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Fuzzy Variables With Nomal Type Membership Functions

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정상형 소속함수를 갖는 퍼지변수

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Summary

This article has used the concept of a fuzzy variable in Nahamias' sense. These fuzzy variables are of form normal type of order 2 as in the normal density.

In this article we will focus the normal type fuzzy variable of order $p\,(\geq 2)$ for any positive even integer. It is shown that if p is a positive even integer, the sum of normal fuzzy variables of order p is also a normal fuzzy variable of order p. This result is a extension of Nahmias' concept.

1. Introduction

Nahmias (1978) introduced the concept of fuzzy variable as a possible theoretical framework from which a rigorous theory may be constructed about fuzziness. After he introduced an elegant definition for fuzzy variables in terms of pattern space, theoretical application of fuzzy variables have been studied. Zadeh (1978) used the term fuzzy variable in generalizing the formal definition of a variable. The approach Nahmias proposed is analogous to the sample space concept of statistics. As an application, Cai K.Y. et al. (1991) adopted the fuzzy variables as a basis for a theory of fuzzy reliability.

In this article we will interpret fuzzy variable in Nahmias' Sense and extend his results to a general case. In section 2 of this paper, we briefly review certain properties of fuzzy variables. In section 3, we define a normal fuzzy variable of order p. It is shown that the sum of normal fuzzy variables of order p is also a normal fuzzy variable.

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2. Pattern space and fuzzy variable

Following Nahmias, we define a scale σ on the class of all subsets of a base set Γ.

Definition 2.1 For a base ste Γ, suppose that ζ is the class of all subsets of Γ. Suppose a scale, σ, is defined on ζ and satisties the following properties :

ⅰ) $\sigma(\phi) 0$ and $\sigma(\Gamma)=1$

ⅱ) For any arbitrary collection of a subsets A of Γ, $\sigma(\cup_a A_a) = \sup_a \sigma(A_a)$.

Then σ is a scale and the triple (Γ, ζ, σ) referred to as pattern space.

The scale σ is analogous to a probability measure p and the concept of the scale is eqivalent to Zadeh's possibility distribution and a fuzzy measure.

We may think of the scales as assigning a grade of membership to each point in the pattern space.

Definition 2.2 A fuzzy variable X is a real valued function from Γ to R.

A fuzzy variable X can be imagined as some quantitative representation of an object.

Definition 2.3 The membership function of a fuzzy variable X, denoted by

$$\mu_x(x) = \sigma\{r \in \Gamma | X(r) = x\}, \quad x \in R.$$

Note that

$$\sup_x \mu_x(x) = \sigma\{\cup_x \{r | X(r) = x\}\} = \sigma(\Gamma) = 1$$

It has been shown that the value of $\mu_x(x)$ at point x can be interpreted as the possibility that $X = x$ holds.

To obtain the membership function of $g(X)$ where X is a fuzzy variable and $g : R \to R$ is any function, Nahmias proved Zadeh's extension principle :

$$\mu_{g(x)}(z) = \sup_{u : g(u) = z} \mu_x(u)$$

As special case of this transformation, we give the following examples :

(ⅰ) $\mu_{\alpha x}(Z) = \mu_x(\frac{z}{a})$ for all $\alpha \neq 0$, all z.

(ⅱ) $\mu_{x^2}(z) = \mu_x(\sqrt{z}) \vee \mu_x(-\sqrt{z})$ for $z \geq 0$.

The notation $\vee$ denotes the maximum.

The following definition was introduced by Nahmias.

Definition 2.4 The sets $A_1, A_2, \cdots, A_n \subset \zeta$ are said to be mutually unrelated if for any permatation of the set $\{1, 2, \cdots, n\}$, denoted by $i_1, \cdots, i_k$ for $1 < k \leq n$,

$$\sigma(A_{i_1} \cap \cdots \cap A_{ik}) = \min(\sigma(A_{i_1}), \cdots, \sigma(A_{ik})).$$

Definition 2.5 Given a pattern space (Γ, ζ, σ), the fuzzy variables $X_1, \cdots X_n$ are said to be mutually unrelated if for any subset $\{i_1, i_2, \cdots, i_k\}$ of $\{1, 2, \cdots, n\}$, the sets $(X_{i_1} = x_1), \cdots, (X_{ik} = x_k)$ are unrelated for all $x_1, \cdots, x_k \in R$.

A collection of fuzzy variables is mutually unrelated if every finite subcollection has the property that the scale of the intersection can be computed via the minimum of the scale of each term.

Rao and Rashed (1981) preferred an alternative terminology 'min-relatedness' in stead of 'unrelatedness'. The concept of unrelatedness in fuzzy variable has a direct analogy in the independence of events or random variables.

Another useful notion for fuzzy variables is modal value.

The following definition is natural.

Definition 2.6 Let X be a fuzzy variable with membership function μ_x. A real number m is said to be a modal value of X if $\mu_x(m) = 1$.

Definition 2.7 A fuzzy variable X is said to be unimodal, if there exists unique $a \in R$ such that $\mu_x(a) = 1$.

Definition 2.8 A fuzzy variable X is convex if its membership function is quasi-concave. That is, $\mu_x(\lambda_a + (1-\lambda)\ b) \geq \min\ (\mu_x(a),\ \mu_x(b))$ for all a, $b \in R$ and $0 \leq \lambda \leq 1$.

Using the concept of unrelated fuzzy variables, Nahmias derived Zadeh's extension principle for the sum of two fuzzy variables. Nahmias derived the following theorem.

Theorem 2.1 If X and Y are fuzzy variables then

i) $\mu_{x+y}(z) = \sup_x \sigma(\{X=x\} \wedge \{Y=z-x\})$,
ynd if X and Y are unrelated this reduces to

ii) $\mu_{x+y}(z) = \sup\ [\mu_x(x) \wedge \mu_y(z-x)]$
$= \sup_x \min[\mu_x(x),\ \mu_y(z-x)]$.

Another binary operations for unrelate fuzzy variables of Theorem 2.1. Some examples are :

$\mu_{x-y}(z) = \sup_x \min(\mu_x(x),\ \mu_y(z+x))$,

$\mu_{xy}(z) = \sup_x \min(\mu_x(x),\ \mu_y(\frac{z}{x}))$.

3. Nomal fuzzy variables of order p

In this section we would like to study a normal type fuzzy variables of order p. At first, we introduce the notion of normal fuzzy variable of order p.

Definition 3.1 Let p be a positive even integer. A fuzzy variable X is of the normal class of order p if the membership function is of form

$$\mu_x(x) = \exp(-(\frac{x-a}{b})^p)$$

where $a \in R$ and $b > 0$.

At this time we say that a fuzzy variable X is a normal fuzzy variable of order p with parameters (a, b ; p).

Hereafter we use the notation N(a, b ; p) for the normal fuzzy variable of order p.

We note that a normal fuzzy varibable by Nahmias is a special case of p=2.

Theorem 3.1 Let X and Y be unrelated normal fuzzy variable N(a, b ; p) and N(c, d ; p) respectively.

The the fuzzy variable Z=X+Y has a unimodal value a+c.

Proof. We will show that $\mu_z(a+c) = 1$.

From Theorem 2.1, for any real z,

$\mu_z(z) = \sup_x \min[\mu_x(x), \mu_y(z-x)]$.

Since $\min[\mu_x(x), \mu_y(z-x)]$ is convex, the supremum is obtained when z=a+c and x=a.

That is $\mu_z(a+c) = 1$.

So, we have the required result.

Theorem 3.2 If X and Y are unrelated normal fuzzy variable $N(a_1, b_1 ; p)$ and $N(a_2, b_2 ; p)$ respectively, then Z=X+Y is also a normal fuzzy variable $N(a_1+a_2, b_1+b_2 ; p)$.

Proof. Since X and y are unrelated fuzzy variables, $\mu_z(z) = \sup_x\ \min(\mu_x(x),\ \mu_y(z-x))$ by Theorem 2.1. Assume $a_2 > a_1$ and $b_2 > b_1 > 0$ without loss of generality.

From the fact that μ_x and μ_y are convex, the function $\min(\mu_x(x),\ \mu_y(z-x))$ has its maximum at the point x' satisfying the following equation :

$\mu_x(x') = \mu_y(z-x')$.

So, the following equation is derived :

$(\frac{x'-a}{b_1})^p = (\frac{z-x'-a}{b_2})^p$ for positive even integer p.

Since p is a positive even integer, the above equation gives real x' satisfying

$$\frac{x'-a_1}{b_1} = \pm \frac{z-x'-a_2}{b_2}.$$

So, the real solutions x' are

$$x'_- = \frac{1}{b_1+b_2}\ [b_1(z-a_2) + b_2 a_1]$$

and

$$x'_- = \frac{1}{b_1-b_2}\ [b_1(z-a_2) + b_2 a_1].$$

From the condition $b_1-b_2 < b_1+b_2$ and the convexity of the function min $(\mu_x(x),\ \mu_y(z-x))$,

$$\mu_z(z)=\mu_x(x'_+)=\exp(-(\frac{z-a_1-a_2}{b_1+b_2})^p).$$

Hence we obtain the result.

Corollary 3.3 If a fuzzy variable X is a normal fuzzy variable N(a, b ; p), then for some a≠0, g(X) = aX is also a normal fuzzy variable N(αa, αb ; p).

Proof. Let $g(X)=\alpha X$.

From the expression i) of section 2,

$$\mu_{g(x)}(x)=\mu_x(\frac{x}{a})=\exp(-(\frac{\frac{x}{\alpha}-a}{b})^p).$$

$$=\exp(-(\frac{x-\alpha a}{ab})^p)$$

Hence $g(X)=\alpha X$ is a normal fuzzy variable N(αa, αb ; p).

Using theorem 3.2 and Corollary 3.3, we have the following.

Theorem 3.4 If X_i, i=1, 2, ⋯, n are unrelated normal fuzzy varizble N(a_i, b_i ; p) respectively, then $\sum_{i=1}^{n}\alpha_i X_i$ is normaly fuzzy variable N($\sum_{i=1}^{n}\alpha_i a_i$, $\sum_{i=1}^{n}\alpha_i b_i$; p) for nonzero α_i, i=1, ⋯, n.

Now we will state special cases of theorem 3.4

Lemma 3.5 Let $X_1, X_2 \cdots, X_n$ be unrelated normal fuzzy variables N(a_i, b_i ; p) respectively.

Then $\frac{1}{n}\sum_{i=1}^{n} X_i$ is a normal fuzzy variable N($\bar{a}$, $\bar{b}$; p) where

$$\bar{a}=\frac{1}{n}\sum_{i=1}^{n} a_i,\quad \bar{b}=\frac{1}{n}\sum_{i=1}^{n} b_i.$$

Proof. In Theorem 3.4, set $\alpha_i=\frac{1}{m}$ for i=1, ⋯, n.

Lemma 3.6 Let $X_1, X_2, \cdots, X_n$ be unrelated normal fuzzy variables from N(a, b ; p). Then $\frac{1}{n}\sum_{i=1}^{n} X_i$ is also a fuzzy variable N(a, b ; p).

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〈국문초록〉

정상형 소속함수를 갖는 퍼지변수

본 논문에서는 Nahmias가 제안한 퍼지변수, 특히 정상형 소속함수를 갖는 퍼지변수에 대해 이 퍼지변수의 소속함수가 정상형이며 또 위수가 2이상인 짝수 위수 p를 갖는 경우의 여러 성질을 다루었으며, 이 정상형 퍼지변수들의 합 또한 위수가 p인 정상형 퍼지변수가 됨을 보이므로 Nahmias의 결과를 확장하였다.