A NOTE OF THE LIE DERIVATIVE

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〈國文抄錄〉

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본 論文에서는, 첫째로, 實數에서의 derivation을 定意하고, $C^{\infty}(a)$에서 R로 가는 寫像을 모아 놓은 集合을 $\mathcal{D}(a)$라 했을 때, $\mathcal{D}(a)$의 몇 가지 性質을 調査하고 $\mathcal{D}(a)$가 벡터공간(Vector Space)이 됨을 보였으며, 접공간(Tangent Space)에 대한 性質들을 調査하였다. 둘째로, X에 대한 Y의 리微分(Lie derivative) $L_X Y$가 Bracket [X, Y]와 같음을 보이고, $L_X Y$는 $L_{F_*(X)} F_*(Y)$에 F-related 됨을 보였다.

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I . INTRODUCTION.

The Theory of the derivative have been treated as an important problems in differential geometry.

In particular, it is a matter of interested to the study of the properties of the Lie derivative on C^{∞}-manifold.

The purpose of the present paper, we introduce some properties of the most basic tools used in the study of Lie derivative on C^{∞}-manifold and the bracket of C^{∞}-vector fields $\mathbf{X}$ and $\mathbf{Y}$.

In chapter II, making use of the definition of derivation $\mathbf{D}(a)$ on $C^{\infty}(a)$ in R, if D is a derivation of $\mathbf{D}(a)$, then γD is also derivation of $\mathbf{D}(a)$. Furthermore, D_1 and D_2 are derivation of $\mathbf{D}(a)$ on $C^{\infty}(a)$ into R, then $D_1 + D_2$ is a derivation of $\mathbf{D}(a)$. Thus $\mathbf{D}(a)$ is a vector space.

Let M and N be a C^{∞}-manifold. If a function F is a C^{∞}-mapping of M into N and if $F^* : C^{\infty}(F(p)) \to C^{\infty}(p)$ defined by $F^*(f) = f \circ F$ and $F_* : \mathbf{T}_p(M) \to \mathbf{T}_{F(p)}(N)$ defined by $F_*(\mathbf{X}_p)f = \mathbf{X}_p(F^* f)$, then the differential of F, F_* is homomorphism.

In chapter III, let $\theta : R \times M \to M$ be a C^{∞}-mapping satisfies any two conditions, then θ is C^{∞}-action (or one parameter group) of M.

For C^{∞}-vector field $\mathbf{X}$, there is infinitesimal generator of θ such that

$$\mathbf{X}_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f(\theta_{\Delta t}(p) - f(p)]$$

Thus the map θ_{t*} is a mapping of $\mathbf{T}(M)$ into $\mathbf{T}(M)$ defined by $\theta_{t*}(\mathbf{X}_p) = \mathbf{X}_{\theta_t(p)}$

Finally, we have proved Lie derivative of $\mathbf{Y}$ with respect to $\mathbf{X}$ such that

$(L_{\mathbf{X}}\mathbf{Y})_p = \lim_{\tau \to o} \frac{1}{t}[\theta_{-t*}(\mathbf{Y}_{\theta(t,p)}) - \mathbf{Y}_p]$ is equal to bracket $[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}$ and so Lie derivative $L_{\mathbf{X}}\mathbf{Y}$ is F-related to $L_{F_*(\mathbf{X})}F_*(\mathbf{Y})$.

Throughout the present paper, by the manifolds and vector fields we mean C^∞-manifold and C^∞ vector fields, respectively. The dimension of manifold M is n unless explicity stated otherwise.

II. DERIVATION ON C^∞-MAP

Let $\mathbf{a} = (a^1, a^2, \cdots, a^n)$ be any point of $\mathbf{R}^n$.

We define $\mathbf{T}_a(\mathbf{R}^n)$, the tangent space attached to $\mathbf{a}$, as follows. It consist of all pairs of $(a, x) = \overrightarrow{\mathbf{a}x}$ and if such a pair denoted by $\mathbf{X}_a$, there exists the mapping $\varphi_a : \mathbf{T}_a(\mathbf{R}^n) \to V^n$ is defined by $\varphi_a(\mathbf{X}_a) = (x^1 - a^1, x^2 - a^2, \cdots, x^n - a^n)$ also have the following properties:

$$(1)\ \mathbf{X}_a + \mathbf{Y}_a = \varphi_a^{-1}(\varphi_a(\mathbf{X}_a) + \varphi_a(\mathbf{Y}_a))$$
$$(2)\ \alpha\mathbf{X}_a = \varphi_a^{-1}(\alpha\varphi_a(\mathbf{X}_a))$$

for $\mathbf{X}_a$, $\mathbf{Y}_a \in \mathbf{T}_a(\mathbf{R}^n)$ and $\alpha \in \mathbf{R}$

If $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n$ be the natural basis of V^n and $E_{1a}, E_{2a}, \cdots, E_{na}$ be the natural basis of $\mathbf{T}_a(\mathbf{R}^n)$, then $E_{1a} = \varphi_a^{-1}(\mathbf{e}^1), E_{2a} = \varphi^{-1}{}_a(\mathbf{e}^2), \cdots, E_{na} = \varphi_a^{-1}(\mathbf{e}^n)$

Definition 2.1 Let $\mathbf{X}_a = \sum_{i=1}^{n} \alpha^i E_{ia}$ be the expression for a vector of $\mathbf{T}_a(\mathbf{R}^n)$. For the differential map f defined on open subset of $\mathbf{R}^n$, the *directional derivative* Δf of f at a in the "direction of $\mathbf{X}_a$" defined by

$$\Delta f = \sum_{i=1}^{n} \alpha^i \frac{\partial f}{\partial x^i}.$$

Since Δf depend on f, $\mathbf{a}$ and $\mathbf{X}_a$, we shall write it as $\mathbf{X}_a{}^* f$ Thus $\mathbf{X}_a{}^* f = \sum_{i=1}^{n} \alpha^i \left(\frac{\partial f}{\partial x^i}\right)_a$. We may take any C^∞-function defined in a neighborhood of $\mathbf{a}$. Then for each $f \in C^{\infty}(a)$, we have $\mathbf{X}_a{}^* : C^{\infty}(a) \to \mathbf{R}$ is defined by $\mathbf{X}_a{}^* = \sum_{i=1}^{n} a^i (\frac{\partial}{\partial x^i})$.

Property 2.2 If $\alpha, \beta \in \mathbf{R}$ and $f, g \in C^{\infty}(a)$, then we have two fundamental properties of derivatives followings;

$$(1)\ \mathbf{X}_a{}^*(\alpha f + \beta g) = \alpha(\mathbf{X}_a{}^* f) + \beta(\mathbf{X}_a{}^* g) \quad - \text{(linearity)}$$

$$(2)\ \mathbf{X}_a{}^*(fg) = (\mathbf{X}_a{}^* f)\, g(a) + f(a)(\mathbf{X}_a{}^* g) \quad - \text{(Leibniz rule)}$$

Let $\mathcal{D}(a)$ denote all mappings of $C^{\infty}(a)$ to $\mathbf{R}$ with linearity and Leibniz rule.

Then the elements of $\mathcal{D}(a)$ is called *derivations* on $C^{\infty}(a)$ into **R.**

Lemma 2.3 If D is a derivation of $\mathcal{D}(a)$, then γD is also derivation of $\mathcal{D}(a)$

Proof. Let $D \in \mathcal{D}(a)$, $\alpha, \beta, \gamma \in \mathbf{R}$ and $f, g \in C^{\infty}(a)$. To show the map $\gamma D : C^{\infty}(a) \to \mathbf{R}$ is linear. Using(1) of property **2.2**

$$\begin{aligned}(\gamma D)(\alpha f + \beta g) &= \gamma[D(\alpha f + \beta g)] \\ &= \gamma[(\alpha(Df) + \beta(Dg)] \\ &= \gamma\alpha(Df) + \gamma\beta(Dg) \\ &= \alpha(\gamma D)f + \beta(\gamma D)g\end{aligned}$$

By means of the property **2.2**

$$\begin{aligned}(\gamma D)(fg) &= \gamma[D(fg)] \\ &= \gamma[(Df)g(a) + f(a)(Dg)] \\ &= \gamma[(Df)g(a) + f(a)\gamma(Dg)] \\ &= ((\gamma D)f)g(a) + f(a)((\gamma D)g)\end{aligned}$$

Lemma 2.4. If D_1, D_2 are derivation of $\mathcal{D}(a)$, then $D_1 + D_2$ is a derivation of $\mathcal{D}(a)$.

Proof. Let α, β be a real numbers and let f, g be a C^{∞}-function.

Then

$$
\begin{aligned}
(D_1 + D_2)(\alpha f + \beta g) &= D_1(\alpha f + \beta g) + D_2(\alpha f + \beta g) \\
&= [\alpha(D_1 f) + \beta(D_1 g)] + [\alpha(D_2 f) + \beta(D_2 g)] \\
&= \alpha[(D_1 f) + (D_2 f)] + \beta[(D_1 g) + (D_2 g)] \\
&= \alpha(D_1 + D_2)f + \beta(D_1 + D_2)g
\end{aligned}
$$

It follows that the map $D_1 + D_2 : C^{\infty}(a) \rightarrow \mathbf{R}$ is linear

$$
\begin{aligned}
(D_1 + D_2)(fg) &= D_1(fg) + D_2(fg) \\
&= [(D_1 f)g(a) + f(a)(D_1 g)] + [(D_2 f)g(a) + f(a)(D_2 g)] \\
&= [(D_1 f)g(a) + (D_2 f)g(a)] + [f(a)(D_1 g) + f(a)(D_2 g)] \\
&= [(D_1 f) + (D_2 f)]g(a) + f(a)[(D_1 g) + (D_2 g)] \\
&= [(D_1 + D_2)f]g(a) + f(a)[(D_1 + D_2)g]
\end{aligned}
$$

Thus $D_1 + D_2$ satisfies the Leibniz rule for differentiation of products.

Therorm 2.5 $D(a)$ is a vector space.

Proof. By Lemma **2.3, 2.4,** we have the result.

Let U is an open set of manifold M. Then for any $p \in U$, $\varphi : U \rightarrow \mathbf{R}^n$ defined by $\varphi(p) = (x^1, x^2, \cdots, x^n)$ is a homeomorphism on U and the pair (U, φ) is called a *coordinate neighborhood*

Definition 2.6. Let f be a real-valued function on an open set U of a n-dimensional manifold M. Then $f : U \rightarrow \mathbf{R}$ is a C^{∞}-function if each $p \in U$ lies in a coordinate neighborhood (U, φ) such that $f \circ \varphi(x^1, x^2, \cdots, x^n)$ is a C^{∞} on $\varphi(U)$.

Definition 2.7. Let M and N be a C^{∞}-manifolds. A function F is a C^{∞}-mapping of M into N, if for every $p \in M$, there exist (U, φ) of p and (V, Ψ) of $F(p)$ with $F(U) \subset V$ such that

$$\Psi \circ F \circ \varphi^{-1}(U) : \varphi(U) \to \Psi(V)$$

is the C^∞-function in Euclidean Sense.

Furthermore, we call F homeomorphism if $\Psi \circ F \circ \varphi^{-1}$ is homeomorphism. A C^∞-mapping $F : M \to N$ between C^∞-manifolds is called a diffeomorphism if it is a homeomorphism and F and F^{-1} are C^∞-mappings.

Definition 2.8. We define the tangent space $\mathbf{T}_p(M)$ to M at p to be the set of all mapping $\mathbf{X}_p : C^\infty(p) \to \mathbf{R}$ satisfying all $\alpha, \beta \in \mathbf{R}$ and $f, g \in C^\infty(p)$ the two conditions;

(1) $\mathbf{X}_p(\alpha f + \beta g) = \alpha(\mathbf{X}_p f) + \beta(\mathbf{X}_p g)$

(2) $\mathbf{X}_p(fg) = (\mathbf{X}_p f)\, g(p) + f(p)(\mathbf{X}_p g)$

with the vector space operations in $\mathbf{T}_p(M)$ definde by

$$(\mathbf{X}_p + \mathbf{Y}_p)f = \mathbf{X}_p f + \mathbf{Y}_p f, \quad (\alpha \mathbf{X}_p)f = \alpha(\mathbf{X}_p f)$$

Any $\mathbf{X}_p \in \mathbf{T}_p(M)$ is called a tangent vector to M at p.

Let $F : M \to N$ be a C^∞-map of manifolds. Then for $p \in M$, the map $F^* : C^\infty(F(p)) \to C^\infty(p)$ defined by $F^*(f) = f \circ F$ and $F_* : \mathbf{T}_p(M) \to \mathbf{T}_{F(p)}(N)$ defined by $F_*(\mathbf{X}_p)f = \mathbf{X}_p(F^* f)$ which gives $F_*(\mathbf{X}_p)$ as a map of $C^\infty(F(p))$ to $\mathbf{R}$.

We have

Theorem 2.9. F_* is a homomorphism.

Proof. Let $\mathbf{X}_p \in \mathbf{T}_p(M)$ and $f, g \in C^\infty(F(p))$. We must prove that the map $F_*(\mathbf{X}_p) : C^\infty(F(p)) \to \mathbf{R}$ is a vector at $F(p)$, that is, a linear map

satisfying the Leibniz rule, we have

$$\begin{aligned} F_*(\mathbf{X}_p)(fg) &= \mathbf{X}_p F^*(fg) \\ &= \mathbf{X}_p[(f \circ F)(g \circ F)] \\ &= \mathbf{X}_p(f \circ F)\, g(F(p)) + f(F(p))\mathbf{X}_p(g \circ F) \\ &= \mathbf{X}_p(F^*(f))\, g(F(p)) + f(F(p))\mathbf{X}_p(F^*(g)) \\ &= (F_*(\mathbf{X}_p)f)\, g(F(p)) + f(F(p))(F_*(\mathbf{X}_p)g) \end{aligned}$$

Thus $F_* : \mathbf{T}_p(M) \to \mathbf{T}_{F(p)}(M)$.

Further F_* is a homomorphism.

$$\begin{aligned} F_*(\alpha\mathbf{X}_p + \beta\mathbf{Y}_p)f &= (\alpha\mathbf{X}_p + \beta\mathbf{Y}_p)(F \circ f) \\ &= \alpha\mathbf{X}_p(F \circ f) + \beta\mathbf{Y}_p(F \circ f) \\ &= \alpha F_*(\mathbf{X}_p)f + \beta F_*(\mathbf{Y}_p)f \\ &= [\alpha F_*(\mathbf{X}_p) + \beta F_*(\mathbf{Y}_p)]f \end{aligned}$$

Remark. The homomorphism $F_* : \mathbf{T}_p(M) \to \mathbf{T}_{F(p)}(N)$ is called the *differential* of F.

III. SOME PROPERTIES OF THE LIE DERIVATIVE OF Y

Definition 3.1. Let M be a C^∞-manifold and let $\theta : R \times M \to M$ be a C^∞-mapping which satisfies the two conditions;

(1) $\theta(0,p) = p$ for every $p \in M$

(2) $\theta_t \circ \theta_s(p) = \theta_{t+s}(p) = \theta_s \circ \theta_t(p)$ for every $s, t \in R$ and $p \in M$ where $\theta_t(p) = \theta(t, p)$

Then θ is called a C^{∞}-*action* or *one parameter group* of M.

For each one parameter group $\theta : R \times M \to M$, there exists a unique C^{∞}-vector field $\mathbf{X}$, which is called the *infinitesimal generaotr* of θ such that

$$\mathbf{X}_p f = \lim_{\Delta t \to 0} 1 \Delta t [\, f(\theta_{\Delta t}(p) - f(p) \,]$$

Threm 3.2. Let θ_{t_*} is a map $\mathbf{T}(M)$ to $\mathbf{T}(M)$. If $\theta : R \times M \to M$ is a C^{∞}-action of R. Then $\theta_{t_*}(\mathbf{X}_p) = \mathbf{X}_{\theta_t(p)}$.

Proof. Let $f \in C^{\infty}(\theta_t(p))$ for some $(t,p) \in R \times M$.

$$\theta_{t_*}(\mathbf{X}_p) f = \mathbf{X}_p(f \circ \theta_t)$$

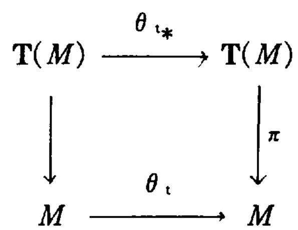
$$= \lim_{\Delta t \to 0} 1 \Delta t [\, (f \circ \theta_t)(\theta_{\Delta t}(p)) - f \circ \theta_t(p)]$$

Since $\theta_t \circ \theta_{\Delta t} = \theta_{t+\Delta t} = \theta_{\Delta t} \circ \theta_t$,

$$\theta_{t_*}(\mathbf{X}_p) f = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\, (f \circ \theta_{\Delta t})(\theta_t(p)) - f(\theta_t(p))]$$

$$= \mathbf{X}_{\theta_t(p)} f$$

Remark. For all $f \in R$, $\theta_t : M \to M$ and θ_{t_*} is a map of $\mathbf{T}(M)$ to $\mathbf{T}(M)$, then we have the following diagram which commutes



where $\pi : \mathbf{T}(M) \to M$ is the tangent vector bunble of M.

Definition 3.3. If $\mathbf{X}$ and $\mathbf{Y}$ are C^{∞}-vector fields, then the product of $\mathbf{X}$ and $\mathbf{Y}$ defined by $[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}$ is called the bracket of $\mathbf{X}$ and $\mathbf{Y}$, where $\mathbf{XY}$ is an operator on C^{∞}-function on M.

Definition 3.4. The vector field $\mathbf{L_X Y}$, called the Lie derivative of $\mathbf{Y}$ with respect to $\mathbf{X}$ is defined at each $p \in M$ by either of the following limits.

$$(\mathbf{L_X Y})_p = \lim_{t \to 0} \frac{1}{t}[\theta_{-t_*}(\mathbf{Y}_{\theta(t,p)}) - \mathbf{Y}_p]$$

$$= \lim_{t \to 0} \frac{1}{t}[\mathbf{Y}_p - \theta_{t_*}\mathbf{Y}_{\theta(-t,p)}]$$

where

$$\theta_{-t_*} : \mathbf{T}_{\theta(t,p)}(M) \to \mathbf{T}_p(M)$$

Remark. Let f be a C^{∞}-function on any open set U containing p on M, and let V be a neighborhood of p in U. Then we can take a function $g(q,t)$ defined on a $V \times I_{\delta}$ such that

$$f(\theta_t(q)) = f(q) + tg(q,t) \quad \text{and}$$

$$\mathbf{X}_p f = g(q,0) \quad \text{for} \quad q \in V$$

Theorem 3.5. If $\mathbf{X}$ and $\mathbf{Y}$ are C^{∞}-vector fields on M. Then $L_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$.

Proof. By definition of Lie derivative.

$$(L_{\mathbf{X}}\mathbf{Y})_p f = (\lim_{t \to 0} \frac{1}{t}[\mathbf{Y}_p - \theta_{t_*}(\mathbf{Y}_{\theta_{-t}(p)})])f$$

This differential quotient and that of the following expression, whoes limit is the derivaive of a C^{∞}-function of t, are equal for all $t \to 0$;

$$(L_X\mathbf{Y})_p f = \lim_{t\to 0}\frac{1}{t}[\mathbf{Y}_p f - \mathbf{Y}_{\theta_{-t}(p)}(f\circ\theta_t)]$$

Make use of the function $f(\theta_t(p)) = f(p) + tg(p, t)$ and $g(p, t)$ by g_t

$$(L_X\mathbf{Y})_p f = \lim_{t\to 0}\frac{1}{t}[\mathbf{Y}_p f - \mathbf{Y}_{\theta_{-t}(p)}(f + tg_t)]$$

Replace t by $-t$

$$\begin{aligned}(L_X\mathbf{Y})_p f &= \lim_{t\to 0} -\frac{1}{t}[\mathbf{Y}_p f - \mathbf{Y}_{\theta_t(p)}(f - tg_t)]\\ &= \lim_{t\to 0}\frac{1}{t}[\mathbf{Y}_{\theta_t(p)} f - \mathbf{Y}_p f] - \lim_{t\to 0}\mathbf{Y}_{\theta_t(p)} g(t)\\ &= \lim_{t\to 0}\frac{1}{t}[(\mathbf{Y}f)(\theta_t(p)) - (\mathbf{Y}f)(p)] - \lim_{t\to 0}\mathbf{Y}_{\theta_t(p)} g(t)\end{aligned}$$

Using the formula $g_0 = g(p, 0) = \mathbf{X}f(p)$ and the definition of the infinitesimal generator of θ

$$\begin{aligned}(L_X\mathbf{Y})_p f &= \mathbf{X}_p(\mathbf{X}f) - \mathbf{Y}_p(\mathbf{X}f)\\ &= [\mathbf{X}, \mathbf{Y}]_p f\end{aligned}$$

Corollary 3.6. If $\mathbf{X}$ and $\mathbf{Y}$ are C^∞-vector fields, then $L_X\mathbf{Y} = -L_Y\mathbf{X}$, $L_X\mathbf{X} = 0$

Proof. Since $L_X\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$ and $[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}]$,

$$L_X\mathbf{Y} = [\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}] = -L_Y\mathbf{X}$$

therefore

$$L_X\mathbf{Y} = -L_Y\mathbf{X}$$

Since $[\mathbf{X}, \mathbf{X}] = -[\mathbf{X}, \mathbf{X}]$. $[\mathbf{X}, \mathbf{X}] = 0$

therefore

$$L_X\mathbf{X} = [\mathbf{X}, \mathbf{X}] = \mathbf{0}$$

Let $F: M \rightarrow N$ be a C^{∞}-mapping and suppose that $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{Y}_1$, $\mathbf{Y}_2$ are vectpr fields om M, N, respectively. If for $i = 1, 2$ $F_*(\mathbf{X}_i) = \mathbf{Y}_i$, then $[\mathbf{X}_1, \mathbf{X}_2]$ and $[\mathbf{Y}_1, \mathbf{Y}_2]$ is called F-related.

Theorem 3.7. If $[\mathbf{X}_1, \mathbf{X}_2]$ and $[\mathbf{Y}_1, \mathbf{Y}_2]$is F-related, then $L_{\mathbf{X}}\mathbf{Y}$ is F-related to $L_{F_*(\mathbf{X})}F_*(\mathbf{Y})$.

Proof. Using the porperties of F-related, that is, $F_*[\mathbf{X}_1, \mathbf{X}_2] = [F_*(\mathbf{X}_1)F_*(\mathbf{X}_2)]$ By the theorem 3.5,

$$\begin{aligned} F_*(L_{\mathbf{X}}\mathbf{Y}) &= F_*[\mathbf{X}, \mathbf{Y}] \\ &= [F_*(\mathbf{X}), F_*(\mathbf{Y})] \\ &= L_{F_*(\mathbf{X})}F_*(\mathbf{Y}) \end{aligned}$$

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