博士學位論文

Liouville type theorem for $(\mathcal{F}, \mathcal{F}')_p$-harmonic maps on foliations

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Liouville type theorem for $(\mathcal{F}, \mathcal{F}')_p$-harmonic maps on foliations

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Contents

⟨**Abstract**⟩

Liouville Type Theorem for $(\mathcal{F}, \mathcal{F}')_p$-Harmonic Maps on Foliations

In this thesis, we introduce the concept of $(\mathcal{F}, \mathcal{F}')_p$-harmonic maps on foliated Riemannian manifolds. Furthermore, the first and second variational formulas for $(\mathcal{F}, \mathcal{F}')_p$-harmonic map are investigated explicitly according to the transversal p-energy. Simultaneously, the generalized Weitzenböck type formula is given and the Liouville type theorem for $(\mathcal{F}, \mathcal{F}')_p$-harmonic map is illustrated precisely.

1 Introduction

Let (M,g) and (M',g') be Riemannian manifolds and let $\phi:(M,g)\to(M',g')$ be a smooth map. Then ϕ is said to be *harmonic* if the tension field $\tau(\phi)=\mathrm{tr}_g(\nabla d\phi)$ vanishes or ϕ is a critical point of the energy functional $E(\phi)$ which is defined by

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2\mu_M,$$

where μ_M is the volume element of M ([5]). In recent years, many geometers are interested in harmonic maps on foliated Riemannian manifolds.

Let $(M,g,\mathcal{F})$ and $(M',g',\mathcal{F}')$ be foliated Riemannian manifolds and let $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ be a smooth foliated map, i.e., ϕ is a smooth leaf-preserving map. Then ϕ is said to be *transversally harmonic* if the transversal tension field $\tau_b(\phi)$ vanishes, where $\tau_b(\phi)=\mathrm{tr}_Q(\nabla_{\mathrm{tr}}d_T\phi)$, $d_T\phi=d\phi|_Q$ and Q is the normal bundle of $\mathcal{F}$. A transversally harmonic map was introduced by J. Konderak and R. Wolak ([12]) and the properties of such maps were considered in ([3,12,13,18]). It is well known that the transversally harmonic map ϕ is not a critical point of the transversal energy $E_B(\phi)$ which is defined by ([10])

$$E_B(\phi)=\frac{1}{2}\int_M |d_T\phi|^2\mu_M.$$

So S. Dragomir and A. Tommasoli ([4]) defined a new harmonic map, called $(\mathcal{F},\mathcal{F}')$-*harmonic map*, which is a critical point of the transversal energy $E_B(\phi)$. But two definitions are equivalent when $\mathcal{F}$ is minimal. In this thesis, we study $(\mathcal{F},\mathcal{F}')_p$-harmonic maps which are generalizations of $(\mathcal{F},\mathcal{F}')$-harmonic maps. In fact, a smooth foliated map ϕ is said to be $(\mathcal{F}, \mathcal{F}')_p$-*harmonic* if ϕ is a critical point of the *transversal p-energy* $E_{B,p}(\phi)$ which is defined by

$$E_{B,p}(\phi) = \frac{1}{p}\int_M |d_T\phi|^p \mu_M.$$

Trivially, $(\mathcal{F}, \mathcal{F}')_p$-harmonic maps are p-harmonic maps for point foliations and $(\mathcal{F}, \mathcal{F}')_2$-harmonic map is just $(\mathcal{F}, \mathcal{F}')$-harmonic map ([4]). For p-harmonic map, see ([14,17,23]).

This thesis is organized as follows. In Chapter 2, we review some basic facts on foliated Riemannian manifolds. In Chapter 3, the first and second variational formulas for the transversal p-energy are given, respectively. At the same time, the transversally stability is considered. In Chapter 4, we investigate the generalized Weitzenböck type formula and its application. In Chapter 5, we study the Liouville type theorem for $(\mathcal{F}, \mathcal{F}')_p$-harmonic maps. The Liouville theorem states that harmonic maps are constant under some conditions. The classical Liouville theorem is that any bounded harmonic function defined on the whole plane must be constant ([16]). Many geometers discussed the Liouville type theorem on Riemannian manifolds ([8,22,26]) and on foliated Riemannian manifolds ([6,9]), respectively.

2 Basic facts on foliated Riemannian manifolds

Definition 2.1 A family $\mathcal{F} \equiv \{L_\alpha\}_{\alpha \in A}$ of connected subsets of a manifold M^{p+q} is called a p-dimensional (or codimension q) *foliation* if

(1) $M = \cup_\alpha L_\alpha$,

(2) $\alpha \neq \beta \Longrightarrow L_\alpha \cap L_\beta = \varnothing$,

(3) for any point in M, there exist a C^r–chart (local coordinate system) (φ_U, U), such that if $U \cap L_\alpha \neq \varnothing$, then $\varphi_U(U \cap L_\alpha) = A_c \cap \varphi(U)$, where

$$A_c = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q | y = \text{constant}\},$$

(4) on $U_i \cap U_j \neq \varnothing$, the coordinate change $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ has the form

$$\varphi_j \circ \varphi_i^{-1}(x, y) = (\varphi_{ij}(x, y), \gamma_{ij}(y)),$$

where $\gamma_{ij} : \mathbb{R}^q \to \mathbb{R}^q$ is a diffeomorphism.

Here (φ_U, U) is called a *distinguished* (or *foliated*) chart.

Roughly speaking, a foliation corresponds to a decomposition of a manifold into a union of connected submanifolds of dimension p called *leaves*.

Examples 2.2 (1) $M = \mathbb{R}^n$ and $L_c = \{\mathbb{R}^p \times c\}$ with $c \in \mathbb{R}^{n-p}$.

(2) $M = \mathbb{R}^2 - \{0\}$ and $L_r = \{(x, y) | x^2 + y^2 = r^2\}$.

(3) $M = \mathbb{R}^2$ and $L_a = \{(x, y) | y = x^2 + a\}$.

(4) $M = \mathbb{R}^2$ and $L_a = \{(x, y) | y = \ln|\sec x| + a\}$. Equivalently, L_a is the solution of

$\frac{dy}{dx} = \tan x$.

(5) A manifold M with Euler characteristic $\chi(M) = 0$ admits a nonzero vector field X and the integral curves of X is a 1-dimensional foliation.

(6) Consider a closed 1-form $\omega = adx + bdy, a, b \in \mathbb{R}$ on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then we obtain a family of lines which defines a foliation in T^2. In this case, each leaf is $\mathbb{R}$ ([28]).

(7) A submersion $f : M \to B$ is a map of manifolds with a surjective derivative map at every point of M. Then for $b \in B$, $L_b = f^{-1}(b)$ is a connected submanifold of M. All these submanifolds have the same dimension.

Let $(M, g, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a Riemannian metric g. Let TM be the tangent bunlde of M, L the tangent bundle of $\mathcal{F}$ and then L is the integrable subbundle of TM. i.e., $X, Y \in \Gamma L \Longrightarrow [X, Y] \in \Gamma L$. Let $Q = TM/L$ be the corresponding normal bundle of $\mathcal{F}$. Then the metric g defines a splitting σ in the exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0 \ ,$$

where $\pi : TM \to Q$ is the natural projection and $\sigma : Q \to L^\perp$ is a bundle map satisfying $\pi \circ \sigma = id$. Thus $g = g_L \oplus g_{L^\perp}$ induces a metric g_Q on Q that is

$$g_Q(s, t) = g(\sigma(s), \sigma(t)) \quad \forall s, t \in \Gamma Q.$$

So we have an identification $L^\perp$ with Q via the isometric splitting $(Q, g_Q) \cong (L^\perp, g_{L^\perp})$.

Definition 2.3 A Riemannian metric g_Q on Q of a foliation $\mathcal{F}$ is *holonomy invariant* if $\theta(X)g_Q = 0$ for any $X \in \Gamma L$, where $\theta(X)$ is the transverse Lie derivative, i.e.,

$$Xg_Q(s,t) = g_Q(\pi[X,Y_s],t) + g_Q(s,\pi[X,Y_t]), \quad \forall X \in \Gamma L, \quad \forall s,t, \in \Gamma Q,$$

where $Y_s = \sigma(s)$ for any $s \in \Gamma Q$.

Definition 2.4 A foliation $\mathcal{F}$ is *Riemannian* if there exists a holonomy invariant metric g_Q on Q. A metric g is *bundle-like* (with respect to $\mathcal{F}$) if the induced metric g_Q is holonomy invariant.

Theorem 2.5 ([24]) *Let $\mathcal{F}$ be a foliation on (M,g). Then the following conditions are equivalent.*

(a) *$\mathcal{F}$ is Riemannian and g is bundle-like.*

(b) *There exists an orthonomal adapted frame $\{E_i, E_a\}$ such that*

$$g(\nabla^M_{E_a}E_i, E_b) + g(\nabla^M_{E_b}E_i, E_a) = 0,$$

where ∇^M be the Levi-Civita connection on M.

(c) *All geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.*

Definition 2.6 ([24]) The *transverse Levi-Civita connection* ∇^Q on the normal bundle Q is defined by

$$\nabla^Q_X s = \begin{cases} \pi([X,Y_s]) & \forall X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.1}$$

where ∇^M be the Levi-Civita connection associated to the Riemannian metric g and $Y_s = \sigma(s)$.

Then the transverse Levi-Civita connection ∇^Q is metrical and torsion-free with respect to $g_Q = g_{L^\perp}$. That is, $\nabla^Q_X g_Q = 0$ for all $X \in \Gamma TM$ and

$$T^Q(Y,Z) = \nabla^Q_Y \pi(Z) - \nabla^Q_Z \pi(Y) - \pi[Y,Z] = 0$$

for any $Y, Z \in \Gamma TM$, where T^Q is the transversal torsion tensor field of ∇^Q.

Let R^Q be the transversal curvature tensor of $\nabla^Q \equiv \nabla$, which is defined by

$$R^Q(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \forall X, Y \in \Gamma TM.$$

It is trivial that $i(X)R^Q = 0$ for any $X \in \Gamma L$, where $i(X)$ is the interior product. In fact, $R^Q(X,Y)s = 0$ for any $Y \in \Gamma TM$ and $s \in \Gamma Q$ ([Proposition 3.6, 25]).

Definition 2.7 The *transversal sectional curvature* K^Q, *transversal Ricci operator* Ric^Q and *transversal scalar curvature* σ^Q with respect to ∇ are respectively, defined by

$$K^Q(s,t) = \frac{g_Q(R^Q(s,t)t,s)}{g_Q(s,s)g_Q(t,t) - g_Q(s,t)^2}, \quad \forall s, t, \in \Gamma Q$$

$$Ric^Q(s) = \sum_{a=1}^q R^Q(s, E_a)E_a, \quad \sigma^Q = \sum_{a=1}^q g_Q(Ric^Q(E_a), E_a),$$

where $\{E_a\}$ is a local orthonomal basic frame of Q.

Definition 2.8 The *mean curvature form* κ of $\mathcal{F}$ is given by

$$\kappa(X) = g_Q(\sum_{i=1}^p \pi(\nabla^M_{E_i} E_i), X), \quad \forall X \in \Gamma Q,$$

where $\{E_i\}_{i=1,\cdots,p}$ is a local orthonormal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Definition 2.9 A differential form ω is *basic* if

$$i(X)\omega = 0,\ \theta(X)\omega = 0, \quad \forall X \in \Gamma L.$$

Locally, the basic r−form ω is expressed by

$$\omega = \sum_{a_1<\cdots<a_r} \omega_{a_1\cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where $\frac{\partial \omega_{a_1\cdots a_r}}{\partial x^j} = 0$ for all $j = 1, \cdots, p$.

Let $\Omega_B^r(\mathcal{F})$ be the space of all basic r–forms. Then ([1])

$$\Omega_B^*(M) = \Omega_B^*(\mathcal{F}) \oplus \Omega_B^*(\mathcal{F})^\perp.$$

Denote ω_B by the basic part of the form ω.

Now, we define the star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ naturally associated to g_Q. The relationship between $\bar{*}$ and $*$ is characterized by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}),$$

$$*\phi = \bar{*}\phi \wedge \chi_{\mathcal{F}}$$

for $\phi \in \Omega_B^r(\mathcal{F})$, where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator ([24]). Let ν be the transversal volume form, i.e., $*\nu = \chi_{\mathcal{F}}$ and $\langle \cdot, \cdot \rangle$ be the inner product on $\Omega_B^r(\mathcal{F})$, which is defined by $\langle \phi, \psi \rangle \nu = \phi \wedge \bar{*}\psi$ for any $\phi, \psi \in \Omega_B^r(\mathcal{F})$. Then the global inner product $\langle\langle \cdot, \cdot \rangle\rangle_B$ on $\Omega_B^r(\mathcal{F})$ is given by

$$\langle\langle \phi, \psi \rangle\rangle_B = \int_M \langle \phi, \psi \rangle \mu_M$$

for any $\phi, \psi \in \Omega_B^r(\mathcal{F})$, where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form of M. With respect to this global inner product $\langle\langle \cdot, \cdot \rangle\rangle_B$, the formal adjoint operator $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$

of $d_B = d|_{\Omega_B^*(\mathcal{F})}$ is given by

$$\delta_B \phi = (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa \wedge) \bar{*} \phi.$$

Therefore, we have the following definition.

Definition 2.10 The *basic Laplacian* Δ_B acting on $\Omega_B^*(\mathcal{F})$ is given by

$$\Delta_B = d_B \delta_B + \delta_B d_B,$$

where δ_B is the formal adjoint operator of $d_B = d|_{\Omega_B^*(\mathcal{F})}$, which are locally given by

$$d_B = \sum_{a=1}^{q} \theta^a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_{a=1}^{q} i(E_a) \nabla_{E_a} + i(\kappa_B^\sharp),$$

where $\kappa_B^\sharp$ is the g_Q–dual vector of κ_B, $\{E_a\}_{a=1,\cdots,q}$ is a local orthonormal basic frame of Q and $\{\theta^a\}$ is its g_Q–dual 1–form.

Theorem 2.11 ([1]) *For a Riemannian foliation $\mathcal{F}$ on a closed manifold, κ_B is closed, i.e., $d_B \kappa_B = 0$.*

We define $\nabla_{\rm tr}^* \nabla_{\rm tr} : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$ by

$$\nabla_{\rm tr}^* \nabla_{\rm tr} = -\sum_{a=1}^{q} \nabla^2_{E_a, E_a} + \nabla_{\kappa_B^\sharp},$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in \Gamma TM$.

Proposition 2.12 ([7]) *The operator $\nabla_{\rm tr}^* \nabla_{\rm tr}$ is positive definite and formally self adjoint on the space of basic forms, i.e.,*

$$\int_M \langle \nabla_{\rm tr}^* \nabla_{\rm tr} \varphi, \psi \rangle \mu_M = \int_M \langle \nabla_{\rm tr} \varphi, \nabla_{\rm tr} \psi \rangle \mu_M,$$

where $\langle \nabla_{\rm tr} \varphi, \nabla_{\rm tr} \psi \rangle = \sum_{a=1}^{q} \langle \nabla_{E_a} \varphi, \nabla_{E_a} \psi \rangle$.

Definition 2.13 ([11]) A vector field $Y \in M$ is an *infinitesimal automorphism* of $\mathcal{F}$ if

$$[Y, Z] \in \Gamma L, \quad \forall Z \in \Gamma L.$$

Let $V(\mathcal{F})$ be the space of all infinitesimal automorphaisms and let $\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) | Y \in V(\mathcal{F})\}$. It is trivial that an element s of $\bar{V}(\mathcal{F})$ satisfies $\nabla_X s = 0$ for all $X \in \Gamma L$. Hence the metric defined by (2.4) induces an identification ([19])

$$\bar{V}(\mathcal{F}) \cong \Omega_B^1(\mathcal{F}).$$

For the later use, we recall the transversal divergence theorem on a foliated Riemannian manifold ([28]).

Theorem 2.14 **(Transversal divergence theorem)** *Let $(M, g, \mathcal{F})$ be a closed, oriented Riemannian manifold with a transversally oriented foliation $\mathcal{F}$ and a bundle-like metric g with respect to $\mathcal{F}$. Then*

$$\int_M div_\nabla(\bar{X})\mu_M = \int_M g_Q(\bar{X}, \kappa_B^\sharp)\mu_M$$

for all $X \in V(\mathcal{F})$, where $div_\nabla(\bar{X})$ denotes the transversal divergence of $\bar{X}$ with respect to the connection ∇ of (2.1).

Now we define the bundle map $A_Y : \Lambda^r Q^* \to \Lambda^r Q^*$ for any $Y \in V(\mathcal{F})$ by ([11])

$$A_Y \phi = \theta(Y)\phi - \nabla_Y \phi.$$

It is well-known ([11]) that for any $s \in \Gamma Q$

$$A_Y s = -\nabla_{Y_s} \bar{Y},$$

where Y_s is the vector field such that $\pi(Y_s) = s$. In fact, $A_Y s = \theta(Y)s - \nabla_Y s = \nabla_Y s - \nabla_{Y_s}\bar{Y} - \nabla_Y s = -\nabla_{Y_s}\bar{Y}$. Thus, A_Y depends only on $\bar{Y} = \pi(Y)$. Now, we recall the generalized Weitzenböck type formula on $\Omega_B^*(\mathcal{F})$ ([7]).

Theorem 2.15 (Generalized Weitzenböck type formula) *On a foliated Riemannian manifold $(M, g, \mathcal{F})$, we have*

$$\Delta_B \phi = \nabla_{\rm tr}^* \nabla_{\rm tr} \phi + F(\phi) + A_{\kappa_B^\sharp}\phi, \quad \phi \in \Omega_B^r(\mathcal{F}), \tag{2.2}$$

where $F(\phi) = \sum_{a,b} \theta^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi$. If ϕ is a basic 1–form, then $F(\phi)^\sharp = Ric^Q(\phi^\sharp)$.

3 Variational formulas for the transversal p-energy

Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds. Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth *foliated map*, i.e., $d\phi(L) \subset L'$. Then we define $d_T\phi : Q \to Q'$ by

$$d_T\phi := \pi' \circ d\phi \circ \sigma.$$

Then $d_T\phi$ is a section in $Q^* \otimes \phi^{-1}Q'$, where $\phi^{-1}Q'$ is the pull-back bundle on M. Let ∇^ϕ and $\tilde{\nabla}$ be the connections on $\phi^{-1}Q'$ and $Q^* \otimes \phi^{-1}Q'$, respectively.

Definition 3.1 The map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is called *transversally totally geodesic* if

$$\tilde{\nabla}_{\rm tr} d_T\phi = 0,$$

where $(\tilde{\nabla}_{\rm tr} d_T\phi)(X, Y) = (\tilde{\nabla}_X d_T\phi)(Y)$ for any $X, Y \in \Gamma Q$.

Note that if $\phi : M \to M'$ is transversally totally geodesic, then for any transversal geodesic γ in M, $\phi \circ \gamma$ is also transversal geodesic. From now on, we use ∇ instead of all induced connections if we have no confusion.

Definition 3.2 The *transversal p-tension field* $\tau_{b,p}(\phi)$ of ϕ is defined by

$$\tau_{b,p}(\phi) = {\rm tr}_Q(\nabla_{\rm tr}(|d_T\phi|^{p-2} d_T\phi)) = \sum_{a=1}^{q} (\nabla_{E_a} |d_T\phi|^{p-2} d_T\phi)(E_a),$$

where $|d_T\phi|^2 = \sum_{a=1}^{q} g_{Q'}(d_T\phi(E_a), d_T\phi(E_a))$.

From Definition 3.2, we get

$$\begin{aligned}
\tau_{b,p}(\phi) &= \sum_{a=1}^{q}(\nabla_{E_a}|d_T\phi|^{p-2}d_T\phi)(E_a)\\
&= \sum_{a=1}^{q}(\nabla_{E_a}|d_T\phi|^{p-2}d_T\phi(E_a) - |d_T\phi|^{p-2}d_T\phi(\nabla_{E_a}E_a))\\
&= \sum_{a=1}^{q}(|d_T\phi|^{p-2}\nabla_{E_a}d_T\phi(E_a) - |d_T\phi|^{p-2}d_T\phi(\nabla_{E_a}E_a) + E_a(|d_T\phi|^{p-2})d_T\phi(E_a))\\
&=|d_T\phi|^{p-2}\tau_b(\phi) + (p-2)|d_T\phi|^{p-3}d_T\phi(\mathrm{grad_Q}(|d_T\phi|))\\
&=|d_T\phi|^{p-2}\{\tau_b(\phi) + (p-2)d_T\phi(\mathrm{grad_Q}(\ln|d_T\phi|))\},
\end{aligned}$$

where $|d_T\phi| \neq 0$ and $\tau_b(\phi) = \mathrm{tr}_Q(\nabla_{\mathrm{tr}}d_T\phi)$ is the transversal tension field ([10]). It follows that $\tau_{b,2}(\phi) = \tau_b(\phi)$.

Definition 3.3 Let Ω be a compact domain of M. Then the *transversal p-energy* of ϕ on $\Omega \subset M$ is defined by

$$E_{B,p}(\phi;\Omega) = \frac{1}{p}\int_{\Omega}|d_T\phi|^p\mu_M,$$

where μ_M is the volume element of M.

Definition 3.4 The map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be $(\mathcal{F}, \mathcal{F}')_p$*-harmonic* if ϕ is a critical point of the transversal p-energy $E_{B,p}(\phi)$.

In particular, a $(\mathcal{F}, \mathcal{F}')_2$-harmonic map is called a $(\mathcal{F}, \mathcal{F}')$-harmonic map. Some properties of $(\mathcal{F}, \mathcal{F}')$-harmonic map have been discussed in ([4]). Next, we consider the first variational formula for the transversal p-energy. Let $V \in \phi^{-1}Q'$. Obviously, V may be considered as a vector field on Q' along ϕ. Then there is a 1-parameter family of foliated maps ϕ_t with $\phi_0 = \phi$ and $\frac{d\phi_t}{dt}|_{t=0} = V$. Then the family $\{\phi_t\}$ is said to be a *foliated variation* of ϕ with the *normal variation vector field* V. Then we have the following theorem.

Theorem 3.5 (The first variational formula) *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Let $\{\phi_t\}$ be a smooth foliated variation of ϕ supported in a compact domain Ω. Then*

$$\frac{d}{dt} E_{B,p}(\phi_t; \Omega)|_{t=0} = -\int_\Omega \langle V, \tilde{\tau}_{b,p}(\phi)\rangle \mu_M,$$

where $\tilde{\tau}_{b,p}(\phi) = \tau_{b,p}(\phi) - |d_T\phi|^{p-2} d_T\phi(\kappa_B^\sharp)$, $V = \frac{d\phi_t}{dt}|_{t=0}$ is the normal variation vector field of $\{\phi_t\}$ and $\langle\cdot,\cdot\rangle$ is the pull-back metric on $\phi^{-1}Q'$.

Proof. Let Ω be a compact domain of M and let $\{\phi_t\}$ be a foliated variation of ϕ supported in Ω with the normal variation vector field $V \in \phi^{-1}Q'$. Fix $x \in M$. Let $\{E_a\}$ be a local orthonormal basic frame on Q such that $(\nabla E_a)(x) = 0$. Define $\Phi : M \times (-\epsilon, \epsilon) \to M'$ by $\Phi(x, t) = \phi_t(x)$. Obviously, $d\Phi(E_a) = d_T\phi_t(E_a)$ and $d\Phi(\frac{\partial}{\partial t}) = \frac{d\phi_t}{dt}$. Moreover, we have $\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = \nabla_{\frac{\partial}{\partial t}} E_a = \nabla_{E_a} \frac{\partial}{\partial t} = 0$. Hence at x,

$$\begin{aligned}
\frac{d}{dt} E_{B,p}(\phi_t; \Omega) &= \frac{1}{p}\frac{d}{dt}\int_\Omega (\sum_{a=1}^q \langle d\Phi(E_a), d\Phi(E_a)\rangle)^{\frac{p}{2}} \mu_M \\
&= \int_\Omega \sum_{a=1}^q |d_T\Phi|^{p-2} \langle \nabla_{\frac{\partial}{\partial t}} d\Phi(E_a), d\Phi(E_a)\rangle \mu_M \\
&= \int_\Omega \sum_{a=1}^q |d_T\Phi|^{p-2} \langle \nabla_{E_a} d\Phi(\frac{\partial}{\partial t}), d\Phi(E_a)\rangle \mu_M
\end{aligned}$$

$$= \int_\Omega \sum_{a=1}^q \{E_a\langle d\Phi(\frac{\partial}{\partial t}), |d_T\Phi|^{p-2} d\Phi(E_a)\rangle - \langle d\Phi(\frac{\partial}{\partial t}), (\nabla_{E_a}|d_T\Phi|^{p-2}d\Phi)(E_a)\rangle\}\mu_M$$

$$= \int_\Omega \sum_{a=1}^q E_a\langle \frac{d\phi_t}{dt}, |d_T\phi_t|^{p-2} d_T\phi_t(E_a)\rangle\mu_M - \int_\Omega \langle \frac{d\phi_t}{dt}, \tau_{b,p}(\phi_t)\rangle\mu_M,$$

where $|d_T\Phi|^2 = \sum_{a=1}^q \langle d\Phi(E_a), d\Phi(E_a)\rangle = |d_T\phi_t|^2$.

If we choose a normal vector field X_t with

$$\langle X_t, Z\rangle = \langle \frac{d\phi_t}{dt}, |d_T\phi_t|^{p-2} d_T\phi_t(Z)\rangle$$

for any vector field Z, then

$$\mathrm{div}_\nabla(X_t) = \sum_{a=1}^q E_a\langle \frac{d\phi_t}{dt}, |d_T\phi_t|^{p-2} d_T\phi_t(E_a)\rangle.$$

So by the transversal divergence theorem (Theorem 2.14), we have

$$\begin{aligned}
\frac{d}{dt}E_{B,p}(\phi_t;\Omega) &= \int_\Omega \mathrm{div}_\nabla(X_t)\mu_M - \int_\Omega \langle \frac{d\phi_t}{dt}, \tau_{b,p}(\phi_t)\rangle\mu_M \\
&= \int_\Omega \langle X_t, \kappa_B^\sharp\rangle\mu_M - \int_\Omega \langle \frac{d\phi_t}{dt}, \tau_{b,p}(\phi_t)\rangle\mu_M \\
&= -\int_\Omega \langle \frac{d\phi_t}{dt}, \tau_{b,p}(\phi_t) - |d_T\phi_t|^{p-2} d_T\phi_t(\kappa_B^\sharp)\rangle\mu_M \\
&= -\int_\Omega \langle \frac{d\phi_t}{dt}, \tilde\tau_{b,p}(\phi_t)\rangle\mu_M,
\end{aligned}$$

which proves (3.5) by $t = 0$. □

Corollary 3.6 *The map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is a $(\mathcal{F}, \mathcal{F}')_p$-harmonic map if and only if $\tilde\tau_{b,p}(\phi) = 0$.*

Since $E_{B,2}(\phi) = E_B(\phi)$ is the transversal energy, we have the following.

Corollary 3.7 ([10]) *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Let $\{\phi_t\}$ be a smooth foliated variation of ϕ supported in a compact domain Ω. Then*

$$\frac{d}{dt} E_B(\phi_t; \Omega)|_{t=0} = -\int_\Omega \langle V, \tilde{\tau}_b(\phi)\rangle \mu_M,$$

where $\tilde{\tau}_b(\phi) = \tau_b(\phi) - d_T\phi(\kappa_B^\sharp)$, $V = \frac{d\phi_t}{dt}|_{t=0}$ is the normal variation vector field of $\{\phi_t\}$ and $\langle \cdot, \cdot \rangle$ is the pull-back metric on $\phi^{-1}Q'$.

Now, we consider the second variational formula for the transversal p-energy. Let $V, W \in \phi^{-1}Q'$. Then there exists a family of foliated maps $\phi_{t,s}(-\epsilon < s, t < \epsilon)$ satisfying

$$\begin{cases} V = \frac{\partial \phi_{t,s}}{\partial t}|_{(t,s)=(0,0)}, \\ W = \frac{\partial \phi_{t,s}}{\partial s}|_{(t,s)=(0,0)}, \\ \phi_{0,0} = \phi. \end{cases} \tag{3.1}$$

The family $\{\phi_{t,s}\}$ is said to be a *foliated variation* of ϕ with the *normal variation vector fields* V and W.

Theorem 3.8 (The second variational formula) *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a $(\mathcal{F}, \mathcal{F}')_p$-harmonic map. Then for the normal variation vector fields V and W of the foliated variation $\{\phi_{t,s}\}$,*

$$\begin{aligned} &\frac{\partial^2}{\partial t \partial s} E_{B,p}(\phi_{t,s}; \Omega)|_{(t,s)=(0,0)} \\ &\quad = \int_\Omega |d_T\phi|^{p-2} \langle \nabla_{\rm tr} V, \nabla_{\rm tr} W\rangle \mu_M - \int_\Omega |d_T\phi|^{p-2} \langle {\rm tr_Q} R^{Q'}(V, d_T\phi) d_T\phi, W\rangle \mu_M \\ &\qquad + (p-2) \int_\Omega |d_T\phi|^{p-4} \langle \nabla_{\rm tr} V, d_T\phi\rangle \langle \nabla_{\rm tr} W, d_T\phi\rangle \mu_M, \end{aligned}$$

where

$$\begin{cases} \mathrm{tr}_Q R^{Q'}(V, d_T\phi)d_T\phi = \sum_{a=1}^{q} R^{Q'}(V, d_T\phi(E_a))d_T\phi(E_a), \\ \langle \nabla_{\mathrm{tr}} V, d_T\phi \rangle = \sum_{a=1}^{q} \langle \nabla_{E_a} V, d_T\phi(E_a) \rangle. \end{cases}$$

Proof. Let $\Phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M'$ be a smooth map which is defined by $\Phi(x,t,s) = \phi_{t,s}(x)$. Then $d\Phi(E_a) = d_T\phi_{t,s}(E_a)$, $d\Phi(\frac{\partial}{\partial s}) = \frac{\partial \phi_{t,s}}{\partial s}$ and $d\Phi(\frac{\partial}{\partial t}) = \frac{\partial \phi_{t,s}}{\partial t}$. Trivially, $[X, \frac{\partial}{\partial t}] = [X, \frac{\partial}{\partial s}] = 0$ for any vector field $X \in TM$. For convenience, we put $f = |d_T\phi_{t,s}|^{p-2}$. By making use of the first variational formula, it turns out that

$$\frac{\partial}{\partial s} E_{B,p}(\phi_{t,s}; \Omega) = -\int_\Omega \langle d\Phi(\frac{\partial}{\partial s}), \tilde{\tau}_{b,p}(\phi_{t,s}) \rangle \mu_M. \tag{3.2}$$

Differentiating (3.2) with respect to t, we get

$$\frac{\partial^2}{\partial t \partial s} E_{B,p}(\phi_{t,s}; \Omega) = -\int_\Omega \langle \nabla_{\frac{\partial}{\partial t}} d\Phi(\frac{\partial}{\partial s}), \tilde{\tau}_{b,p}(\phi_{t,s}) \rangle \mu_M - \int_\Omega \langle d\Phi(\frac{\partial}{\partial s}), \nabla_{\frac{\partial}{\partial t}} \tilde{\tau}_{b,p}(\phi_{t,s}) \rangle \mu_M.$$

Since ϕ is a $(\mathcal{F}, \mathcal{F}')_p$-harmonic map, from Corollary 3.6, we have that at $(t,s) = (0,0)$,

$$\frac{\partial^2}{\partial t \partial s} E_{B,p}(\phi_{t,s}; \Omega)|_{(0,0)} = -\int_\Omega \langle W, \nabla_{\frac{\partial}{\partial t}} \tilde{\tau}_{b,p}(\phi_{t,s})|_{(0,0)} \rangle \mu_M.$$

By choosing a local orthonormal basic frame field E_a with $\nabla E_a(x) = 0$ at some point $x \in M$, we have that at x,

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial t}} \tilde{\tau}_{b,p}(\phi_{t,s}) \\ &= \nabla_{\frac{\partial}{\partial t}} \tau_{b,p}(\phi_{t,s}) - \nabla_{\frac{\partial}{\partial t}} f d\Phi(\kappa_B^\sharp) \\ &= \sum_{a=1}^{q} \nabla_{\frac{\partial}{\partial t}} \{(\nabla_{E_a} f d\Phi)(E_a)\} - f \nabla_{\kappa_B^\sharp} d\Phi(\frac{\partial}{\partial t}) - \frac{\partial f}{\partial t} d\Phi(\kappa_B^\sharp) \end{aligned}$$

$$= \sum_{a=1}^{q}\{\nabla_{E_a}\nabla_{\frac{\partial}{\partial t}} f d\Phi(E_a) + R^{\Phi}(\frac{\partial}{\partial t}, E_a) f d\Phi(E_a)\} - f\nabla_{\kappa_B^\sharp} d\Phi(\frac{\partial}{\partial t}) - \frac{\partial f}{\partial t} d\Phi(\kappa_B^\sharp)$$

$$= \sum_{a=1}^{q}\{\nabla_{E_a}\nabla_{E_a} f d\Phi(\frac{\partial}{\partial t}) + \nabla_{E_a}(\frac{\partial f}{\partial t} d\Phi(E_a) - E_a(f) d\Phi(\frac{\partial}{\partial t})) + R^{\Phi}(\frac{\partial}{\partial t}, E_a) f d\Phi(E_a)\}$$

$$- f\nabla_{\kappa_B^\sharp} d\Phi(\frac{\partial}{\partial t}) - \frac{\partial f}{\partial t} d\Phi(\kappa_B^\sharp). \tag{3.3}$$

From (3.3), we have

$$\int_\Omega \langle \nabla_{\frac{\partial}{\partial t}} \tilde{\tau}_{b,p}(\phi_{t,s}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$= \int_\Omega \sum_{a=1}^{q} \langle \nabla_{E_a}\nabla_{E_a} f d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$+ \int_\Omega \sum_{a=1}^{q} \langle R^{Q'}(d\Phi(\frac{\partial}{\partial t}), d\Phi(E_a)) f d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$+ \int_\Omega \sum_{a=1}^{q} \langle \nabla_{E_a} \frac{\partial f}{\partial t} d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M - \int_\Omega \sum_{a=1}^{q} \langle \nabla_{E_a} E_a(f) d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$- \int_\Omega \langle f\nabla_{\kappa_B^\sharp} d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M - \int_\Omega \langle \frac{\partial f}{\partial t} d\Phi(\kappa_B^\sharp), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$= - \int_\Omega \langle \nabla_{\rm tr}^* \nabla_{\rm tr} f d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M + \int_\Omega \langle \nabla_{\kappa_B^\sharp} f d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$+ \int_\Omega \sum_{a=1}^{q} \langle R^{Q'}(d\Phi(\frac{\partial}{\partial t}), d\Phi(E_a)) f d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$+ \int_\Omega \sum_{a=1}^{q} E_a \langle \frac{\partial f}{\partial t} d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M - \int_\Omega \sum_{a=1}^{q} \langle \frac{\partial f}{\partial t} d\Phi(E_a), \nabla_{E_a} d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$- \int_\Omega \sum_{a=1}^{q} E_a \langle E_a(f) d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M + \int_\Omega \sum_{a=1}^{q} \langle E_a(f) d\Phi(\frac{\partial}{\partial t}), \nabla_{E_a} d\Phi(\frac{\partial}{\partial s})\rangle \mu_M$$

$$- \int_\Omega \langle f\nabla_{\kappa_B^\sharp} d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M - \int_\Omega \langle \frac{\partial f}{\partial t} d\Phi(\kappa_B^\sharp), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M. \tag{3.4}$$

Let $X_{t,s}$ and $Y_{t,s}$ be two normal vector fields such that

$$\begin{cases} \langle X_{t,s}, Z\rangle = \langle \frac{\partial f}{\partial t} d\Phi(Z), d\Phi(\frac{\partial}{\partial s})\rangle, \\ \langle Y_{t,s}, Z\rangle = \langle Z(f) d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \end{cases}$$

for any vector field Z on M, respectively. Then

$$\begin{cases} \operatorname{div}_{\nabla}(X_{t,s}) = \sum_{a=1}^{q} E_a\langle \frac{\partial f}{\partial t} d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle, \\ \operatorname{div}_{\nabla}(Y_{t,s}) = \sum_{a=1}^{q} E_a\langle E_a(f) d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle. \end{cases} \tag{3.5}$$

By (3.5) and the transversal divergence theorem (Theorem 2.14), we have

$$\begin{aligned} &\int_\Omega \sum_{a=1}^{q} E_a\langle \frac{\partial f}{\partial t} d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M - \int_\Omega \sum_{a=1}^{q} E_a\langle E_a(f) d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M \\ &= \int_\Omega \operatorname{div}_{\nabla}(X_{t,s})\mu_M - \int_\Omega \operatorname{div}_{\nabla}(Y_{t,s})\mu_M \\ &= \int_\Omega \langle X_{t,s}, \kappa_B^\sharp\rangle \mu_M - \int_\Omega \langle Y_{t,s}, \kappa_B^\sharp\rangle \mu_M \\ &= \int_\Omega \langle \frac{\partial f}{\partial t} d\Phi(\kappa_B^\sharp), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M - \int_\Omega \langle \kappa_B^\sharp(f) d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M. \end{aligned} \tag{3.6}$$

From (3.4) and (3.6), we get

$$\begin{aligned} &\frac{\partial^2}{\partial t \partial s} E_{B,p}(\phi_{t,s};\Omega) \\ &= \int_\Omega \langle \nabla_{\rm tr}^* \nabla_{\rm tr} f d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M \\ &\quad - \int_\Omega \sum_{a=1}^{q} f\langle R^{Q'}(d\Phi(\frac{\partial}{\partial t}), d\Phi(E_a)) d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M \\ &\quad + \int_\Omega \sum_{a=1}^{q} \langle \frac{\partial f}{\partial t} d\Phi(E_a), \nabla_{E_a} d\Phi(\frac{\partial}{\partial s})\rangle \mu_M - \int_\Omega \sum_{a=1}^{q} \langle E_a(f) d\Phi(\frac{\partial}{\partial t}), \nabla_{E_a} d\Phi(\frac{\partial}{\partial s})\rangle \mu_M \\ &= \int_\Omega f\langle \nabla_{\rm tr} d\Phi(\frac{\partial}{\partial t}), \nabla_{\rm tr} d\Phi(\frac{\partial}{\partial s})\rangle \mu_M \\ &\quad - \int_\Omega \sum_{a=1}^{q} f\langle R^{Q'}(d\Phi(\frac{\partial}{\partial t}), d\Phi(E_a)) d\Phi(E_a), d\Phi(\frac{\partial}{\partial s})\rangle \mu_M \\ &\quad + \int_\Omega \sum_{a=1}^{q} \langle \frac{\partial f}{\partial t} d\Phi(E_a), \nabla_{E_a} d\Phi(\frac{\partial}{\partial s})\rangle \mu_M. \end{aligned} \tag{3.7}$$

Since

$$\frac{\partial f}{\partial t} = (p-2)|d_T\phi_{t,s}|^{p-4} \sum_{b=1}^{q} \langle \nabla_{E_b} d\Phi(\frac{\partial}{\partial t}), d_T\phi_{t,s}(E_b)\rangle, \tag{3.8}$$

the proof of Theorem 3.8 follows from (3.7) and (3.8) at $(t,s)=(0,0)$. □

Corollary 3.9 ([4]) *Let $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ be a $(\mathcal{F},\mathcal{F}')$-harmonic map. Then*

$$\frac{\partial^2}{\partial t\partial s}E_B(\phi_{t,s};\Omega)|_{(t,s)=(0,0)}=\int_\Omega\langle\nabla_{\rm tr}V,\nabla_{\rm tr}W\rangle\mu_M-\int_\Omega\langle{\rm tr_Q}R^{Q'}(V,d_T\phi)d_T\phi,W\rangle\mu_M.$$

Definition 3.10 *Let $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ be a $(\mathcal{F},\mathcal{F}')_p$-harmonic map. Then ϕ is said to be transversally stable if $I(V,V)\geq 0$, where*

$$I(V,W):=\frac{\partial^2}{\partial t\partial s}E_{B,p}(\phi_{t,s})|_{(t,s)=(0,0)}$$

for the normal variation vector fields V and W as in (3.1).

It is easy to obtain the following theorem from Theorem 3.8.

Theorem 3.11 *Let $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ be a $(\mathcal{F},\mathcal{F}')_p$-harmonic map with compact M. If the transversal sectional curvature of M' is non-positive, then ϕ is transversally stable.*

Proof. By Theorem 3.8, we have

$$\begin{aligned}I(V,V)&=\int_M|d_T\phi|^{p-2}\{|\nabla_{\rm tr}V|^2-\langle R^{Q'}(V,d_T\phi)d_T\phi,V\rangle\}\mu_M\\&\quad+(p-2)\int_M|d_T\phi|^{p-4}\langle\nabla_{\rm tr}V,d_T\phi\rangle^2\mu_M.\end{aligned}\tag{3.9}$$

Since $K^{Q'}\leq 0$, from (3.9), we get

$$\langle R^{Q'}(V,d_T\phi)d_T\phi,V\rangle=\sum_{a=1}^q\langle R^{Q'}(V,d_T\phi(E_a))d_T\phi(E_a),V\rangle=\sum_{a=1}^q K^{Q'}(V,d_T\phi(E_a))\leq 0.$$

It means that $I(V,V)\geq 0$. This completes the proof. □

Corollary 3.12 *Let* $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ *be a* $(\mathcal{F}, \mathcal{F}')$*-harmonic map with compact* M*. If the transversal sectional curvature of* M' *is non-positive, then* ϕ *is transversally stable.*

4 The generalized Weitzenböck type formula

Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds and let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Let $\Omega_B^r(E) = \Omega_B^r(\mathcal{F}) \otimes E$ be the space of E–valued basic r–forms, where $E = \phi^{-1}Q'$. Let ∇ be the induced connection on $\Omega_B^r(E)$. Then the transversal curvature tensor R on $\Omega_B^r(E)$ is given by

$$R(X,Y)(\omega \otimes s) = R^Q(X,Y)\omega \otimes s + \omega \otimes R^E(X,Y)s.$$

Now we define $d_\nabla : \Omega_B^r(E) \to \Omega_B^{r+1}(E)$ by

$$d_\nabla(\omega \otimes s) = d_B\omega \otimes s + (-1)^r \omega \wedge \nabla s,$$

and let δ_∇ be the formal adjoint of d_∇. Locally,

$$d_\nabla = \sum_{a=1}^q \theta^a \wedge \nabla_{E_a}, \quad \delta_\nabla = -\sum_{a=1}^q i(E_a)\nabla_{E_a} + i(\kappa_B^\sharp), \tag{4.1}$$

where $i(X)(\omega \otimes s) = i(X)\omega \otimes s$ for any $X \in \Gamma TM$. The Laplacian Δ on $\Omega_B^*(E)$ is defined by

$$\Delta = \delta_\nabla d_\nabla + d_\nabla \delta_\nabla.$$

Moreover, the operators A_X and $\theta(X)$ are extended to $\Omega_B^r(E)$ as

$$A_X(\omega \otimes s) = A_X\omega \otimes s,$$

$$\theta(X)(\omega \otimes s) = \theta(X)\omega \otimes s + (-1)^r \omega \wedge \nabla_X s$$

for any $X \in \Gamma TM$. Hence $\theta(X)\Psi = (d_\nabla i(X) + i(X)d_\nabla)\Psi$ for any $X \in \Gamma TM$ and $\Psi \in \Omega_B^*(E)$. Trivially, $\Psi \in \Omega_B^*(E)$ if and only if $i(X)\Psi = 0$ and $\theta(X)\Psi = 0$ for all

$X \in \Gamma L$. Then the generalized Weitzenböck type formula (2.2) is extended to $\Omega_B^*(E)$ as follows ([10]): for any $\Psi \in \Omega_B^r(E)$,

$$\Delta\Psi = \nabla_{\rm tr}^*\nabla_{\rm tr}\Psi + A_{\kappa_B^\sharp}\Psi + F(\Psi),$$

where $F(\Psi) = \sum_{a,b=1}^{q} \theta^a \wedge i(E_b)R(E_b, E_a)\Psi$. Moreover, we have

$$\frac{1}{2}\Delta_B|\Psi|^2 = \langle\Delta\Psi, \Psi\rangle - |\nabla_{\rm tr}\Psi|^2 - \langle A_{\kappa_B^\sharp}\Psi, \Psi\rangle - \langle F(\Psi), \Psi\rangle. \tag{4.2}$$

From (4.2), we get the following theorem.

Theorem 4.1 *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then the generalized Weitzenböck type formula is given by*

$$\begin{aligned}\frac{1}{2}\Delta_B|d_T\phi|^{2p-2} =& \langle\Delta|d_T\phi|^{p-2}d_T\phi, |d_T\phi|^{p-2}d_T\phi\rangle - |\nabla_{\rm tr}|d_T\phi|^{p-2}d_T\phi|^2 \\ &- \langle A_{\kappa_B^\sharp}|d_T\phi|^{p-2}d_T\phi, |d_T\phi|^{p-2}d_T\phi\rangle - |d_T\phi|^{2p-4}\langle F(d_T\phi), d_T\phi\rangle, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\langle F(d_T\phi), d_T\phi\rangle =& \sum_a g_{Q'}(d_T\phi({\rm Ric}^{\rm Q}(E_a)), d_T\phi(E_a)) \\ &- \sum_{a,b} g_{Q'}(R^{Q'}(d_T\phi(E_b), d_T\phi(E_a))d_T\phi(E_a), d_T\phi(E_b)).\end{aligned} \tag{4.4}$$

Proof. Since $|d_T\phi|^{p-2}d_T\phi \in \Omega_B^1(E)$, the proof of (4.3) follows from (4.2) directly. The equation (4.4) follows from ([Theorem 5.1, 10]). □

Lemma 4.2 *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a $(\mathcal{F}, \mathcal{F}')_p$-harmonic map. Then*

$$d_\nabla|d_T\phi|^{p-2}d_T\phi = d_B|d_T\phi|^{p-2} \wedge d_T\phi, \quad \delta_\nabla|d_T\phi|^{p-2}d_T\phi = 0.$$

Proof. Note that for any vector fields X, Y in $Q \cong L^{\perp}$, we know that

$$\nabla_X d_T\phi(Y) - \nabla_Y d_T\phi(X) = d_T\phi([X,Y]).$$

From (4.1), we have

$$(d_\nabla d_T\phi)(X,Y) = (\nabla_X d_T\phi)(Y) - (\nabla_Y d_T\phi)(X) = 0,$$

that is, $d_\nabla d_T\phi = 0$. Therefore, we have

$$d_\nabla |d_T\phi|^{p-2} d_T\phi = d_B|d_T\phi|^{p-2} \wedge d_T\phi + |d_T\phi|^{p-2} d_\nabla d_T\phi = d_B|d_T\phi|^{p-2} \wedge d_T\phi.$$

and

$$\begin{aligned}\delta_\nabla |d_T\phi|^{p-2} d_T\phi &= -\sum_{a=1}^{q} i(E_a)\nabla_{E_a}|d_T\phi|^{p-2}d_T\phi + i(\kappa_B^\sharp)|d_T\phi|^{p-2}d_T\phi \\ &= -\sum_{a=1}^{q}(\nabla_{E_a}|d_T\phi|^{p-2}d_T\phi)(E_a) + i(\kappa_B^\sharp)|d_T\phi|^{p-2}d_T\phi \\ &= -\tau_{b,p}(\phi) + |d_T\phi|^{p-2} i(\kappa_B^\sharp) d_T\phi \\ &= -\tilde{\tau}_{b,p}(\phi).\end{aligned}$$

Since ϕ is a $(\mathcal{F},\mathcal{F}')_p$-harmonic map, $\delta_\nabla |d_T\phi|^{p-2} d_T\phi = 0$ follows from Corollary 3.6. This completes the proof. □

Lemma 4.3 *Let $\phi : (M,g,\mathcal{F}) \to (M',g',\mathcal{F}')$ be a $(\mathcal{F},\mathcal{F}')_p$-harmonic map. Then*

$$\begin{aligned}&|d_T\phi|\Delta_B|d_T\phi|^{p-1} - \langle \delta_\nabla d_\nabla |d_T\phi|^{p-2} d_T\phi, d_T\phi\rangle \\ &\quad + \langle d_\nabla i(\kappa_B^\sharp) d_T\phi, |d_T\phi|^{p-2} d_T\phi\rangle - |d_T\phi|^{p-1}\kappa_B^\sharp(|d_T\phi|) \\ &\quad \le -|d_T\phi|^{p-2}\langle F(d_T\phi), d_T\phi\rangle.\end{aligned}$$

Proof. Since ϕ is a $(\mathcal{F},\mathcal{F}')_p$-harmonic map, from Theorem 4.1 and Lemma 4.2, we have

$$\begin{aligned}\frac{1}{2}\Delta_B|d_T\phi|^{2p-2} =&\langle\delta_\nabla d_\nabla|d_T\phi|^{p-2}d_T\phi,|d_T\phi|^{p-2}d_T\phi\rangle - |\nabla_{\rm tr}|d_T\phi|^{p-2}d_T\phi|^2\\ &- |d_T\phi|^{p-2}\langle d_\nabla i(\kappa_B^\sharp)d_T\phi,|d_T\phi|^{p-2}d_T\phi\rangle + |d_T\phi|^{2p-3}\kappa_B^\sharp(|d_T\phi|)\\ &- |d_T\phi|^{2p-4}\langle F(d_T\phi),d_T\phi\rangle. \end{aligned}\tag{4.5}$$

By a simple calculation, we have

$$\frac{1}{2}\Delta_B|d_T\phi|^{2p-2} = |d_T\phi|^{p-1}\Delta_B|d_T\phi|^{p-1} - |d_B|d_T\phi|^{p-1}|^2.\tag{4.6}$$

From (4.5) and (4.6), we get

$$\begin{aligned}|d_T\phi|^{p-1}\Delta_B|d_T\phi|^{p-1} =&|d_B|d_T\phi|^{p-1}|^2 - |\nabla_{\rm tr}|d_T\phi|^{p-2}d_T\phi|^2 + \langle\delta_\nabla d_\nabla|d_T\phi|^{p-2}d_T\phi,|d_T\phi|^{p-2}d_T\phi\rangle\\ &- |d_T\phi|^{p-2}\langle d_\nabla i(\kappa_B^\sharp)d_T\phi,|d_T\phi|^{p-2}d_T\phi\rangle + |d_T\phi|^{2p-3}\kappa_B^\sharp(|d_T\phi|)\\ &- |d_T\phi|^{2p-4}\langle F(d_T\phi),d_T\phi\rangle. \end{aligned}\tag{4.7}$$

By the first Kato's inequality ([2]), we have

$$|\nabla_{\rm tr}|d_T\phi|^{p-2}d_T\phi| \geq |d_B|d_T\phi|^{p-1}|.\tag{4.8}$$

Therefore, the result follows from (4.7) and (4.8). □

The following conclusion is achieved as the application of the generalized Weitzenböck type formula.

Theorem 4.4 *Let $(M,g,\mathcal{F})$ be a closed foliated Riemannian manifold of non-negative transversal Ricci curvature. Let $(M',g',\mathcal{F}')$ be a foliated Riemannian manifold of non-*

positive transversal sectional curvature. If $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is a $(\mathcal{F}, \mathcal{F}')_p$-harmonic map, then ϕ is transversally totally geodesic. Furthermore,

(1) *If the transversal Ricci curvature of $\mathcal{F}$ is positive somewhere, then ϕ is transversally constant.*

(2) *If the transversal sectional curvature of $\mathcal{F}'$ is negative, then ϕ is either transversally constant or $\phi(M)$ is a transversally geodesic closed curve.*

Proof. By the hypothesis and (4.4), we know $\langle F(d_T\phi), d_T\phi\rangle \geq 0$. Since ϕ is a $(\mathcal{F}, \mathcal{F}')_p$-harmonic map, from Lemma 4.3, we have

$$|d_T\phi|\Delta_B|d_T\phi|^{p-1} \leq \langle \delta_\nabla d_\nabla |d_T\phi|^{p-2} d_T\phi, d_T\phi\rangle - \langle d_\nabla i(\kappa_B^\sharp) d_T\phi, |d_T\phi|^{p-2} d_T\phi\rangle + |d_T\phi|^{p-1}\kappa_B^\sharp(|d_T\phi|). \tag{4.9}$$

Integrating (4.9), we have

$$\int_M \langle |d_T\phi|, \Delta_B |d_T\phi|^{p-1}\rangle \mu_M \leq \int_M \langle \delta_\nabla d_\nabla |d_T\phi|^{p-2} d_T\phi, d_T\phi\rangle \mu_M - \int_M \langle d_\nabla i(\kappa_B^\sharp) d_T\phi, |d_T\phi|^{p-2} d_T\phi\rangle \mu_M + \int_M |d_T\phi|^{p-1}\kappa_B^\sharp(|d_T\phi|)\mu_M. \tag{4.10}$$

Since $d_\nabla(d_T\phi) = 0$, we get

$$\int_M \langle \delta_\nabla d_\nabla |d_T\phi|^{p-2} d_T\phi, d_T\phi\rangle \mu_M = \int_M \langle d_\nabla |d_T\phi|^{p-2} d_T\phi, d_\nabla d_T\phi\rangle \mu_M = 0. \tag{4.11}$$

Since ϕ is a $(\mathcal{F}, \mathcal{F}')_p$-harmonic map, from Lemma 4.2, we obtain

$$\int_M \langle d_\nabla i(\kappa_B^\sharp) d_T\phi, |d_T\phi|^{p-2} d_T\phi\rangle \mu_M = \int_M \langle i(\kappa_B^\sharp) d_T\phi, \delta_\nabla |d_T\phi|^{p-2} d_T\phi\rangle \mu_M = 0. \tag{4.12}$$

Now, we choose a bundle-like metric g such that $\delta_B \kappa_B = 0$. Then we have

$$\begin{aligned}\int_M |d_T\phi|^{p-1}\kappa_B^\sharp(|d_T\phi|)\mu_M =& \frac{1}{p}\int_M \kappa_B^\sharp(|d_T\phi|^p)\mu_M \\ =& \frac{1}{p}\int_M \langle \kappa_B, d_B|d_T\phi|^p\rangle \mu_M \\ =& \frac{1}{p}\int_M \langle \delta_B\kappa_B, |d_T\phi|^p\rangle\mu_M \\ =& 0. \end{aligned} \tag{4.13}$$

From (4.10)~(4.13), we get

$$\int_M \langle |d_T\phi|, \Delta_B |d_T\phi|^{p-1}\rangle \mu_M \leq 0. \tag{4.14}$$

On the other hand, we know that

$$\begin{aligned}\int_M \langle |d_T\phi|, \Delta_B|d_T\phi|^{p-1}\rangle\mu_M &= \int_M \langle d_B|d_T\phi|, d_B|d_T\phi|^{p-1}\rangle \mu_M \\ &= (p-1)\int_M |d_T\phi|^{p-2}|d_B|d_T\phi||^2\mu_M \\ &\geq 0. \end{aligned} \tag{4.15}$$

Then from (4.14) and (4.15), we get

$$0 = \int_M \langle |d_T\phi|, \Delta_B|d_T\phi|^{p-1}\rangle\mu_M = (p-1)\int_M |d_T\phi|^{p-2}|d_B|d_T\phi||^2\mu_M, \tag{4.16}$$

which yields $d_T\phi = 0$ or $d_B|d_T\phi| = 0$. If $d_B|d_T\phi| \neq 0$, then $d_T\phi = 0$, i.e., ϕ is transversally constant. Trivially, ϕ is transversally totally geodesic. If $d_T\phi \neq 0$, then $d_B|d_T\phi| = 0$. It means that $|d_T\phi|$ is constant. From (4.7), we have

$$\begin{aligned}\langle |d_T\phi|, \Delta_B|d_T\phi|^{p-1}\rangle =& -|d_T\phi|^{p-2}|\nabla_{\rm tr} d_T\phi|^2 - \langle d_\nabla i(\kappa_B^\sharp)d_T\phi, |d_T\phi|^{p-2}d_T\phi\rangle \\ & - |d_T\phi|^{p-2}\langle F(d_T\phi), d_T\phi\rangle. \end{aligned} \tag{4.17}$$

From (4.16), (4.17) and Lemma 4.2, we get

$$
\begin{aligned}
0 &= \int_M |d_T\phi|\Delta_B|d_T\phi|^{p-1}\mu_M \\
&= -\int_M |d_T\phi|^{p-2}|\nabla_{\rm tr} d_T\phi|^2\mu_M - \int_M \langle d_\nabla i(\kappa_B^\sharp)d_T\phi, |d_T\phi|^{p-2}d_T\phi\rangle\mu_M \\
&\quad - \int_M |d_T\phi|^{p-2}\langle F(d_T\phi), d_T\phi\rangle\mu_M \\
&= -\int_M |d_T\phi|^{p-2}|\nabla_{\rm tr} d_T\phi|^2\mu_M - \int_M \langle i(\kappa_B^\sharp)d_T\phi, \delta_\nabla |d_T\phi|^{p-2}d_T\phi\rangle\mu_M \\
&\quad - \int_M |d_T\phi|^{p-2}\langle F(d_T\phi), d_T\phi\rangle\mu_M \\
&= -\int_M |d_T\phi|^{p-2}|\nabla_{\rm tr} d_T\phi|^2\mu_M - \int_M |d_T\phi|^{p-2}\langle F(d_T\phi), d_T\phi\rangle\mu_M. \qquad (4.18)
\end{aligned}
$$

Since $|\nabla_{\rm tr} d_T\phi|^2 \geq 0$ and $\langle F(d_T\phi), d_T\phi\rangle \geq 0$, from (4.18), we have

$$
|\nabla_{\rm tr} d_T\phi|^2 + \langle F(d_T\phi), d_T\phi\rangle = 0. \qquad (4.19)
$$

Thus, $\nabla_{\rm tr} d_T\phi = 0$, i.e., ϕ is transversally totally geodesic.

Furthermore, from (4.4) and (4.19), we get

$$
\begin{cases}
g_{Q'}(d_T\phi({\rm Ric}^Q(E_a)), d_T\phi(E_a)) = 0, \\
g_{Q'}(R^{Q'}(d_T\phi(E_a), d_T\phi(E_b))d_T\phi(E_a), d_T\phi(E_b)) = 0
\end{cases} \qquad (4.20)
$$

for any indices a and b. If ${\rm Ric}^Q$ is positive at some point, then $d_T\phi = 0$, i.e., ϕ is transversally constant, which proves (1). For the statement (2), if the rank of $d_T\phi \geq 2$, then there exists a point $x \in M$ such that at least two linearly independent vectors at $\phi(x)$, say, $d_T\phi(E_1)$ and $d_T\phi(E_2)$. Since the transversal sectional curvature $K^{Q'}$ of $\mathcal{F}'$ is negative,

$$
g_{Q'}(R^{Q'}(d_T\phi(E_1), d_T\phi(E_2))d_T\phi(E_2), d_T\phi(E_1)) < 0,
$$

which contradicts (4.20). Hence the rank of $d_T\phi < 2$, that is, the rank of $d_T\phi$ is zero or one everywhere. If the rank of $d_T\phi$ is zero, then ϕ is transversally constant. If the rank of $d_T\phi$ is one, then $\phi(M)$ is closed transversally geodesic. □

Corollary 4.5 *Let $(M, g, \mathcal{F})$ be a closed foliated Riemannian manifold of non-negative transversal Ricci curvature. Let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold of non-positive transversal sectional curvature. If $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is a $(\mathcal{F}, \mathcal{F}')$-harmonic map, then ϕ is transversally totally geodesic. Furthermore,*

(1) *If the transversal Ricci curvature of $\mathcal{F}$ is positive somewhere, then ϕ is transversally constant.*

(2) *If the transversal sectional curvature of $\mathcal{F}'$ is negative, then ϕ is either transversally constant or $\phi(M)$ is a transversally geodesic closed curve.*

5 Liouville type theorem for $(\mathcal{F},\mathcal{F}')_p$-harmonic maps

In this chapter, we investigate the Liouville type theorem for $(\mathcal{F},\mathcal{F}')_p$-harmonic maps on foliated Riemannian manifolds. Let μ_0 be the infimum of the eigenvalues of the basic Laplacian Δ_B acting on L^2-basic functions on M. Then the following theorem is obtained.

Theorem 5.1 *Let $(M,g,\mathcal{F})$ be a complete foliated Riemannian manifold with coclosed mean curvature form κ_B and all leaves be compact. Let $(M',g',\mathcal{F}')$ be a foliated Riemannian manifold with non-positive transversal sectional curvature $K^{Q'}$. Assume that the transversal Ricci curvature* $\mathrm{Ric}^{\mathrm{Q}}$ *of M satisfies* $\mathrm{Ric}^{\mathrm{Q}} \geq -\frac{4(p-1)}{p^2}\mu_0$ *for all $x \in M$ and* $\mathrm{Ric}^{\mathrm{Q}} > -\frac{4(p-1)}{p^2}\mu_0$ *at some point x_0. Then any $(\mathcal{F},\mathcal{F}')_p$-harmonic map $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ of $E_{B,p}(\phi)<\infty$ is transversally constant.*

Proof. Let M be a complete foliated Riemannian manifold such that $\mathrm{Ric}^{\mathrm{Q}} \geq -C$ for all x and $\mathrm{Ric}^{\mathrm{Q}} > -C$ at some point x_0, where $C = \frac{4(p-1)}{p^2}\mu_0$. Since $K^{Q'} \leq 0$ and $\mathrm{Ric}^{\mathrm{Q}} \geq -C$, from (4.4), we have

$$\langle F(d_T\phi), d_T\phi\rangle \geq \sum_{a=1}^{q} g_{Q'}(d_T\phi(\mathrm{Ric}^{\mathrm{Q}}(E_a)), d_T\phi(E_a)) \geq -C|d_T\phi|^2.$$

Since ϕ is a $(\mathcal{F},\mathcal{F}')_p$-harmonic map, from Lemma 4.3, we have

$$\begin{aligned}
&|d_T\phi|\Delta_B|d_T\phi|^{p-1} - \langle \delta_\nabla d_\nabla |d_T\phi|^{p-2}d_T\phi, d_T\phi\rangle \\
&\quad + \langle d_\nabla i(\kappa_B^\sharp)d_T\phi, |d_T\phi|^{p-2}d_T\phi\rangle - |d_T\phi|^{p-1}\kappa_B^\sharp(|d_T\phi|) \\
&\quad \leq -|d_T\phi|^{p-2}\sum_a g_{Q'}(d_T\phi(\mathrm{Ric}^{\mathrm{Q}}(E_a)), d_T\phi(E_a)) \leq C|d_T\phi|^p. \qquad (5.1)
\end{aligned}$$

Let $B_l = \{y \in M | \rho(y) \le l\}$, where $\rho(y)$ is the distance between leaves through a fixed point x_0 and y. Let ω_l be the Lipschitz continuous basic function such that

$$\begin{cases} 0 \le \omega_l(y) \le 1 \quad \text{for any } y \in M \\ \operatorname{supp} \omega_l \subset B_{2l} \\ \omega_l(y) = 1 \quad \text{for any } y \in B_l \\ \lim_{l \to \infty} \omega_l = 1 \\ |d\omega_l| \le \frac{\alpha}{l} \quad \text{almost everywhere on } M, \end{cases}$$

where α is positive constant ([26]). Therefore, $\omega_l \phi$ has compact support for any basic form $\phi \in \Omega_B^*(\mathcal{F})$. Multiplying (5.1) by ω_l^2 and integrating by parts, this yields

$$\begin{aligned} &\int_M \langle \omega_l^2 |d_T\phi|, \Delta_B |d_T\phi|^{p-1} \rangle \mu_M - \int_M \langle \omega_l^2 d_T\phi, \delta_\nabla d_\nabla |d_T\phi|^{p-2} d_T\phi \rangle \mu_M \\ &\quad + \int_M \langle d_\nabla i(\kappa_B^\sharp) d_T\phi, \omega_l^2 |d_T\phi|^{p-2} d_T\phi \rangle \mu_M - \int_M \langle \omega_l^2 |d_T\phi|^{p-1}, \kappa_B^\sharp(|d_T\phi|) \rangle \mu_M \\ &\quad \le -\sum_{a=1}^q \int_M \omega_l^2 |d_T\phi|^{p-2} g_{Q'}(d_T\phi(\mathrm{Ric}^Q(E_a)), d_T\phi(E_a)) \mu_M \\ &\quad \le C \int_M \omega_l^2 |d_T\phi|^p \mu_M. \end{aligned} \tag{5.2}$$

By Lemma 4.2, we have

$$\begin{aligned} \delta_\nabla(\omega_l^2 |d_T\phi|^{p-2} d_T\phi) &= -\sum_{a=1}^q i(E_a) \nabla_{E_a}(\omega_l^2 |d_T\phi|^{p-2} d_T\phi) + i(\kappa_B^\sharp)(\omega_l^2 |d_T\phi|^{p-2} d_T\phi) \\ &= -\sum_{a=1}^q i(E_a)(E_a(\omega_l^2)|d_T\phi|^{p-2} d_T\phi + \omega_l^2 \nabla_{E_a} |d_T\phi|^{p-2} d_T\phi) \\ &\quad + \omega_l^2 i(\kappa_B^\sharp)(|d_T\phi|^{p-2} d_T\phi) \end{aligned}$$

$$= -\sum_{a=1}^{q} i(E_a)(E_a(\omega_l^2)|d_T\phi|^{p-2}d_T\phi) - \sum_{a=1}^{q}\omega_l^2 i(E_a)\nabla_{E_a}|d_T\phi|^{p-2}d_T\phi$$

$$+ \omega_l^2 i(\kappa_B^\sharp)(|d_T\phi|^{p-2}d_T\phi)$$

$$= -i(d_B\omega_l^2)|d_T\phi|^{p-2}d_T\phi + \omega_l^2\delta_\nabla|d_T\phi|^{p-2}d_T\phi$$

$$= -2\omega_l i(d_B\omega_l)|d_T\phi|^{p-2}d_T\phi. \tag{5.3}$$

Since

$$|i(X)d_T\phi|^2 = \langle X^\flat \wedge i(X)d_T\phi, d_T\phi\rangle$$

$$= -\langle i(X)(X^\flat \wedge d_T\phi), d_T\phi\rangle + |X|^2|d_T\phi|^2$$

$$= -|X^\flat \wedge d_T\phi|^2 + |X|^2|d_T\phi|^2 \tag{5.4}$$

for any vector X, we get

$$|i(X)d_T\phi|^2 \le |X|^2|d_T\phi|^2. \tag{5.5}$$

From (5.3) and (5.5), we have

$$\left|\int_M \langle d_\nabla i(\kappa_B^\sharp)d_T\phi, \omega_l^2|d_T\phi|^{p-2}d_T\phi\rangle\mu_M\right| = \left|\int_M \langle i(\kappa_B^\sharp)d_T\phi, -2\omega_l i(d_B\omega_l)|d_T\phi|^{p-2}d_T\phi\rangle\mu_M\right|$$

$$\le 2\int_M \omega_l|i(\kappa_B^\sharp)d_T\phi||i(d_B\omega_l)|d_T\phi|^{p-2}d_T\phi|\mu_M$$

$$\le 2\int_M \omega_l|\kappa_B||d_B\omega_l||d_T\phi|^p\mu_M$$

$$\le 2\frac{\alpha}{l}\max\{|\kappa_B|\}\int_M \omega_l|d_T\phi|^p\mu_M.$$

If we let $l \to \infty$, then

$$\lim_{l\to\infty}\int_M \langle d_\nabla i(\kappa_B^\sharp)d_T\phi, \omega_l^2|d_T\phi|^{p-2}d_T\phi\rangle\mu_M = 0. \tag{5.6}$$

At the same time, we have

$$
\begin{aligned}
&\int_M \langle \omega_l^2 |d_T\phi|^{p-1}, \kappa_B^\sharp(|d_T\phi|)\rangle \mu_M \\
&\quad = \frac{1}{p}\{\int_M \kappa_B^\sharp(\omega_l^2|d_T\phi|^p)\mu_M - \int_M 2\omega_l|d_T\phi|^p\langle\kappa_B, d_B\omega_l\rangle\mu_M\} \\
&\quad = \frac{1}{p}\{\int_M \langle\kappa_B, d_B(\omega_l^2|d_T\phi|^p)\rangle\mu_M - \int_M 2\omega_l|d_T\phi|^p\langle\kappa_B, d_B\omega_l\rangle\mu_M\} \\
&\quad = \frac{1}{p}\{\int_M \langle\delta_B\kappa_B, \omega_l^2|d_T\phi|^p\rangle\mu_M - \int_M 2\omega_l|d_T\phi|^p\langle\kappa_B, d_B\omega_l\rangle\mu_M\} \\
&\quad = -\frac{2}{p}\int_M \omega_l|d_T\phi|^p\langle\kappa_B, d_B\omega_l\rangle\mu_M.
\end{aligned}
$$

The last equality in the above follows from $\delta_B\kappa_B = 0$. By the Cauchy-Schwarz inequality, i.e., $|\langle\kappa_B, d_B\omega_l\rangle| \le |\kappa_B||d_B\omega_l|$, we get

$$
-\frac{\alpha}{l}\max\{|\kappa_B|\}\int_M \omega_l|d_T\phi|^p\mu_M \le \int_M \omega_l|d_T\phi|^p\langle\kappa_B, d_B\omega_l\rangle\mu_M \le \frac{\alpha}{l}\max\{|\kappa_B|\}\int_M \omega_l|d_T\phi|^p\mu_M.
$$

So by letting $l \to \infty$, $\int_M \omega_l|d_T\phi|^p\langle\kappa_B, d_B\omega_l\rangle\mu_M \to 0$, which means

$$
\lim_{l\to\infty}\int_M \langle\omega_l^2|d_T\phi|^{p-1}, \kappa_B^\sharp(|d_T\phi|)\rangle\mu_M = 0. \tag{5.7}
$$

By the Cauchy-Schwarz inequality, we know that

$$
\begin{aligned}
&\int_M \langle\omega_l^2|d_T\phi|, \Delta_B|d_T\phi|^{p-1}\rangle\mu_M \\
&\quad = \int_M \langle d_B(\omega_l^2|d_T\phi|), d_B|d_T\phi|^{p-1}\rangle\mu_M \\
&\quad = \frac{A_1}{p}\int_M \omega_l^2|d_B|d_T\phi|^{\frac{p}{2}}|^2\mu_M + A_1\int_M\langle|d_T\phi|^{\frac{p}{2}}d_B\omega_l, \omega_l d_B|d_T\phi|^{\frac{p}{2}}\rangle\mu_M \\
&\quad \ge \frac{A_1}{p}\int_M \omega_l^2|d_B|d_T\phi|^{\frac{p}{2}}|^2\mu_M - A_1\int_M \omega_l|d_T\phi|^{\frac{p}{2}}|d_B\omega_l||d_B|d_T\phi|^{\frac{p}{2}}|\mu_M,
\end{aligned} \tag{5.8}
$$

where $A_1 = \frac{4(p-1)}{p}$. It is well known ([20]) that for a basic function f on M, we get from (5.4)

$$|d_\nabla(f d_T\phi)| = |d_B f \wedge d_T\phi| \le |d_B f||d_T\phi|.$$

Hence we have

$$\begin{aligned}\left|\int_M \langle \omega_l^2 d_T\phi, \delta_\nabla d_\nabla |d_T\phi|^{p-2} d_T\phi\rangle \mu_M\right| &= \left|\int_M \langle d_\nabla(\omega_l^2 d_T\phi), d_\nabla |d_T\phi|^{p-2} d_T\phi\rangle \mu_M\right| \\ &\le \int_M |d_\nabla(\omega_l^2 d_T\phi)||d_\nabla |d_T\phi|^{p-2} d_T\phi|\mu_M \\ &\le 2\int_M |\omega_l d_B\omega_l||d_B|d_T\phi|^{p-2}||d_T\phi|^2 \\ &\le A_2 \int_M \omega_l |d_B\omega_l||d_T\phi|^{\frac{p}{2}}|d_B|d_T\phi|^{\frac{p}{2}}|, \qquad (5.9)\end{aligned}$$

where $A_2 = \frac{4(p-2)}{p}$. From (5.8) and (5.9), we get

$$\begin{aligned}&\int_M \langle \omega_l^2 |d_T\phi|, \Delta_B |d_T\phi|^{p-1}\rangle \mu_M - \int_M \langle \omega_l^2 d_T\phi, \delta_\nabla d_\nabla |d_T\phi|^{p-2} d_T\phi\rangle \mu_M \\ &\ge -(A_1 + A_2)\int_M \omega_l |d_B\omega_l||d_T\phi|^{\frac{p}{2}}|d_B|d_T\phi|^{\frac{p}{2}}|\mu_M + \frac{A_1}{p}\int_M \omega_l^2 |d_B|d_T\phi|^{\frac{p}{2}}|^2\mu_M \\ &\ge (A_1 + A_2)\int_M (\varepsilon |d_B\omega_l|^2 |d_T\phi|^p + \frac{1}{\varepsilon}\omega_l^2 |d_B|d_T\phi|^{\frac{p}{2}}|^2)\mu_M + \frac{A_1}{p}\int_M \omega_l^2 |d_B|d_T\phi|^{\frac{p}{2}}|^2 \mu_M,\end{aligned} \qquad (5.10)$$

where ε is a positive constant. So by letting $l \to \infty$, from (5.2), (5.6), (5.7), (5.10) and Fatou's inequality,

$$C\int_M |d_T\phi|^p \mu_M \ge (\frac{A_1 + A_2}{\varepsilon} + \frac{A_1}{p})\int_M |d_B|d_T\phi|^{\frac{p}{2}}|^2 \mu_M.$$

Since $E_{B,p}(\phi) < \infty$, we know that $d_B|d_T\phi|^{\frac{p}{2}} \in L^2$. Hence by the Hölder inequality,

$$\int_M \omega_l |d_B\omega_l||d_T\phi|^{\frac{p}{2}}|d_B|d_T\phi|^{\frac{p}{2}}|\mu_M \le (\int_M |d_T\phi|^p |d_B\omega_l|^2 \mu_M)^{\frac{1}{2}}(\int_M \omega_l^2 |d_B|d_T\phi|^{\frac{p}{2}}|^2 \mu_M)^{\frac{1}{2}}.$$

If we let $l \to \infty$, then

$$\lim_{l\to\infty} \int_M \omega_l |d_T\phi|^{\frac{p}{2}} |d_B\omega_l| |d_B|d_T\phi|^{\frac{p}{2}}| \mu_M = 0. \tag{5.11}$$

From (5.8) and (5.11), we have

$$\lim_{l\to\infty} \int_M \langle \omega_l^2 |d_T\phi|, \Delta_B |d_T\phi|^{p-1} \rangle \mu_M \geq \frac{A_1}{p} \int_M |d_B|d_T\phi|^{\frac{p}{2}}|^2 \mu_M. \tag{5.12}$$

On the other hand, by the Rayleigh quotient theorem, we have

$$\frac{\int_M \langle d_B|d_T\phi|^{\frac{p}{2}}, d_B|d_T\phi|^{\frac{p}{2}} \rangle \mu_M}{\int_M |d_T\phi|^p \mu_M} \geq \mu_0. \tag{5.13}$$

From (5.2), (5.6), (5.7), (5.11), (5.12) and (5.13), by $l \to \infty$, we get

$$\begin{aligned} \frac{A_1}{p}\mu_0 \int_M |d_T\phi|^p \mu_M &\leq \frac{A_1}{p} \int_M |d_B|d_T\phi|^{\frac{p}{2}}|^2 \mu_M \\ &\leq -\sum_{a=1}^q \int_M |d_T\phi|^{p-2} g_{Q'}(d_T\phi(\mathrm{Ric^Q}(E_a)), d_T\phi(E_a)) \mu_M \\ &\leq C \int_M |d_T\phi|^p \mu_M. \end{aligned} \tag{5.14}$$

Since $C = \frac{4(p-1)}{p^2}\mu_0 = \frac{A_1}{p}\mu_0$, (5.14) implies that

$$\sum_{a=1}^q \int_M |d_T\phi|^{p-2} g_{Q'}(d_T\phi((\mathrm{Ric^Q} + C)(E_a)), d_T\phi(E_a)) \mu_M = 0. \tag{5.15}$$

Since $\mathrm{Ric^Q} > -C$ at some point x_0, then $d_T\phi = 0$ by (5.15). It means that ϕ is transversally constant. □

Corollary 5.2 *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold with coclosed mean curvature form κ_B and all leaves be compact. Let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold with non-positive transversal sectional curvature $K^{Q'}$. Assume that the transversal Ricci curvature $\mathrm{Ric^Q}$ of M satisfies $\mathrm{Ric^Q} \geq -\frac{4(p-1)}{p^2}\mu_0$ for* *all $x \in M$ and $\mathrm{Ric}^{\mathrm{Q}} > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0. Then any $(\mathcal{F}, \mathcal{F}')_q$-harmonic map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ with $2 \le q \le p$ of $E_{B,q}(\phi) < \infty$ is transversally constant.*

Proof. For $2 \le q \le p$, we have $\frac{4(q-1)}{q^2} \ge \frac{4(p-1)}{p^2}$. So the proof is trivial. □

The following corollary can be obtained readily when $p = 2$.

Corollary 5.3 *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold with coclosed mean curvature form κ_B and all leaves be compact. Let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold with non-positive transversal sectional curvature $K^{Q'}$. Assume that the transversal Ricci curvature $\mathrm{Ric}^{\mathrm{Q}}$ of M satisfies $\mathrm{Ric}^{\mathrm{Q}} \ge -\mu_0$ for all $x \in M$ and $\mathrm{Ric}^{\mathrm{Q}} > -\mu_0$ at some point x_0. Then any $(\mathcal{F}, \mathcal{F}')$-harmonic map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ of $E_B(\phi) < \infty$ is transversally constant.*

Remark 5.4 Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then ϕ is said to be *transversally p-harmonic* if the transversal p-tension field $\tau_{b,p}(\phi)$ of ϕ vanishes. In general, $(\mathcal{F}, \mathcal{F}')_p$-harmonic map and transversally p-harmonic map are not equivalent. However, based on Theorem 3.5, we know that $(\mathcal{F}, \mathcal{F}')_p$-harmonic map is transversally p-harmonic map if $\mathcal{F}$ is minimal. The Liouville type theorem for the transversally p-harmonic map is still open for $p > 2$. When $p = 2$, the Liouville type theorem for the transversally harmonic map is proved by X. S. Fu and S. D. Jung in ([6]).

Remark 5.5 Theorem 5.1 can be viewed as the generalization of Theorem 1.4 in ([14]) from Riemannian manifold to foliated Riemannian manifold.

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