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ON APPROXIMATE SOLUTION FOR STOCHASTIC DIFFERENTIAL INCLUSION

YONG SIK YUN AND SANG UK RYU

Department of Mathematics, Cheju National University, Korea

ABSTRACT. For the stochastic differential inclusion on infinite dimensional space of the form $dX_t \in \sigma(X_t)dW_t + b(X_t)dt$, where σ, b are set-valued maps, W is an infinite dimensional Hilbert space valued Q-Wiener process, we prove the existence of solution under the assumption that σ and b are closed convex set-valued satisfying the Lipschitz property using approximation.

1. INTRODUCTION

Let H and U be two separable Hilbert spaces and denote by $L = L(U, H)$ the set of all linear bounded operators from U into H. The set L is a linear space and, equipped with the operator norm, becomes a Banach space. However if both spaces are infinite dimensional, then L is not a separable space. Let Q be a symmetric nonnegative operator in $L(U)$ and $W(t), t \geq 0$, be a U-valued Q-Wiener process. Let $U_0 = Q^{1/2}U$ and $L_2^0 = L_2(U_0, H)$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a right-continuous increasing family $(\mathfrak{F}_t)_{t\geq 0}$ of sub $\sigma-$fields of $\mathfrak{F}$ each containing all P-null sets. We consider the following stochastic differential inclusion (1.1) on infinite dimensional Hilbert space H and our aim is to show the existence of solution.

$$dX_t \in \sigma(X_t)dW_t + b(X_t)dt, \tag{1.1}$$

where $\sigma : H \to \mathcal{P}(L_2^0)$, $b : H \to \mathcal{P}(H)$ are set-valued maps. For finite dimensional case, the study of the existence and properties of solution for these stochastic differential inclusions have been developed by many authors ([1], [4]). Furthermore the results for the viable solutions have been made ([2], [6], [7]). We had proved also the existence of solution for the stochastic differential inclusion (1.1) on finite

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dimensional space under the condition that σ and b satisfy the Lipschitz condition ([5]).

2. Preliminaries

We prepare the definition of solution for stochastic differential inclusion and some results for the stochastic differential equation on infinite dimensional Hilbert space. We consider two Hilbert spaces H and U, and a symmetric nonnegative operator $Q \in L(U)$. We consider first the case when Tr $Q < +\infty$. Then there exists a complete orthonormal system $\{e_k\}$ in U, and a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k,\ k = 1, 2, \cdots$.

Definition 2.1. An U-valued stochastic process $W(t), t \geq 0$, is called a Q-Wiener process if

(i) $W(0) = 0$,

(ii) W has continuous trajectories,

(iii) W has independent increments,

(iv) $\mathfrak{L}(W(t) - W(s)) = \mathcal{N}(0, (t-s)Q)$, the Gaussian distribution, $t \geq s \geq 0$.

If a process $W(t), t \in [0, T]$ satisfies (i) - (iii) and (iv) for $t, s \in [0, T]$, then we say that W is a Q-Wiener process on $[0, T]$. Using the Kolmogorov extension theorem, for arbitrary trace class symmetric nonnegative operator Q on a separable Hilbert space U there exists a Q-Wiener process $W(t), t \geq 0$ ([3], Proposition 4.2) .

For an $L = L(U, H)$-valued elementary processes Φ one defines the stochastic integral by the formula

$$\int_0^t \Phi(s)dW(s) = \sum_{m=0}^{k-1} \Phi_m(W_{t_{m+1}\wedge t} - W_{t_m \wedge t})$$

and denote it by $\Phi \cdot W(t), t \in [0, T]$.

It is useful, at this moment, to introduce the subspace $U_0 = Q^{1/2}(U)$ of U which, endowed with the inner product

$$< u, v >_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < u, e_k >< v, e_k > = < Q^{-1/2}u, Q^{-1/2}v >,$$

is a Hilbert space.

In the construction of the stochastic integral for more general processes an important role will be played by the space of all Hilbert-Schmidt operators $L_2^0 = L_2(U_0, H)$ from U_0 into H. The space L_2^0 is also a separable Hilbert space, equipped with the norm

$$\begin{aligned}||\Psi||^2_{L_2^0} &= \sum_{h,k=1}^{\infty} |< \Psi_{g_h}, f_k >|^2 = \sum_{h,k=1}^{\infty} \lambda_h |< \Psi_{e_h}, f_k >|^2 \\ &= ||\Psi Q^{1/2}||^2 = \text{Tr}\,[\Psi Q \Psi^*]\end{aligned}$$

where $\{g_j\}$, with $g_j = \sqrt{\lambda_j} e_j, j = 1, 2, \cdots$, $\{e_j\}$ and $\{f_j\}$ are complete orthonormal bases in U_0, U and H respectively. Clearly, $L \subset L_2^0$, but not all operators from L_2^0 can be regarded as restrictions of operators from L. The space L_2^0 contains genuinely unbounded operators on U ([3]).

Let $\Phi(t), t \in [0, T]$, be a measurable L_2^0-valued process; we define the norms

$$\begin{aligned}|||\Phi|||_t &= \{E \int_0^t ||\Phi(s)||^2_{L_2^0} ds\}^{1/2} \\ &= \{E \int_0^t \text{Tr}\,(\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^* ds\}^{1/2}, \quad t \in [0, T].\end{aligned}$$

Proposition 2.2. ([3], Proposition 4.5) If a process Φ is elementary and $|||\Phi|||_t < \infty$ then the process $\Phi \cdot W$ is a continuous, square integrable H-valued martingale on $[0, T]$ and

$$E|\Phi \cdot W(t)|^2 = |||\Phi|||_t^2, \quad 0 \le t \le T.$$

Let us consider the stochastic differential inclusion on infinite dimensional space

$$dX_t \in \sigma(X_t) dW_t + b(X_t) dt, \tag{1.1}$$

with initial value $X_0 = x$, where $\sigma : H \to \mathcal{P}(L_2^0)$, $b : H \to \mathcal{P}(H)$ are set-valued maps and x is an H-valued $\mathfrak{F}_0$-measurable random variable.

Definition 2.3. A stochastic process $X = \{X_t,\ t \in [0, T]\} \in L^p(\Omega \to C([0, T] \to H)), p \ge 2$, is said to be a solution of (1.1) on $[0, T]$ with the initial condition $X_0 = x$ if there are predictable random processes $f : \Omega \times [0, T] \to L_2^0$, $g : \Omega \times [0, T] \to H$ such that $f(t) \in \sigma(X_t)$, $g(t) \in b(X_t)$ for every $t \in [0, T]$ almost surely and

$$X_t = x + \int_0^t f(s)\, dW_s + \int_0^t g(s)\, ds,$$

where

$$L^p(\Omega \to C([0,T] \to H)) = \left\{ X \mid X \text{ is predictable, continuous, and } E\Big[\sup_{0\le s\le T} |X_s|_H^p\Big] < \infty \right\}.$$

For the stochastic differential equation

$$\text{(2.1)} \qquad \begin{cases} dX = f(t,X)dW_t + g(t,X)dt, \\ X(0) = x, \end{cases}$$

where $f : [0,T] \times H \to L_2^0$, $g : [0,T] \times H \to H$ are $\mathfrak{B}([0,T]) \otimes \mathfrak{B}(H)$-measurable and x is H-valued $\mathfrak{F}_0$-measurable, a predictable H-valued process $X(t), t \in [0,T]$, is said to be a solution of (2.1) if, for arbitrary $t \in [0,T]$,

$$X(t) = x + \int_0^t f(s, X(s))dW_s + \int_0^t g(s, X(s))ds, \quad P - \text{a.s.}$$

The following theorem is well known.

Theorem 2.4. ([3], Theorem 7.4) Assume that there exists a constant $C > 0$ such that:

(i) $\|f(t,x) - f(t,y)\|_{L_2^0} + |g(t,x) - g(t,y)| \le C|x-y|,\ x,y \in H,\ t \in [0,T]$.

(ii) $\|f(t,x)\|_{L_2^0}^2 + |g(t,x)|^2 \le C^2(1+|x|^2),\ x,y \in H,\ t \in [0,T]$.

Then we have:

(i) There exists a mild solution X to (2.1) unique, up to equivalence, among the processes satisfying

$$P\left(\int_0^T |X(s)|^2 ds\right) < +\infty.$$

Moreover it has a continuous modification.

(ii) For any $p \ge 2$ there exists a constant $C_{p,T} > 0$ such that

$$\sup_{t\in[0,T]} E[|X(t)|^p] \le C_{p,T}(1+|x|^p).$$

(iii) For any $p > 2$ there exists a constant $\hat{C}_{p,T} > 0$ such that

$$E\left[\sup_{t\in[0,T]} |X(t)|^p\right] \le \hat{C}_{p,T}(1 + |x|^p).$$

3. Main Result

For a Banach space X with the norm $||\cdot||$ and for non-empty sets A, A' in X, we denote $||A|| = \sup\{||a|| \mid a \in A\}$, $d(a, A') = \inf\{d(a, a') \mid a' \in A'\}$, $d(A, A') = \sup\{d(a, A') \mid a \in A\}$ and $d_H(A, A') = \max\{d(A, A'), d(A', A)\}$, a Hausdorff metric. We can prove the existence of solution for stochastic differential inclusion (1.1) under Lipschitz condition using approximation.

Theorem 3.1. Assume that $\sigma : H \to \mathcal{P}(L_2^0)$, $b : H \to \mathcal{P}(H)$ are closed convex set-valued which are Lipschitz, i.e., there exists constants $L > 0$ and $K > 0$ such that

$$\begin{cases} d_H(\sigma(x), \sigma(y)) \le L|x - y|, \; d_H(b(x), b(y)) \le L|x - y| \\ ||\sigma(x)|| \le K(1 + |x|), \; ||b(x)|| \le K(1 + |x|). \end{cases}$$

Then there exists a solution $X \in \Lambda^p = L^p(\Omega \to C([0,T] \to H))$ for the stochastic differential inclusion (1.1).

Proof. For arbitrary ξ_t^0 and η_t^0, define $(X_t^n), (\xi_t^n)$, and (η_t^n) as the following by induction.

$$X_t^n = x + \int_0^t \xi_s^n dW_s + \int_0^t \eta_s^n ds,$$
$$\xi_t^{n+1} = P_{\sigma(X_t^n)}\xi_t^n, \; \eta_t^{n+1} = P_{b(X_t^n)}\eta_t^n,$$

where $P_A x$ is the nearest point of A from x for closed convex set A, i.e., $|x - P_A x| = d(x, A) = \inf\{|x-y| : y \in A\}$. We claim that (X_t^n) converges and the limit becomes a solution. Since

$$\begin{aligned} ||\xi_t^{n+1} - \xi_t^n||_{L_2^0} &\le d_H(\sigma(X_t^n), \sigma(X_t^{n-1})) \\ &\le L|X_t^n - X_t^{n-1}| \\ &\le L\left|\int_0^t (\xi_s^n - \xi_s^{n-1}) dW_s + \int_0^t (\eta_s^n - \eta_s^{n-1}) ds\right|, \end{aligned}$$

we have

$$
\begin{aligned}
&E\left[\sup_{0\le s\le t}||||\xi_s^{n+1}-\xi_s^n|||_{L_2^0}||^p\right]^{1/p}\\
&\le LE\left[\sup_{0\le s\le t}\left|\int_0^s(\xi_v^n-\xi_v^{n-1})dW_v\right|^p\right]^{1/p}+LE\left[\sup_{0\le s\le t}\left|\int_0^s(\eta_v^n-\eta_v^{n-1})dv\right|^p\right]^{1/p}\\
&\le LC_1E\left[\left\{\int_0^t||\xi_s^n-\xi_s^{n-1}||_{L_2^0}^2ds\right\}^{p/2}\right]^{1/p}+LE\left[\left(\int_0^t|\eta_s^n-\eta_s^{n-1}|ds\right)^p\right]^{1/p}\\
&\quad\text{(by Burkholder's inequality)}\\
&\le LC_1\left|\left|\int_0^t||\xi_s^n-\xi_s^{n-1}||_{L_2^0}^2ds\right|\right|_{p/2}^{1/2}+L\left|\left|\int_0^t|\eta_s^n-\eta_s^{n-1}|ds\right|\right|_p\\
&\le LC_1\left\{\int_0^t||||\xi_s^n-\xi_s^{n-1}||_{L_2^0}^2||_{p/2}ds\right\}^{1/2}+L\int_0^t||\eta_s^n-\eta_s^{n-1}||_pds\\
&=LC_1\left\{\int_0^t||||\xi_s^n-\xi_s^{n-1}||_{L_2^0}||_p^2ds\right\}^{1/2}+L\int_0^t||\eta_s^n-\eta_s^{n-1}||_pds.
\end{aligned}
$$

By the same way, we have also that

$$
\begin{aligned}
&E\left[\sup_{0\le s\le t}|\eta_s^{n+1}-\eta_s^n|^p\right]^{1/p}\\
&\qquad=LC_1\left\{\int_0^t||||\xi_s^n-\xi_s^{n-1}||_{L_2^0}||_p^2ds\right\}^{1/2}+L\int_0^t||\eta_s^n-\eta_s^{n-1}||_pds,
\end{aligned}
$$

since $|\eta_t^{n+1}-\eta_t^n|\le d_H(b(X_t^n),b(X_t^{n-1}))$. Take $M>0$ be such that

$$
\frac{2LC_1}{2M+1}+\frac{2L}{M+1}\le 1,\ \ 2LC_1\sqrt{t}\le e^{Mt},\ \text{and}\ \ 2Lt\le e^{Mt}.
$$

Then, by the induction, we have that

$$
\begin{aligned}
(3.1)\qquad &\left|\left|\sup_{0\le s\le t}||\xi_s^{n+1}-\xi_s^n||_{L_2^0}\right|\right|_p\\
&\qquad\le\frac{e^{Mt}}{2^n}\left\{\sup_{0\le s\le t}||||\xi_s^1-\xi_s^0||_{L_2^0}||_p+\sup_{0\le s\le t}||\eta_s^1-\eta_s^0||_p\right\},
\end{aligned}
$$

(3.2)
$$\left\|\sup_{0\le s\le t}|\eta_s^{n+1}-\eta_s^n|\right\|_p \le \frac{e^{Mt}}{2^n}\left\{\sup_{0\le s\le t}\|\|\|\xi_s^1-\xi_s^0\|_{L_2^0}\|_p+\sup_{0\le s\le t}\|\eta_s^1-\eta_s^0\|_p\right\}.$$

Indeed, in case of $n=1$,

$$\begin{aligned}\left\|\sup_{0\le s\le t}\|\xi_s^2-\xi_s^1\|_{L_2^0}\right\|_p &\le LC_1\sqrt{t\sup_{0\le s\le t}\|\|\|\xi_s^1-\xi_s^0\|_{L_2^0}\|_p^2}+Lt\sup_{0\le s\le t}\|\eta_s^1-\eta_s^0\|_p\\ &\le \frac{e^{Mt}}{2}\left\{\sup_{0\le s\le t}\|\|\|\xi_s^1-\xi_s^0\|_{L_2^0}\|_p+\sup_{0\le s\le t}\|\eta_s^1-\eta_s^0\|_p\right\}.\\ &\left(\because LC_1\sqrt{t}\le\frac{e^{Mt}}{2},\ Lt\le\frac{e^{Mt}}{2}\right)\end{aligned}$$

For η, we can prove similarly. Assume that the above inequalities hold for $n-1$. Then

$$\begin{aligned}&\left\|\sup_{0\le s\le t}\|\xi_s^{n+1}-\xi_s^n\|_{L_2^0}\right\|_p\\ &\le LC_1\left\{\int_0^t\left(\frac{e^{Ms}}{2^{n-1}}\right)^2\phi(t)^2ds\right\}^{1/2}+L\int_0^t\frac{e^{Ms}}{s^{n-1}}\,ds\\ &=LC_1\phi(t)\frac{1}{2^{n-1}}\left\{\frac{1}{2M+1}(e^{2Mt}-1)\right\}^{1/2}+\frac{L}{2^{n-1}}\frac{1}{M+1}(e^{Mt}-1)\phi(t)\\ &\le\frac{e^{Mt}}{2^n}\phi(t)\left\{\frac{2LC_1}{2M+1}+\frac{2L}{M+1}\right\}\\ &\le\frac{e^{Mt}}{2^n}\phi(t),\end{aligned}$$

where $\phi(t)=\sup_{0\le s\le t}\|\|\|\xi_s^1-\xi_s^0\|_{L_2^0}\|_p+\sup_{0\le s\le t}\|\eta_s^1-\eta_s^0\|_p$. For η, we can prove similarly. Thus the above inequalities (3.1) and (3.2) hold for every $n=1,2,\cdots$.

Since

$$\sum_{n=0}^{\infty}\left\|\sup_{0\le s\le t}\|\xi_s^{n+1}-\xi_s^n\|_{L_2^0}\right\|_p<\infty,$$
$$\sum_{n=0}^{\infty}\left\|\sup_{0\le s\le t}|\eta_s^{n+1}-\eta_s^n|\right\|_p<\infty,$$

(ξ_t^n) and (η_t^n) are converge in L^p. Denoting the limits by (ξ_t) and (η_t), respectively, we have that

$$\lim_{n\to\infty}\left\|\sup_{0\le s\le t}\|\xi_s^n-\xi_s\|_{L_2^0}\right\|_p=0,$$

$$\lim_{n\to\infty}\left\|\sup_{0\le s\le t}|\eta_s^n-\eta_s|\right\|_p=0.$$

Putting

$$X_t=x+\int_0^t\xi_s\,dW_s+\int_0^t\eta_s\,ds,$$

we have

$$\left\|\sup_{0\le s\le t}|X_s^n-X|\right\|_p\le C_1\left\{\int_0^t\|\|\|\xi_s^n-\xi_s\|_{L_2^0}\|_p^2\,ds\right\}^{1/2}+\int_0^t\|\eta_s^n-\eta_s\|_p\,ds.$$

Letting $n\to\infty$, the right hand side tends to 0. Thus (X_s^n) converges to (X_s) in L^p. Furthermore, we have

$$\begin{aligned}d(\xi_s,\sigma(X_s))&\le\|\xi_s-\xi_s^n\|_{L_2^0}+d(\xi_s^n,\sigma(X_s))\\&\le\|\xi_s-\xi_s^n\|_{L_2^0}+d(\sigma(X_s^{n-1}),\sigma(X_s))\\&\le\|\xi_s-\xi_s^n\|_{L_2^0}+L|X_s^{n-1}-X_s|,\end{aligned}$$

and thus

$$\begin{aligned}&\left\|\sup_{0\le s\le t}d(\xi_s,\sigma(X_s))\right\|_p\\&\qquad\le\left\|\sup_{0\le s\le t}\|\xi_s-\xi_s^n\|_{L_2^0}\right\|_p+L\left\|\sup_{0\le s\le t}|X_s^{n-1}-X_s|\right\|_p.\end{aligned}$$

Since the right hand side converges to 0, $\xi_s\in\sigma(X_s)$, a.e. Similarly, we can prove that $\eta_s\in b(X_s)$, a.e. Hence (X_t) is a solution. □

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Department of Information and Mathematics, College of Natural Sciences, Cheju National University, 1 Ara1-dong, Jeju-shi, Jeju-do, 690-756, Korea

E-mail address: yunys@cheju.ac.kr and ryusu81@cheju.ac.kr

무한차원 공간에서의 확률포함방정식

윤 용 식 · 유 상 욱
제주대학교 정보수학과

집합치 함수들로 주어지는 다음과 같은 무한차원인 가분 힐버트공간에서의 확률포함방정식의 해의 존재성에 관하여 연구하였다.

$$dX_t \in \sigma(X_t)\,dW_t + b(X_t)\,dt$$

여기서 σ와 b는 집합치 함수이고 W_t는 Wiener process이다. 위와 같이 주어진 확률포함방정식의 해는 σ와 b가 폐 볼록 집합치함수이면서 동시에 Lipschitz인 가정하에서 존재한다. 존재정리는 해에 수렴하는 적당한 함수열을 찾아 근사법으로 증명되어진다. 본 논문은 존재하는 해들이 유계라는 것과 폐집합이 된다는 것 등의 여러 가지 해들의 성질을 연구하는 기초가 될 것이고 나아가 해들의 연속성을 증명하는데 이용되어질 것이라고 사료된다.