Common Knowledge in Game Theory; A review article

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I. Introduction

Knowledge and interactive knowledge are crucial elements in analyzing some economic phenomenons. Given some competitive situations in economic environment, the rational agents should reflect what the others know, and what the others know what they know, and so on, before finally choosing how to act. People in financial markets usually act on the basis of incomplete information they have. One anticipating that the stock price will go up may participate in the stock market as a buyer. Before actually buying the promising stock, he should consider the other seller's information that the stock price will go down. Having taking this into account, the prospective buyer must ask whether she buy or not.

Game theory is mathematics of competition and cooperation. It studies the strategic interaction between players in competitive situations. To model the strategic interaction between players, we should be interested in the player's knowledge about another player's knowledge. Accordingly the assumptions on players' knowledge lying behind the solution concepts in Nash equilibrium and rationalizability is very important in modeling the strategic interaction between players.

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We usually say that an event is mutual knowledge in some state if in that state that event is known to each individuals. Common knowledge is more broad and deeper concept than mutual knowledge: An event is common knowledge if each individual knows it, and if each individual knows that all other individuals know it, each individual knows that all other individuals know that all the individuals know it, and so on. Aumann(1976) introduces a formal definition of common knowledge.

The most obvious event of common knowledge is the public events; Every one knows that event, each individual knows that all other individuals know that all the individuals know it, and so on. From the perspective of game theory, each individual player's action spaces are common knowledge. The rationality of the players and the structure of the game are also common knowledge.

The objective of this paper is to review the basic concepts of common knowledge in game theory and its role in predicting the outcome in non-cooperative game theory, and to teach the students the role of common knowledge underlying the solution concepts of Nash equilibrium and rationalizability. We also review the several examples in games in view of common knowledge.

II. Common Knowledge

As we have already explained, an event is common knowledge in some event if each individual not only knows the event, but also knows that all other individuals know that all the individuals know it, and so on.

To define this verbal representation of common knowledge to be in axiomatic form, we need to define both the information function and the knowledge function.

When an economic agent faces certain decision problems, he may not identify the true state w. He usually knows only that the state w is in the subset of the set of the whole state W. In the same way, we define an information function P such that P maps each state w in W into the nonempty subset P(w) of W. We usually assume that this information function has the following properties:

A. For every w in W, w is in P(w).

B. If w′ is in P(w), then P(w′) = P(w).

Above properties of the information function imply that an information function P for the whole set W of state is partitional in the sense that there is a partition of W such that for any w in *W* the set P(w) is the element of the partition containing w.

The next step we do is to define the decision maker's knowledge function for some event E (a subset of *W*) by using the information function P:

K(E) = {w in *W* : P(w) is a subset of E}

The interpretation of this is that for any event E the set K(E) is the set of all states in which the decision maker knows E. We are now in a position to define the common knowledge. To formalize the notion of common knowledge, we assume that there are N individuals in the economy.

DEFINITION 1. Let K_i be the knowledge functions of the individual i in N for the whole set *W* of states. An event E in *W* is common knowledge between individuals in the state w in *W* if w is a member of every set in the infinite sequence $K_i(E)$, $K_j(E)$, $K_i(K_j(E))$, $K_j(K_i(E))$, $K_i(\ K_j(E))$, ..., where i, j in N.

We will give an example which shows that some event E is common knowledge:

W = $\{w_1, w_2, w_3, w_4, w_5, w_6\}$. We assume that there are two individuals 1 and 2. We also denote their partitional information functions by P_1 and P_2 as follows.

$P_1 = \{\{w_1, w_2\}, \{w_3, w_4, w_5\}, \{w_6\}\}$

$P_2 = \{\{w_1\}, \{w_2\ w_3, w_4\}, \{w_5\}, \{w_6\}\}$

Then event E= $\{w_1, w_2, w_3, w_4, w_5\}$ is common knowledge because $K_1(E) = K_2(E) = E$. On the other hand, the event F = $\{w_1, w_2, w_3, w_4\}$ is not a common knowledge between individuals since $K_1(K_2(K_1(E)))$ = null set.

III. Common Knowledge in Strategic-form Game

In non-cooperative game theory, the Nash equilibrium is the strong solution concept of the given game. Common knowledge plays a crucial role in the underlying basis of the Nash equilibrium of a given game. As we have already explained, the structure of the game and the rationality of the players are common knowledge. That is, the finite set of actions of all the players and all the player's payoffs each of which is associated with some strategy profile in the game are common knowledge. The rationality of player is that given the other players' choices of actions, he maximizes his own payoff by choosing action available to him. The structure of the game is depicted in Figure 1.

Figure 1

	L	C	R
U	4, 10	3, 0	1, 4
D	0, 0	2, 9	10, 4

There are two players in the game, named player 1 and player 2 respectively. Player 1's set of actions is {U, L}. His opponent's set of actions is {L, C, R}. Each player's payoff is in the cell of the box, with player 1's payoff coming first.

We will find the equilibrium point by using the common knowledge. Player 2 is rational in the sense that given any mixed strategy of player 1 he wants to maximize his own payoff. Facing any mixed strategy of player 1, player 2 will never choose the action R. The action R is dominated by either the action L or the action C. Under the assumption of common knowledge of the game, Player 1 knows that player 2 is rational. That is, the player 1 knows that player 2 will never play the action R. In this case, player 1's best response is to choose U since U dominates D. Player 2 also knows that player 1 knows that play 2 will never play R. In this case, player 2 knows that player 1 will choose U. Given this information, the best response of player 2 is to select his action L. Consequently (U, L) is the equilibrium point. This example shows that the full force of common knowledge is not needed to solve the given game. We have used only certain level of common knowledge: Each player knows that the other player knows that he is rational. However, Common knowledge (up to infinite level of knowledge) may be needed to solve finite non-cooperative multi-person game. Anyway, above reasoning in solving the example suggest the following proposition. We will present this proposition without formal proof.

PROPOSITION 1: Given the common knowledge about the rationality of the players and the structure of the game, each player chooses an iteratively undominated strategy.

Notice that the equilibrium point (U,L) in the above example is the mutual knowledge since player 1 makes a best response U to his opponent's choice L, and vice versa for play 2. This observation implies that (U,L) is Nash equilibrium outcome. This argument yields the following proposition (Aumann and Bradenburger, 1995):

PROPOSITION 2: Suppose that each player is rational and that the strategy choices of the players are mutual knowledge. Then the choices constitute a pure-strategy Nash equilibrium.

IV. Common knowledge vs. Almost common Knowledge; An example of Rubinstein's electronic mail game

Suppose that the structure of the game is given in Figure 2.

Figure 2. The component games of electronic mail game

	A	B
A	M, M	1, −L
B	−L, 1	0, 0

G_a(probability 1−p)

	A	B
A	0, 0	1, −L
B	−L, 1	M, M

G_b(probability p)

There are two players, each of whom must choose one of the actions, A or B. Nature determines the game where players should be involved. The players are involved in game G_b with probability $p<\frac{1}{2}$; with probability 1−p they are involved in G_a. Faced with any certain game (either G_a or G_b), it is mutually beneficial for the players to coordinate their actions: In G_a the outcome (A, A) best, while in G_b the outcome (B, B) is best. The payoffs are in the cell of Figure 2, where $L>M>1$. Given uncertain situations, the Nash equilibrium of this game is that both players always choose A. The outcome (A, A) is the best one in terms of the expected payoff of each player.

Suppose player 1 is in a position to know which game is true. On the other hand, the player 2 can not obtain this information. Even if player 1 is sure which game is true, he is at the risk of choosing some actions since the choice of nature is not resolved to player 2. If we could devise some mechanisms which make the knowledge of player 1 become common knowledge between players, then both players perfectly coordinate their actions which always yield the same payoffs (M, M) to each player. If the communication means is not perfect in the sense that the means which is open to the players does not allow the game to become common knowledge, then the mathematical induction shows that the Nash equilibrium outcome is (A, A).

Following Rubinstein(1989), suppose that the players are restricted to communicate each other via computers under the following protocol. If the game is G_b, then player 1's computer automatically sends a message to player 2's computer; if the game is G_a, then no message is sent. If a computer receives any message, it sends automatically a confirmation. There is a small probability e>0 that any given message does not arrive at its intended destination. At the end of communication the number of messages received appears on each player's screen.

To formalize this situation in game theoretic world, we define the set of states to be $W = \{(Q_1, Q_2): Q_1 = Q_2, \text{ or } Q_1 = Q_2 + 1\}$. Suppose that player 1 sends q messages. Then the possible state of this case is either (q, q) or (q+1, q). The state (q, q) means that player 1 sends q messages, all of which arrive at player 2's computer, and q^{th} message sent by player 2 is lost. On the other hand, the state (q, q−1) indicates that player 1's q^{th} message goes astray. Thus, player 1's information function is defined as $P_1(q, q) = \{(q, q), (q, q-1)\}$ if $q >= 1$ and $P_1(0, 0) = \{0, 0\}$; in the same way player 2's information function is defined by $P_2(q, q) = \{ (q, q), (q + 1, q)\}$ for all q.

We are now in a position to define the following Bayesian game, referred to as the electronic mail game:

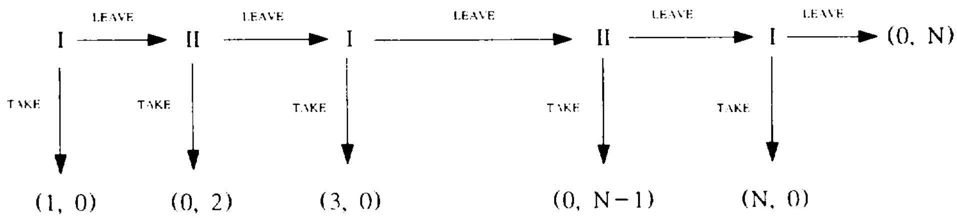
1. The set of states is $W = \{(Q_1, Q_2); Q_1 = Q_2 \text{ or } Q_1 = Q_2 + 1\}$.
2. The signal function T_i of each player I is defined by $T_i(Q_1, Q_2) = Q_i$.
3. Each player's belief (probability) on W is the same; $p_i(0, 0) = 1-p$, $p_i(q + 1, q) = pe(1-e)^{2q}$, and $p_i(q + 1, q + 1) = pe(1-e)^{2q+1}$ for any nonnegative integer q.
4. In each state (Q_1, Q_2) the payoffs are determined by the game $G(Q_1, Q_2)$.

Rubinstein(1989) showed that the electronic mail game has a unique Nash equilibrium, in which both players always choose A. This implies that even if players have almost common knowledge (by communication with very small noise in the network system), players act as if they had no information and play A. The plausible intuition on this game is that given some state, for example {q = 15, q= 15}, the players may coordinate their actions; the outcome of this game is (B, B). But the mathematical induction of this game is that the players always play the Nash equilibrium outcome (A, A). Some surprising discrepancies between the intuition and the mathematical induction arise. This implies that the mathematical induction may not be the reasoning process of human beings.

V. Common knowledge in Extensive form game

Let's consider the following extensive form game with perfect information (for more details, see Reny(1992)), in Figure 3.

Figure 3 Take it or Leave it



The game is called Take it or Leave it, or TOL for short. The game structure is as follows. A referee places one dollar on the table at the beginning of the play. If player I took the dollar, the game ends. If he leaves it, the referee places a second dollar on the table. Then player II has the chance of taking two dollars to end the game, or leaving it. The play in the game proceeds in this way. At the final step, N dollars were given such that if the player I took the N dollars, the game ends. If he leaves it, then N dollars will be retrieved to the referee.

Given the common knowledge about the structure of the game and the rationality of the players, the solution concept of sub-game perfect equilibrium implies that if the player was given any chances to take dollars, he should take the dollars. This backward induction procedure indicates that at the beginning of the play, the player I takes one dollar regardless of the value of N. Even though the logic of backward induction appears impeccable, the intuition of this game is that if N is large enough, the player I will let the play go on instead of taking one dollar to end the game.

Suppose that if player I leaves one dollar intentionally or by slipping his hand at the first stage of the game. Then player II may be in doubt that player I is a rationally maximizing agent. Under this circumstance, Player II will assess the player I's rationality in terms of probability. His assessment whether player I is rational is or not is that with probability p player I is rational, with probability $1-p$ of player II' s irrationality. Notice that p grows as the pot increases (as the game proceeds). At the second stage of the game, player II will determine whether he will take two dollars on the table or not. If he took it, the game ends so that he will get the two dollars. Thus he should compare the expected value of leaving two dollars with certain two dollars at this stage; the expected value of leaving two dollars is $4(1-p)$. This implies that if his belief (expressed in probabilistic assessment) of player I's rationality is less than ½, then he will leave the dollars. Given the third stage of player I's turn to move, player I may think that player II thinks that player I is not a maximizing agent. Player I may conclude that there are some possibilities that player II will leave the dollars.

Suppose that he assesses that probability as $(1-q)$. Then he will leave the dollars if $3<5(1-q)$; that is, if $q<2/5$, he will leave the dollars. In this way, common knowledge of the rationality of the players collapses. The game terminates at some stages. Who will be the winner or loser depends on the belief (probabilistic assessment) of the rationality of each player. Along these lines Reny(1992)proposed the following proposition:

PROPOSITION 3: If player I leaves the first dollar in TOL, then from that point on it is no longer possible for maximizing behavior to be common knowledge.

VI. Conclusion

We have reviewed the common knowledge in game theory at elementary level. Before defining the common knowledge in axiomatic form, we have specified the information functions and the knowledge functions of players. The role of common knowledge was examined by using the several examples in game theory. We have illustrated the crucial role of common knowledge in finding the Nash equilibrium outcome in strategic form game. As Rubinstein(1989) showed in electronic mail game, if players have almost common knowledge, the players act as if there no common knowledge among players. But the intuition on this kind of game is that given almost common knowledge players act as if they have common knowledge. This discrepancies implies that the mathematical induction is not the exact part of the reasoning process of human beings. In the example of the extensive form game, we have shown that the common knowledge of the maximizing behavior may not valid in solving the solution of the game.

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