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The Onset of Convective Instability and Heat Transfer Correlation in Internally Heated Horizontal Fluid Layers

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내부열원에 의해 가열되는 수평 유체층에서의 대류 불안정성 발생 및 열전달 상관관계

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Summary

Buoyancy effects in internally heated horizontal fluid layers with a rigid, adiabatic lower boundary and a rigid, isothermal upper boundary is analyzed theoretically. The onset of thermal convection is analyzed by using the propagation theory, we have developed, and its connection to the fully-developed heat transport is sought. The critical time to mark cellular motion for the deep-pool case is found to increase with a decrease in Prandtl number. Based on the present stability criteria, a new correlation of the Nusselt number is produced as a function of both the Rayleigh number and the Prandtl number. It is shown that the present heat transfer correlation on thermal convection compares reasonably with existing experimental data of water.

Introduction

From the beginning of this century the convective motion driven by buoyancy forces has attracted many researcher's interest. Benard (1901) conducted systematic experiments on the onset of natural convection in a horizontal fluid layer. Later, Lord Rayleigh (1916) showed that the buoyancy-driven convection can occur when the adverse temperature gradient exceeds a certain critical value. Thereafter, many researchers analyzed the onset condition of buoyancy driven convection in fluid layers heated from below or cooled from above. Extensive results for the various systems have been summarized by Chandrasekhar (1961) and Berg et al. (1974).

Kulacki and Goldstein (1975) extended the stability analysis to the horizontal fluid layer heated by internal heat sources. It is well-known that thermal convection problems driven by energy release from distributed volumetric energy sources appears to play an important role in wide variety of engineering applications, such as geothermal reservoirs, chemical reactors and heat removal of nuclear power plants.

When an initially quiescent horizontal fluid layer system is heated rapidly buoyancy-driven motion sets in before the basic temperature field is fully-developed. Therefore, in case of rapid heating the basic temperature profile of pure conduction becomes time-dependent. To analyze this kind of thermal instability in horizontal fluid layers several theoretical methods have been proposed: the amplification theory (Foster, 1965), energy method (Wankat and Homsy, 1977), stochastic model (Jhavary and Homsy, 1982) and propagation theory (Choi et al., 1984). The amplification theory treated the time dependency as an initial value problem. This method is quite popular, but it involves arbitrariness in choosing both an initial condition and its amplification factor to mark the onset of motion.

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The propagation theory predicts the conditions to mark the onset time deterministically, it employs the thermal penetration depth as a length scaling factor and transforms the linearized disturbance equations into the similar forms. Its prediction has been coincident with the various experimental results in deep-pool systems experiencing rapid heating, such as laminar forced convection (Kim et al., 1990; Choi and Kim, 1990), laminar natural convection (Chun and Choi, 1991) and also fluid-saturated porous layer (Yoon and Choi, 1989).

Another important problem in buoyancy-driven phenomena is the heat transfer characteristics in thermally fully-developed state. To analyze this problem Howard (1964) proposed the boundary-layer instability model in which the heat transfer for very high Rayleigh numbers has a close relationship with stability criteria. Based on Howard's concept, Long (1976) and Cheung (1980) introduced backbone equations to predict the heat transport in horizontal fluid layers. By incorporating their stability criteria into the boundary layer instability model Choi et al. (1989) and Kim and Choi (1992) have derived new heat transfer correlations for various systems. Their resulting heat transfer correlations are in good agreement with existing experimental results.

In the present study, the stability criteria of the onset of regular cell-type motion in a horizontal fluid layer with uniform energy sources is analyzed by using our propagation theory. And based on the stability criteria, a new heat transfer correlation is derived and compared with the existing experimental results. Here, it is shown that the propagation theory we have developed can become a theoretical base in understanding buoyancy-driven phenomena.

Stability Analysis

1. Governing Equations

The system considered here is an initially quiescent horizontal fluid layer of depth "d" with an adiabatic lower boundary and isothermal upper boundary. Before heating the fluid layer is maintained at uniform temperature T_0 for time $t < 0$. For time $t \geq 0$ the layer is heated internally with the uniform volumetric heat generation rate S. Here we employ the Cartesian coordinates with the downward distance Z. The schematic diagram of present system is shown in Fig. 1. For this system the governing equations of flow and temperature fields are expressed by employing the Boussinesq approximation, as follows:

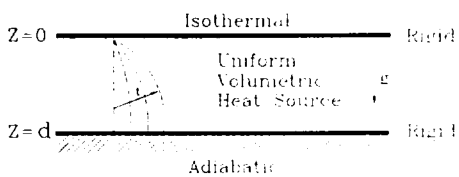

Fig. 1. Schematic diagram of system considered here.

$$\nabla \bullet U = 0 \tag{1}$$

$$\left\{\frac{\partial}{\partial t} + U \bullet \nabla\right\} U = -\frac{1}{\rho_r}\nabla P + \nu\nabla^2 U + g\beta T k \tag{2}$$

$$\left\{\frac{\partial}{\partial t} + U \bullet \nabla\right\} T = \alpha\nabla^2 T + \frac{S}{\rho_r C_p} \tag{3}$$

where U is the velocity vector, T the temperature, P the pressure, μ the viscosity, α the thermal diffusivity, g the gravitational acceleration, ρ the density, C_p the specific heat, β the thermal expansion coefficient. The subscript "r" represents the reference state.

The important parameters to describe the present system are the Prandtl number Pr and the Rayleigh number Ra_I defined by

$$Pr = \frac{\nu}{\alpha} \quad \text{and} \quad Ra_I = \frac{g\beta S d^5}{k\alpha\nu} \tag{4}$$

where k and ν denote the thermal conductivity and the kinematic viscosity, respectively. In case of slow heating the basic temperature profile is parabolic and time-independent and its critical condition is well represented by

$$Ra_{Ic} = 2772 \tag{5}$$

But for a rapid heating system of large Ra_I, the stability problem becomes transient and complicated, and the critical time t_c to mark the onset of buoyancy-driven motion remains unsolved. For this transient stability analysis we define a set of nondimensional variables τ, z, θ_0 by using the scale of time d^2/α, length d and temperature Sd^2/k. Then the basic conduction state is represented in dimensionless form by

$$\frac{\partial\theta_0}{\partial\tau} = \frac{\partial^2\theta_0}{\partial z^2} + 1 \tag{6}$$

with the following initial and boundary conditions,

$$\theta_0(0,z) = \theta_0(\tau,0) = \frac{\partial\theta_0}{\partial z}(\tau,1) = 0 \tag{7}$$

The above equations can be solved by using the conventional separation-of-variables technique, as follows:

$$\theta_0 = z\left(1-\frac{z}{2}\right) - \frac{16}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\left\{\frac{(2n+1)}{2}\pi z\right\} \times \exp\left\{-\frac{(2n+1)^2\pi^2}{4}\tau\right\} \tag{8}$$

For deep-pool systems, the Leveque-type solution can be obtained as follows (Carslaw and Jaeger, 1959):

$$\theta_0 = \tau\left[\left(1+\frac{1}{2}\eta^2\right)\mathrm{erf}\left(\frac{\eta}{2}\right)+\frac{1}{\sqrt{\pi}}\eta\exp\left(-\frac{1}{4}\eta^2\right)-\frac{1}{2}\eta^2\right] \quad (9)$$

where $\eta = z/\sqrt{\tau}$. The above equation is in good agreement with the exact solution (8) in the region of $\tau < 0.1$. Since we are primarily concerned with the deep-pool case of large Ra_I and small τ, the above Leveque type solution (9) represents the basic temperature profile quite well. But for the mathematical convenience in the present stability analysis we simplify the basic temperature profile by using the integral method (Eckert and Robert, 1972) as follows:

$$\theta_0 = \tau\left[1-(1-\zeta)^3\right]\left[1-U_{\zeta-1}\right] \quad (10)$$

where $\zeta = z/\delta_T$. $U_{\zeta-1}$ is the unit step function having the zero value at $\zeta = 1$ and δ_T is the dimensionless thermal penetration depth having the value of $\sqrt{8\tau}$. This approximate solution is in good agreement with the exact one in the region of $\tau \le 0.1$.

2. Stability Equations

Under the linear stability theory disturbances caused by the onset of thermal convection can be formulated, in dimensionless form, in terms of the temperature component θ_1 and the vertical velocity component w_1 by transforming equations (1) ~ (3):

$$\left\{\frac{1}{Pr}\frac{\partial}{\partial\tau}-\overline{\nabla}^2\right\}\overline{\nabla}^2 w_1 = -\overline{\nabla}_1^2\theta_1 \quad (11)$$

$$\frac{\partial\theta_1}{\partial\tau}+Ra_I w_1\frac{\partial\theta_0}{\partial z} = \overline{\nabla}^2\theta_1 \quad (12)$$

where $\overline{\nabla}^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and $\overline{\nabla}_1^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. Here the velocity component has the scale of α/d and the temperature component has the scale of $\alpha\nu/(g\beta d^3)$. The proper boundary conditions are given by

$$w_1 = Dw_1 = \theta_1 = 0 \quad \text{at } z=0 \quad (13.a)$$
$$w_1 = Dw_1 = D\theta_1 = 0 \quad \text{at } z=1 \quad (13.b)$$

Our goal is to find the critical time τ_c for a given Pr and Ra_I by using equations (11) ~ (13).

Based on the normal mode analysis, the amplitude functions w^* and θ^* are constructed as a function of $\zeta(=z/\delta_T)$ only by assuming periodic motion of disturbances in the form of regular cells over the horizontal plane:

$$\left[w_1(\tau,x,y,z),\theta_1(\tau,x,y,z)\right] = \left[\delta_T^2 w^*(\zeta),\theta^*(\zeta)\right]\exp\left[i(a_x x+a_y y)\right] \quad (14)$$

where "i" is the imaginary number. The horizontal wave number "a" has the relation of $a=\left[a_x^2+a_y^2\right]^{1/2}$ By using these relations the stability equation is obtained from equations (11) ~ (13) as

$$\left\{\left(D^2-a^{*2}\right)^2+\frac{4}{Pr}\left(\zeta D^3-a^{*2}\zeta D+2a^{*2}\right)\right\}w^* = -a^{*2}\theta^* \quad (15.a)$$

$$\left(D^2+4\zeta D-a^{*2}\right)\theta^* = Ra^* w^* D\theta_0^* \quad (15.b)$$

where $a^* = a\delta_T$, $Ra^* = Ra_I\delta_T^3\tau$, $D=\frac{d}{d\zeta}$ and $\theta_0^* = \theta_0/\tau$.

It is assumed that a^* and Ra^* are the eigenvalues, and also the onset time of buoyancy-driven convection for a given Ra_I is unique under the principle of exchange of stabilities. The above procedure is the essence of our propagation theory.

3. Solution Procedure

1) In the case of Pr→∞.

The stability equation derived in equation (15) still involves mathematical complexity. This problem can be alleviated by dealing with very high and very low Prandtl numbers. Let us consider the very high Pr case, first. Then the stability equations reduce to

$$\left(D^2+4\zeta D-a^{*2}\right)\left(D^2-a^{*2}\right)^2 w^* = -3a^{*2}Ra^*\left(1-2\zeta+\zeta^2\right)w^* \quad \text{for } 0\le\zeta\le 1 \quad (16.a)$$

$$\left(D^2+4\zeta D-a^{*2}\right)\left(D^2-a^{*2}\right)^2 w^* = 0 \quad \text{for } \zeta\ge 1 \quad (16.b)$$

The above equations are separated, depending on the range of ζ. The boundary conditions can be converted to

$$w^* = Dw^* = \left(D^2-a^{*2}\right)w^* = 0 \quad \text{at } \zeta=0 \quad (17.a)$$
$$w^* = Dw^* = D\left(D^2-a^{*2}\right)w^* = 0 \quad \text{at } \zeta=1/\delta_T \quad (17.b)$$

For a deep-pool system, the condition of $\zeta = 1$ corresponds to the basic thermal penetration depth, and $1/\delta_T$ is practically equivalent to an infinite high value since δ_T is small.

Within the thermal penetration depth ($\zeta\le 1$) the velocity disturbance is approximated by means of rapidly converging power series proposed by Sparrow et al. [22]:

$$w_i^* = \sum_{j=0}^{5} H_j f_j(\zeta) \tag{18.a}$$

$$f_j(\zeta) = \sum_{n=0}^{\infty} b_n^{(j)} \zeta^n \tag{18.b}$$

H_j (j = 0, 1, 2, 3, 4, 5) is an arbitrary coefficient needed in the sixth-order differential equation, and $b_n^{(j)}$ can be obtained by substituting equation (18) into equation (16.a) as the following indicial form:

$$\begin{aligned} b_n^{(j)} = \frac{(n-6)!}{n!}\Big[&\{3a^{*2}(n-2)(n-3)(n-4)(n-5) \\ &-4(n-2)(n-3)(n-4)(n-5)(n-6)\}b_{n-2}^{(j)} \\ &+\{8a^{*2}(n-4)(n-5)(n-6)-3a^{*4}(n-4)(n-5)\}b_{n-4}^{(j)} \\ &+\{a^{*4}-4a^{*2}(n-6)-3a^{*2}Ra^*\}b_{n-6}^{(j)} \\ &+6a^{*2}Ra^* b_{n-7}^{(j)} - 3a^{*2}Ra^* b_{n-8}^{(j)}\Big] \end{aligned} \tag{19.a}$$

$$b_n^{(j)} = \delta_{nj} \quad (n=0,1,2,3,4,5) \ : \text{Kronecker delta} \tag{19.b}$$

$$b_{-1}^{(j)} = b_{-2}^{(j)} = b_{-3}^{(j)} = b_{-4}^{(j)} = b_{-5}^{(j)} = b_{-6}^{(j)} = b_{-7}^{(j)} = b_{-8}^{(j)} = 0 \tag{19.c}$$

Applying the boundary condition of $\zeta = 0$ to equations (18) and (19), the velocity disturbances inside the thermal penetration depth, $\zeta \le 1$, can be expressed in the following form:

$$w_i^* = H_2\left\{f_2(\zeta) + \frac{a^{*2}}{6} f_4(\zeta)\right\} + H_3 f_3(\zeta) + H_5 f_5(\zeta) \tag{20}$$

In order to obtain the velocity disturbance for the region of $\zeta \ge 1$, it is helpful to consider the solution outside the thermal penetration depth in two stages:

$$\left(D^2 + 4\zeta D - a^{*2}\right)Y = 0 \tag{21.a}$$

$$\left(D^2 - a^{*2}\right)^2 w_o^* = Y \tag{21.b}$$

The WKB method can be used to obtain the solution of Y which satisfies the condition $DY = 0$ as $\zeta \to \infty$. Then the solution of Y is given by (Mathews and Walker, 1973):

$$Y \cong \frac{\exp(-\zeta^2)}{\sqrt[4]{4\zeta^2 + 2 + a^{*2}}} \exp\left\{-\int_1^{\zeta} \sqrt{4\xi^2 + 2 + a^{*2}}\, d\xi\right\} \tag{22}$$

The form of Y as is determined by the WKB method is very complicated. In order to find the particular solution of equation (21.b) over the range of $\zeta \ge 1$, ζ is converted as

$$s = \zeta - 1 \tag{23}$$

which provides the convergence in computer calculation. By using the initial values of Y and DY at $\zeta = 1$ the solution of Y is obtained in form of power series. The solution of the velocity disturbance outside the thermal penetration depth can be obtained by inverse-operator technique, as follow:

$$\begin{aligned} w_o^* = {} & H_7 \exp(-a^*s) + H_9 s \exp(-a^*s) \\ & + \frac{H_{10}}{4a^{*2}}\Bigg\{\exp(a^*s)\sum_{n=0}^{\infty}\frac{p_n}{(n+1)(n+2)}s^{n+2} \\ & + \exp(-a^*s)\sum_{n=0}^{\infty}\frac{q_n}{(n+1)(n+2)}s^{n+2} \\ & - \frac{1}{a^*}\exp(a^*s)\sum_{n=0}^{\infty}\frac{p_n}{(n+1)}s^{n+1} + \frac{1}{a^*}\exp(-a^*s)\sum_{n=0}^{\infty}\frac{q_n}{(n+1)}s^{n+1}\Bigg\} \end{aligned} \tag{24.a}$$

$$p_n = -\frac{(n-2)!}{n!}\left\{(2a^* + 4)(n-1)p_{n-1} + 4(n-2+a^*)p_{n-2} + 4a^* p_{n-3}\right\} \tag{24.b}$$

$$p_2 = -\frac{1}{2}\left\{(2a^* + 4)p_1 + 4a^* p_0\right\} \tag{24.c}$$

$$p_1 = DY(1) - a^* Y(1) \tag{24.d}$$

$$p_0 = Y(1) \tag{24.e}$$

$$q_n = -\frac{(n-2)!}{n!}\left\{(4 - 2a^*)(n-1)q_{n-1} + 4(n-2-a^*)q_{n-2} - 4a^* q_{n-3}\right\} \tag{24.f}$$

$$q_2 = -\frac{1}{2}\left\{(4 - 2a^*)q_1 - 4a^* q_0\right\} \tag{24.g}$$

$$q_1 = DY(1) + a^* Y(1) \tag{24.h}$$

$$q_0 = Y(1) \tag{24.i}$$

The above equations (22) and (24) for the velocity disturbance inside and outside the thermal penetration depth are patched at $\zeta = 1$. Here, the velocity, the stress and the temperature are all contineous in a physical sense, and in a mathematical sense the expression for the velocity disturbance is an analytical function. Thus the following relations must be satisfy:

$$D^n w_i^* = D^n w_o^* \quad (n=0,1,2,3,4,5) \quad \text{at } \zeta = 1 \tag{25}$$

The above relations can be expressed in matrix form as

$$\begin{bmatrix} f_2 + (a^{*2}/6)f_4 & f_3 & f_5 & -1 & 0 & 0 \\ Df_2 + (a^{*2}/6)Df_4 & Df_3 & Df_5 & a^{*2} & -1 & 0 \\ D^2f_2 + (a^{*2}/6)D^2f_4 & D^2f_3 & D^2f_5 & -a^{*2} & 2a^* & 0 \\ D^3f_2 + (a^{*2}/6)D^3f_4 & D^3f_3 & D^3f_5 & a^{*3} & -3a^{*2} & 0 \\ D^4f_2 + (a^{*2}/6)D^4f_4 & D^4f_3 & D^4f_5 & -a^{*4} & 4a^{*3} & -Y \\ D^5f_2 + (a^{*2}/6)D^5f_4 & D^5f_3 & D^5f_5 & a^{*5} & -5a^{*4} & -DY \end{bmatrix}_{\zeta=1} \times \left[H_2, H_3, H_5, H_7, H_9, H_{10}\right]^T = 0 \tag{26}$$

To produce nontrivial solution of velocity disturbances, the determinant of 6×6 matrix must be zero. The value of the determinant is determined by the two eigenvalue a^* and Ra^*. Therefore the computer calculation was carried out to obtain Ra^* for a given a^*.

2) In the case of Pr→0

Stability analysis for the very small Prandtl number case is basically similar to the case of Pr→∞. But, in the limiting case of Pr→0, the viscous effects of amplitude function can be ignored in comparison to the convective effects. Also, boundary conditions should be relaxed under the approximation of negligible viscous effects. Therefore, we can't apply the no-slip condition at $\zeta = 0$. Then the stability equations are reduced as:

$$\left(D^2+4\zeta D-a^{*2}\right)\left(\zeta D^3-a^{*2}\zeta D+2a^{*2}\right)w^* = -\frac{3}{4}\Pr Ra^* a^{*2} w^*(1-\zeta)^2 \quad \text{for } 0\le\zeta\le 1 \qquad (27.a)$$

$$\left(D^2+4\zeta D-a^{*2}\right)\left(\zeta D^3-a^{*2}\zeta D+2a^{*2}\right)w^* = 0 \quad \text{for } \zeta\ge 1 \qquad (27.b)$$

with the following boundary conditions

$$w^*=\theta^*=0 \quad \text{at } \zeta=0 \qquad (28.a)$$

$$w^*=Dw^*=D\theta^*=0 \quad \text{as } \zeta\to\infty \qquad (28.b)$$

The inner solution can not be easily obtained as the rapidly converging power series form because of the non-linear characteristic of convective term. Thus Frobenius method is applied in this study as follows:

$$w_i^* = \sum_{n=0}^{\infty} b_n \zeta^{n+c} \qquad (29)$$

By substituting equation (29) into equation (27.a), the indicial equation is obtained as

$$c(c-1)(c-2)^2(c-3)=0 \qquad (30)$$

Now, we can outline the form of the solution for the each induce "c" and obtain the solution as 5 independent series.

$$\begin{aligned} w_i^* = {} & G_0\left\{1-\frac{1}{48}\left(\frac{3}{4}\Pr Ra^* a^{*2}-2a^{*4}\right)\zeta^4+\dots\right\} \\ & +G_1\left\{\zeta-\frac{1}{360}\left(\frac{3}{4}\Pr Ra^* a^{*2}+10a^{*2}-a^{*4}\right)\zeta^5+\dots\right\} \\ & +G_2\left\{\zeta^2-\frac{1}{1440}\left(\frac{3}{4}\Pr Ra^* a^{*2}\zeta^6\right)+\dots\right\} \\ & +G_3\left\{\zeta^3-\frac{1}{30}\left(2-a^{*2}\right)\zeta^5+\dots\right\} \\ & +G_4\left\{\left(\zeta^2-\frac{1}{1440}\frac{4}{3}\Pr Ra^* a^{*2}\zeta^6+\dots\right)\ln\zeta+\left(\frac{a^{*2}}{12}\zeta^4+\dots\right)\right\} \end{aligned} \qquad (31)$$

where coefficient $G_i(i=0,1,2,3,4)$ is an arbitrary constants. In order to satisfy the boundary conditions which the velocity and temperature perturbations do not exist at the rigid-isothermal surface, G_0 and G_4 should be disappeared.

The outer solution in the infinite domain can be obtained by separation equation (27.b) into

$$\left(D^2+4\zeta D-a^{*2}\right)Y=0 \qquad (32.a)$$

$$\left(D^2-a^{*2}\right)(\zeta D-2)w_o^*=Y \qquad (32.b)$$

The asymptotic solution of equation (32.a) is the same as equation (22). By using this solution, we can obtain the outer amplitude function as the previous case. As the first step, the homogeneous solution of equation (32.b) can be produced as

$$w_{o,h}^* = \frac{G_5}{2}\left\{\left(1-a^*\zeta\right)\exp\left(-a^*\zeta\right)+a^{*2}\zeta^2\int_\zeta^\infty \frac{\exp\left(-a^*\xi\right)}{\xi}d\xi\right\} \qquad (33)$$

And, equation (32.a) is transformed into those of s = ζ-1. Then the solution of Y is generated as the forms of exp(a*s)p(s) and exp(-a*s)q(s). p(s) and q(s) are the power series forms as the function of s, whose coefficients are dependent of the asymptotic solution. Consequently equation (32.b) can be written through the operator technique as

$$\left\{(s+1)D^*-2\right\}w_o^* = \frac{G_6}{2a^*}\left\{\exp\left(a^*s\right)\sum_{n=0}^{\infty}\frac{p_n}{n+1}s^{n+1} -\exp\left(-a^*s\right)\sum_{n=0}^{\infty}\frac{q_n}{n+1}s^{n+1}\right\} \qquad (34)$$

with $p_0=q_0=Y(1)$, $p_1=DY(1)-a^*Y(1)$, and $q_1=DY(1)+a^*Y(1)$. For n≥2, the recursion formula for p_n and q_n can be easily constructed, and are identical with equation (24). The particular solution is obtained in the form of

$$w_{o,p}^* = \exp\left(a^*s\right)\sum_{n=0}^{\infty}d_n s^n+\exp\left(-a^*s\right)\sum_{n=0}^{\infty}e_n s^n \qquad (35.a)$$

$$d_0=d_1=0 \qquad (35.b)$$

$$d_2=\frac{G_6}{2a^*}\frac{p_0}{2} \qquad (35.c)$$

$$d_3=\frac{G_6}{2a^*}\frac{1}{6}\left\{p_1-a^*p_0\right\} \qquad (35.d)$$

$$d_{n+2}=\frac{1}{n+2}\left\{\frac{G_6}{2a^*}\frac{p_n}{n+1}-\left(n-1+a^*\right)d_{n+1}-a^*d_n\right\} \quad \text{for } n\ge 2 \qquad (35.e)$$

$$e_0=e_1=0 \qquad (35.f)$$

$$e_2=-\frac{G_6}{2a^*}\frac{q_0}{2} \qquad (35.g)$$

$$e_3=-\frac{G_6}{2a^*}\frac{1}{6}\left\{q_1-a^*q_0\right\} \qquad (35.h)$$

$$e_{n+2}=-\frac{1}{n+2}\left\{\frac{G_6}{2a^*}\frac{q_n}{n+1}-\left(n-1+a^*\right)e_{n+1}-a^*e_n\right\} \quad \text{for } n\ge 2 \qquad (35.i)$$

The outer solution can be obtained as $w_o^* = w_{o,h}^* + w_{o,p}^*$. Since the solution to satisfy all the boundary conditions are found in the whole domain, the following equation to characterize the onset of convection is generated by using equation (25) as the previous case:

$$\begin{bmatrix} G^{(1)} & G^{(2)} & G^{(3)} & -\left(e^{-a^*} - e^{a^*} + a^{*2}E_i\right)/2 & 0 \\ DG^{(1)} & DG^{(2)} & DG^{(3)} & a^* e^{-a^*} - a^{*2}E_i & 0 \\ D^2G^{(1)} & D^2G^{(2)} & D^2G^{(3)} & -a^{*2}E_i & 0 \\ D^3G^{(1)} & D^3G^{(2)} & D^3G^{(3)} & a^{*2}e^{-a^*} & -Y \\ D^4G^{(1)} & D^4G^{(2)} & D^4G^{(3)} & -\left(a^{*3} + a^{*2}\right)e^{-a^*} & -DY + Y \end{bmatrix}_{\zeta=1}$$

$$\times\left[G_1, G_2, G_3, G_5, G_6\right]^T = 0 \qquad (36)$$

where $G^{(i)}(i = 0,1,2,3,4)$ is a infinite series with respect to G_i in equation (31) and $E_i = \int_\zeta^\infty \frac{\exp(-a^*\xi)}{\xi} d\xi$. The value of E_i can be obtained by using IMSL subroutine library. $Pr\,Ra^*$ results from the condition that the determinant of resulting 5×5 square matrix is equal to zero. The minimum value of $Pr\,Ra^*$ in the plot of $Pr\,Ra^*$ vs. a^* is the critical condition to mark the onset of natural convection for extremely small Prandtl number.

4.Stability Results

The marginal stability curves obtained from computer calculation are shown in Figs. 2 and 3. And the critical condition for the onset of buoyancy-driven convection are

$$Ra_c^* = 1062.50 \text{ and } a_c^* = 1.93 \text{ for } Pr \to \infty \qquad (37.a)$$

$$Pr\,Ra_c^* = 435.70 \text{ and } a_c^* = 2.79 \text{ for } Pr \to 0 \qquad (37.b)$$

From the aboves onset time τ_c are expressed as

$$\tau_c = 4.66 Ra_I^{-2/5} \quad \text{for } Pr \to \infty \qquad (38.a)$$

$$\tau_c = 3.26 (Pr\,Ra_I)^{-2/5} \quad \text{for } Pr \to 0 \qquad (38.b)$$

Based on the results for the limiting cases, the stability criteria for a deep-pool system may be roughly constructed as

$$Ra_c^* = 1062.50\left(1 + \frac{0.41}{Pr}\right) \qquad (39)$$

Therefore, the onset time of buoyancy-driven convection may have the following relation :

$$\tau_c = 4.66\left(1 + \frac{0.41}{Pr}\right)^{2/5} Ra_I^{-2/5} \quad \text{for } \tau_c \le 0.1 \qquad (40)$$

Foster (1969) proposed that the onset time of natural convection obtained by using the thermal penetration depth as a length scaling factor should be too short by factor of 4.

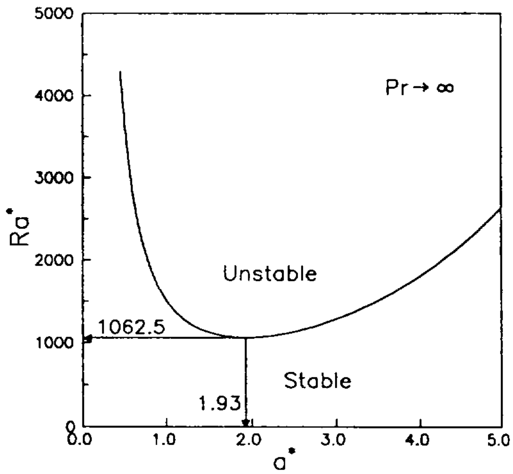

Fig. 2. Maginal stability curve for EMBED Equation

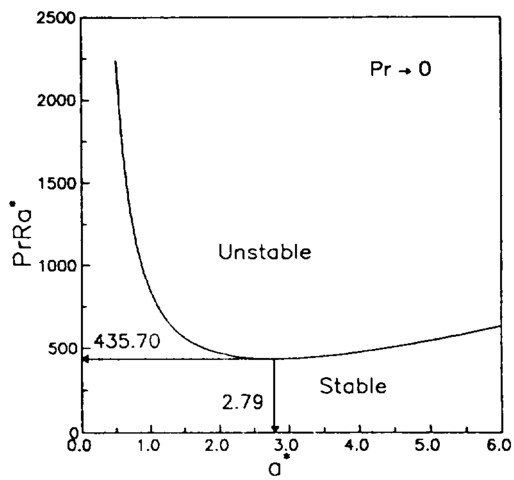

Fig. 3. Maginal stability curve for EMBED Equation

By accepting Foster's concept, we suggest that the disturbances set in at τ_c will lead to manifest convection at $4\tau_c$. Thus, we foretell the onset time when the convective motion can be detectable experimentally, τ_o, as follows:

$$\tau_o = 18.64\left(1 + \frac{0.41}{Pr}\right)^{2/5} Ra_I^{-2/5} \qquad (41)$$

The relationship $\tau_o = 4\tau_c$ can be seen many other systems (Yoon and Choi, 1989; Choi et al., 1988).

Heat Transport

The possibility of connecting the stability criteria to fully-developed turbulent thermal convection has been discussed by Howard (1964). According to Howard's concept called the boundary-layer instability model, the heat transport in fully-developed turbulent state is governed by the narrow region of the heated surface for systems heated isothermally from below. Its modification extending Howard's concept is shown in Fig. 4.

From the boundary layer instability model the Nusselt number, $Nu = \frac{Sd^2}{k\Delta T}$ can be expressed as follows :

$$Nu = \frac{d}{\delta_*} \qquad \text{for } Ra_I \to \infty \tag{42}$$

where δ_* is the conduction thickness. This may be replaced by $\delta_{T,c}$ following Howard's concept. $\delta_{T,c}$ is thermal penetration depth at the onset condition of buoyancy-driven convection. Thus, by using the relation of $Ra_I = RaNu$ equation (42) can be expressed as

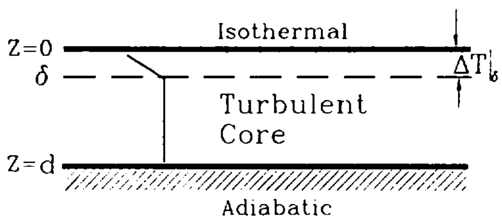

Fig. 4. Schematic diagram of turbulent heat transport model.

$$Nu = \left(\frac{Ra_I}{Ra_{\delta_{T,c}}}\right)^{1/4} \qquad \text{for } Ra_I \to \infty \tag{43}$$

where $Ra_{\delta_{T,c}}$ is represented by

$$Ra_{\delta_{T,c}} = \frac{g\beta\delta_{T,c}^3 \Delta T|_\delta}{\alpha\nu} \tag{44}$$

$\Delta T|_\delta$ is the temperature difference across the boundary layer and can be expressed as

$$\Delta T|_\delta = \frac{S\tau}{k} \tag{45}$$

From the equations (44) and (45) $Ra_{\delta_{T,c}}$ can be substituted by Ra^*. Then the heat transport in the fully-developed state is governed by

$$Nu = \frac{0.1752 Ra_I^{1/4}}{(1+0.41/Pr)^{1/4}} \qquad \text{for } Ra_I \to \infty \tag{46}$$

Long (1976) and Cheung (1980) proposed the backbone equations to predict the heat transport for the horizontal fluid layer heated internally or from below. By modifying the Long and Cheung's results, a new backbone equation to govern the buoyancy-driven heat transport in the present system can be obtained as follows:

$$Nu = 2 + \frac{A\left(Ra_I^{1/4} - Ra_{I,c}^{1/4}\right)}{1 - BRa_I^{-1/12}} \tag{47}$$

where A and B are the undetermined constants. $Ra_{I,c}$ is the minimum value of Ra_I to mark the onset of buoyancy-driven convection, of which value is 2772.

The finite-amplitude heat transfer characteristics slightly over $Ra_{I,c}$ can be obtained by using the shape assumption of Stuart (1964). For the region of $Ra_I \to Ra_{I,c}$, Roberts (1967) expressed the Nusselt number as

$$\frac{2}{Nu} = 1 - \frac{\Gamma}{Ra_I}\left(Ra_I - Ra_{I,c}\right) \tag{48}$$

The constant Γ is obtained from the distribution of disturbance quantities at $Ra = Ra_c$:

$$\Gamma = \frac{2\int_0^1 z w_1\theta_1 dz \int_0^1 w_1\theta_1 dz}{\int_0^1 (w_1\theta_1)^2 dz} = 0.5994 \tag{49}$$

Thus from the equations (48) and (49), we obtain the following relation :

$$\left.\frac{dNu}{dRa_I}\right|_{Ra_I \to Ra_{I,c}} = \frac{2\Gamma}{Ra_{I,c}} \tag{50}$$

Assembling the equations (46), (47) and (50), we can

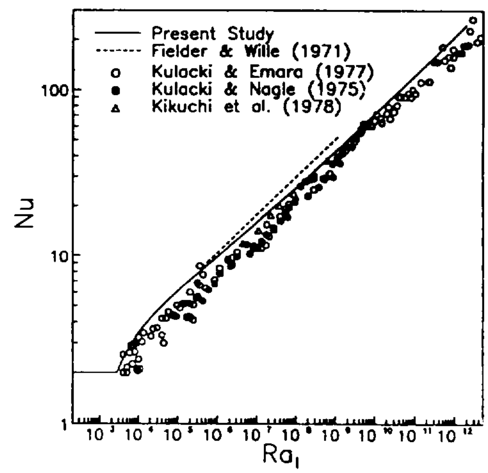

Fig. 5. Comparison between present heat transport correlation and experimenta results.

derive a new heat transfer correlation for the whole range of Ra_I as

$$Nu = 2 + \frac{A\left(Ra_I^{1/4} - 2772^{1/4}\right)}{1 - BRa_I^{-1/12}} \quad \text{for } Ra_I \geq 2772 \qquad (51)$$

where $A = 0.1752\left(1+\frac{0.41}{Pr}\right)^{1/4}$ and $B = 1.9360 -0.5132/\left(1+\frac{0.41}{Pr}\right)^{1/4}$.

The experimental data of water is compared with the above correlation with Pr=7 in Fig. 5. As shown in Fig. 5, the above equation predicts the heat transport quite well.

Conclusion

The onset of regular cell-type motion in a horizontal fluid layer with uniform volumetric energy sources has been analyzed analytically by using linear stability theory. Our propagation theory predicts that the onset time of buoyancy driven motion is a function of the Rayleigh number and Prandtl number. Also, based on the boundary-layer instability model, heat transfer characteristics of the layer are predicted as a function of the Rayleigh number and Prandtl number. These results show that the propagation theory we have developed is a powerful tool in analyzing buoyancy-driven phenomena.

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〈국문초록〉

내부열원에 의해 가열되는 수평 유체층에서의 대류 불안전성 발생 및 열전달 상관관계

밑면이 단열이고 윗면이 등온이 유지되고 있는 유체층이 내부 열원에 의해 가열되고 있는 경우, 수평 유체층에서의 부력효과에 대한 연구가 진행되었다. 자연대류의 발생시점은 본 연구진에 의하여 개발된 전파이론을 적용하여 해석하였고, 자연대류 발생시점과 완전히 전개된 난류상태의 열전달과의 연관성을 살펴보았다. Deep-pool계에서 자연대류 발생을 나타내는 임계시간은 Prandtl수가 감소함에 따라 증가하는 것으로 나타났다. 자연대류 발생에 대한 안정성 해석결과를 바타으로 하여 새로운 Nusselt 수 상관관계를 Rayleigh수와 Prandtl수의 함수로 구하였다. 본 연구에서 구한 열전달 상관식은 기존의 물에 대한 실험결과를 잘 설명하여 주었다.