (ii) in Theorem 2. 2 is weaker than the condition (ii) in Theorem 2. 1.

Theorem 2. 2. Let $H>0$ and let $V : R_+ \times C_H \to R$ be continuous and locally Lipschitz in ϕ, and let $\eta : R_+ \to R_+$ be a function with $\int_0^\infty \eta(s)\,ds = \infty$. Suppose that there exist wedges W_1 and W_2 such that, for all $t \geq 0$ and $\phi \in C_H$,

(i) $V(t, x_t) \leq W_1(|x(t)|)$ and

(ii) $V'(t, x_t) \geq \eta(t) W_2(|x(t)|)$.

If we can choose a continuous initial function such that $V(t_0, \phi) > 0$ for any $t_0 \geq 0$ and $\delta > 0$. Then the zero solution of (1) is unstable.

Proof. Suppose that $x=0$ is stable. For $\epsilon>0$ and $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that $\|\phi\| < \delta$ implies $|x(\phi)| < \epsilon$ for any $t \geq t_0$. Now we may take the initial function ϕ with $\delta/2 < |\phi(s)| < \delta$ for any $s \in [-h, 0]$. Thus

$$V(t, x_t) \leq W_1(|x(\phi)(t)|) < W_1(\epsilon)$$

is bounded above. But

$$V(t, x_t) \geq V(t_0, \phi) + \int_{t_0}^t \eta(s) W_2(|x(s)|)\,ds$$
$$\geq V(t_0, \phi) + V(t_0, \phi) \int_{t_0}^t \eta(s)\,ds \to \infty$$

as $t \to \infty$, which is a contradiction. Hence the proof is complete.

Example 2. 2. Consider a scalar equation

$$x'(t) = a(t)x(t) + b(t) \int_{t-h}^t x(u)\,du, \qquad (3)$$

where $a, b : R_+ \to R$ are continuous such that

$$\eta(t) = 2a(t) - \int_{t-h}^t |b(-s)|\,ds - h|b(t)| \geq 0$$

and $\int_0^\infty \eta(s)\,ds = \infty$. Then the zero solution of (3) is unstable.

Proof. Consider the Liapunov functional

$$V(t, x_t) = x^2(t) - \int_{-h}^0 \int_{t+s}^t |b(u-s)|\,x^2(u)\,du\,ds$$

Then we have

$$\begin{aligned}
V'(t, x_t) &= 2x(t)x'(t) - \int_{-h}^0 \frac{d}{dt}\left\{\int_{t+s}^t |b(u-s)|\,x^2(u)\,du\right\} ds \\
&= 2x(t)\left\{a(t)x(t) + b(t)\int_{t-h}^t x(u)\,du\right\} - \int_{-h}^0 |b(t-s)|\,x^2(t)\,ds \\
&\quad + \int_{-h}^0 |b(t)|\,x^2(t+s)\,ds = 2a(t)x^2(t) + 2b(t)x(t)\int_{t-h}^t x(s)\,ds \\
&\quad - x^2(t)\int_{-h}^0 |b(t-s)|\,ds + |b(t)|\int_{-h}^0 x^2(t+s)\,ds \\
&\geq 2a(t)x^2(t) - x^2(t)\int_{t-h}^t |b(-s)|\,ds - |b(t)|\,x^2(t)h \\
&= \left\{2a(t) - \int_{t-h}^t |b(-s)|\,ds - h|b(t)|\right\} x^2(t),
\end{aligned}$$

which satisfies the conditions in the above theorem. Hence the proof is complete.

References

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〈국문초록〉

유한지연 범함수 미분방정식에서 0해의 불안정성에 관한 정리들

이 논문에서는 0을 해로 갖는 유한지연 범함수 미분방정식에서 0근방에서 시작한 해들이 시간이 지남에 따라 0에서 벌어지는 해들이 형태를 찾아내는 조건들을 제시한다. 이 조건들을 만드는 데 Liapunov의 직접방법을 이용하였다.