J. of Basic Sciences, Cheju Nat'l Univ.
18(2), 91~98, 2005

기초과학연구 제주대학교
18(2), 91~98, 2005

Commuting Pair-Preservers of Nonnegative Integer Matrices

Seok-Zun Song, Jin-Young Oh
Cheju National University, Cheju 690-756, South Korea

Abstract

There are many papers on linear operators that preserve commuting pairs of matrices over Boolean matrices and fuzzy matrices. They gave us the motivation to the research on commuting pair-preservers of matrices over nonnegative integers. So we characterize the linear operators that preserve commuting pairs of matrices over nonnegative integers.

2000 Mathematics Subject Classification: Primary 15A03, 15A04
Key words and phrases: Commuting pairs of matrices, Linear operator

1 Introduction

Partly because of their association with nonnegative real matrices, Boolean matrices [(0, 1)-matrices with the usual arithmetirc, except $1 + 1 = 1$] have been the subject of research by many authors. In 1982, Kim [6] published a compendium of results on the theory and applications of Boolean matrices.

Often, parallels are sought for results known for field-valued matrices, see e.g. deCaen and Gregory [2], Rao and Rao [5, 7], Richman and Schneider [11], Beasley and Pullman [9, 10].

The set of commuting pairs of matrices, $\mathcal{C}$, is the set of (unordered) pairs of matrices (X, Y) such that $XY = YX$. The linear operator T is said to *strongly preserve* $\mathcal{C}$ when $\mathrm{T}(X)\mathrm{T}(Y)=\mathrm{T}(Y)\mathrm{T}(X)$ if and only if $XY = YX$.

In 1976 Watkins [15] proved that if $n \geq 4$, $\mathcal{M}$ is the set of $n \times n$ matrices over an algebraically closed field of characteristic 0, and L is a nonsingular linear operator on $\mathcal{M}$ which preserves commuting pairs, then there exists an invertible matrix S in $\mathcal{M}$, a nonzero scalar c, and a linear functional f such that either $\mathrm{L}(X) = cSXS^{-1} + f(X)I$ or $\mathrm{L}(X) = cSX^tS^{-1} + f(X)I$, for all X in $\mathcal{M}$. In 1978, Beasley [8] extended this to the case $n = 3$. Also in [8], Beasley showed that the same characterization holds if $n \geq 3$ and L strongly preserves commuting pairs. The real symmetric and complex Hermitian cases were first investigated by Chan and Lim [3] in 1982; the same results were established as in the general case, with the exception that the invertible matrix must be orthogonal or unitary. Further extensions and generalizations to more general fields were obtained by Radjavi [4] and Choi, Jafarian, and Radjavi [11]. Song and et al obtained characterizations of the linear operators that preserve the commutativity of matrices over nonnegative reals [12] and general Boolean algebras [14].

Here we investigate the set of linear operators on $\mathcal{M}_n(\mathbb{Z}^+)$ which preserve the set of pairs of commuting matrices, where $\mathbb{Z}^+$ is the nonnegative part of the ring of integers $\mathbb{Z}$.

We obtain characterizations of linear operators that preserve commuting pairs of nonnegative integer matrices.

2 Definitions and Preliminaries

Let $\mathbb{B} = \{0, 1\}$ be the set with the two operations, addition(+) and multiplication($\cdot$) such that

(1) $0 + 0 = 0, 0 \cdot 0 = 0.$

(2) $0 + 1 = 1 + 0 = 1, 0 \cdot 1 = 1 \cdot 0 = 0.$

(3) $1 + 1 =1, 1 \cdot 1 = 1.$

Then $\mathbb{B}$ is called a *Boolean algebra.* A matrix with entries in $\mathbb{B}$ is called a *Boolean matrix.* We let $\mathcal{M}_{m,n}$ denote the set of all $m \times n$ Boolean matrices. The $n \times n$ identity matrix I_n and the $m \times n$ zero matrix $O_{m,n}$ are defined as for a field. The $m \times n$ matrix all of whose entries are zero except its (i,j)th, which is 1, is denoted $E_{i,j}$. We call $E_{i,j}$ a *cell.* We denote the $m \times n$ matrix all of whose entries are 1 by $J_{m,n}$. We omit the subscripts on I, O, and J when they are implied by the context.

EXAMPLE. If A and B are $n \times n$ Boolean matrices, then $A + I$ commutes with $B + I$ whenever A commutes with B. On the other hand, when $E_{1,1}$ does not commute with J. Therefore $X \rightarrow X + I$ preserves commuting pairs of Boolean matrices, but not strongly.

If A and B are in $\mathcal{M}(=\mathcal{M}_{m,n})$, we say B *dominates* A(written B$\geq$A or A$\leq B$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i,j. This provides a reflexive, transitive relation on $\mathcal{M}$.

Linearity of transformations is defined as for vector spaces over fields. A linear transformation on $\mathcal{M}$ is completely determined by its behavior on the set of cells. The number of nonzero entries in a matrix A is denoted $|A|$. A matrix S having at least one nonzero off-diagonal entry is a *line matrix* if all its nonzero entries lie on a line (a row or a column); so $1\leq |S| \leq n$. If the nonzero entries in S are all in a row, we call S a *row matrix* and S^t a *column matrix*. We use R_i (respectively, C_i) to denote the row matrix (respectively, column matrix), respectively with all entries in the ith row (respectively, column) equal 1. We say that cells E and F are *collinear* if there is a line matrix L such that $L \geq E + F$. When X and Y are in $\mathcal{M}$, we define $X \setminus Y$ to be the matrix Z such that $z_{i,j} = 1$ if and only if $x_{i,j} = 1$ and $y_{i,j} = 0$. For example, the matrix in $\mathcal{M}_{n,n}$ having all off-diagonal entries 1 and all diagonal entries 0 is denoted K_n. Thus, $K_n = J \setminus I$. A linear operator T on $\mathcal{M}$ is said to be *nonsingular* if T(X)=O implies that $X = O$. A nonsingular linear operator on $\mathcal{M}$ need not be invertible. If U is any matrix whose first column has all entries 1, then $X \to XU$ is nonsingular but never invertible, unless $m = n = 1$. Similarly, a matrix A is said to be *nonsingular* if $A\mathbf{x} = 0$ implies that $\mathbf{x}= 0$ ($\mathbf{x}$ a column vector). If A has a nonzero entry in each column, then A is nonsingular. Also, when $m = n$, the only invertible matrices are permutation matrices. Therefore, many nonsingular Boolean matrices are not invertible. We let $\mathcal{C}(A)$ denote the commutator semigroup of A, i.e., $\mathcal{C}(A) = \{\mathrm{X} \in \mathcal{M} \mid XA = AX\}$. Then $\mathcal{C}(J)$ consists of O and the matrices X such that both X and X^t are nonsingular. Let $\mathbb{S}$ denote the set of all symmetric matrices in $\mathcal{M}_{n,n}$. We define a *digon matrix* to be the sum of a cell and its transpose. A *star matrix* is the sum of a line matrix and its transpose. Clearly all digon matrices and all star matrices are symmetric. Let $\hat{\mathcal{C}}(J)$ denote the subsemigroup of $\mathcal{C}(J)$ which lies in $\mathbb{S}_n$, that is, $\hat{\mathcal{C}}(J)$ is the commutator of J in $\mathbb{S}$. Then $\hat{\mathcal{C}}(J)$ is the set of all symmetric nonsingular matrices together with O.

Evidently, the following operations strongly preserve the set of commuting pairs of matrices;

(a) transposition ($X \to X^t$) ;

(b) similarity ($X \to SXS^{-1}$ for some fixed invertible matrix S).

3 Linear operators that preserve commuting pairs of nonnegative integer matrices.

In this section, we characterize the linear operators that preserve commuting pairs of nonnegative integer matrices. We let $\mathcal{M}_n(\mathbb{Z}^+)$ denote the set of $n \times n$ matrices over $\mathbb{Z}^+$.
A mapping T: $\mathcal{M}_n(\mathbb{Z}^+) \longrightarrow \mathcal{M}_n(\mathbb{Z}^+)$ is called a *linear operator* if $\mathrm{T}(\alpha A + \beta B) = \alpha\ \mathrm{T}(A) + \beta\ \mathrm{T}(B)$ for all $A, B \in \mathcal{M}_n(\mathbb{Z}^+)$ and for all $\alpha, \beta \in \mathbb{Z}^+$. For $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathcal{M}_n(\mathbb{Z}^+)$, we recall that A *dominates* B, denoted by $A \geq B$, if $b_{i,j} \neq 0$ implies $a_{i,j} \neq 0$. Let T be a linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$. If A and B are matrices in $\mathcal{M}_n(\mathbb{Z}^+)$ with $A \leq B$, we can easily show that $\mathrm{T}(A) \leq \mathrm{T}(B)$. Let $\Delta_n = \{(i,j) \,|\, 1 \leq i,j \leq n\}$. Then for any $(i,j) \in \Delta_n$, we recall that $E_{i,j}$ denotes the $n \times n$ matrix whose (i,j)th entry is 1 and other entries are all 0. We call $E_{i,j}$ a *cell.*

LEMMA 3.1. *Let* T: $\mathcal{M}_n(\mathbb{Z}^+) \longrightarrow \mathcal{M}_n(\mathbb{Z}^+)$ *be a linear operator on* $\mathcal{M}_n(\mathbb{Z}^+)$. *Then the following are equivalent :*

1. T *is bijective.*
2. T *is surjective.*
3. *There exists a permutation* σ *on* Δ_n *such that* $\mathrm{T}(E_{i,j}) = E_{\sigma(i,j)}$

Proof. That 1) implies 2) and 3) implies 1) is straight forward. We now show that 2) implies 3). We assume that T is surjective. Then, for any pair $(i,j) \in \Delta_n$, there exists a matrix $X \in \mathcal{M}_n(\mathbb{Z}^+)$ such that $\mathrm{T}(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is $(r,s) \in \Delta_n$ such that $X = x_{r,s}E_{r,s} + X'$ where (r,s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $\mathrm{T}(E_{r,s}) \neq O$. Since $\mathbb{Z}^+$ has no zero divisors it follows that

$$\mathrm{T}(x_{r,s}E_{r,s}) \leq \mathrm{T}(x_{r,s}E_{r,s}) + \mathrm{T}(X \setminus (x_{r,s}E_{r,s})) = \mathrm{T}(X) = E_{i,j},$$

equivalently,

$$\mathrm{T}(x_{r,s}E_{r,s}) = x_{r,s}\mathrm{T}(E_{r,s}) \leq E_{i,j},$$

and so $\mathrm{T}(E_{r,s}) \leq E_{i,j}$.
It follows from $x_{r,s} \neq 0$ that $\mathrm{T}(E_{r,s}) = b_{r,s}E_{i,j}$ for some nonzero scalar $b_{r,s}$. Let $P_{i,j} = \{E_{r,s} \mid \mathrm{T}(E_{r,s}) \leq E_{i,j}\}$. By the above $P_{i,j} \neq \phi$ for all $(i,j) \in \Delta_n$. By its definition, $P_{i,j} \cap P_{u,v} = \phi$ whenever $(i,j) \neq (u,v)$. That is, $\{P_{i,j}\}$ is the set of n^2 nonempty sets which partition the set of cells. By the pigeonhole principle, we must have that $\mid P_{i,j} \mid = 1$ for all $(i,j) \in \Delta_n$. Necessarily, for each pair (r,s) there is the unique pair (i,j) such that $\mathrm{T}(E_{r,s}) = b_{r,s}E_{i,j}$. Thus, there is some permutation σ on $\{(i,j) \mid i,j = 1,2,\ldots,n\}$ such that $\mathrm{T}(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, for scalars $b_{i,j}$. We now only need to show that $b_{i,j} = 1$, for all i,j. Since T is surjective and $\mathrm{T}(E_{r,s}) \not\leq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$, there is some α such that $\mathrm{T}(\alpha E_{i,j}) = E_{\sigma(i,j)}$. Since T is linear,

$$E_{\sigma(i,j)} = \mathrm{T}(\alpha E_{i,j}) = \alpha\ \mathrm{T}(E_{i,j}) = \alpha b_{i,j} E_{\sigma(i,j)}.$$

That is, $\alpha b_{i,j} = 1$, or $b_{i,j}$ is unit. Since 1 is the only unit element in $\mathbb{Z}^+$, $b_{i,j} = 1$ for all $(i,j) \in \Delta_n$ ■

We denotes $\mathcal{C}_n(\mathbb{Z}^+)$ as the set of commuting pairs of matrices over $\mathbb{Z}^+$; that is, $\mathcal{C}_n(\mathbb{Z}^+) = \{(A,B) \in \mathcal{M}^2(\mathbb{Z}^+) \mid AB = BA\}$.

EXAMPLE 3.2 Let A be given in $\mathcal{M}_n(\mathbb{Z}^+)$. Define an operator T on $\mathcal{M}_n(\mathbb{Z}^+)$ by

$$T(X) = \left(\sum_{i,j=1}^{n} x_{i,j}\right) A$$

for all X=$[x_{i,j}] \in \mathcal{M}_n(\mathbb{Z}^+)$. Then we can easily show that T is a linear operator that preserve commuting pairs of matrices, while it does not preserve non-commuting pairs of matrices. ■

Thus, we are interested in linear operators that

$$(T(A), T(B)) \in \mathcal{C}_n(\mathbb{Z}^+) \text{ if and only if } (A,B) \in \mathcal{C}_n(\mathbb{Z}^+).$$

For a matrix $A \in \mathcal{M}_n(\mathbb{Z}^+)$, A is called *invertible* in $\mathcal{M}_n(\mathbb{Z}^+)$ if there exists a matrix $B \in \mathcal{M}_n(\mathbb{Z}^+)$ such that $AB = BA = I_n$. It is well known [2] that all permutation matrices are only invertible matrices in $\mathcal{M}_n(\mathbb{B})$. Using this fact, we can easily show that all permutation matrices are only invertible matrices in $\mathcal{M}_n(\mathbb{Z}^+)$.

THEOREM 3.3. *Let* T *be a linear operator on* $\mathcal{M}_n(\mathbb{Z}^+)$. *Then* T *is a surjective linear operator which preserves pairs of commuting matrices if and only if there exists an invertible matrix* $U \in \mathcal{M}_n(\mathbb{Z}^+)$ *such that either*

(1) $\mathrm{T}(X) = UXU^t$ *for all* $X \in \mathcal{M}_n(\mathbb{Z}^+)$, *or*

(2) $\mathrm{T}(X) = UX^tU^t$ *for all* $X \in \mathcal{M}_n(\mathbb{Z}^+)$.

Proof. Let T be a surjective linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$ that preserves pairs of commuting matrices. By Lemma 3.1, T is bijective and there exists a permutation σ on Δ_n such that $\mathrm{T}(E_{i,j}) = E_{\sigma(i,j)}$. Note that if $AX = XA$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$, then we have $A = \alpha I_n$ for some $\alpha \in \mathbb{Z}^+$. Thus we have $\mathrm{T}(I_n) = \beta I_n$ for some $\beta \in \mathbb{Z}^+$ because T is bijective. Since T maps a cell onto a cell, $\mathrm{T}(I_n) = I_n$. It follows that there is a permutation γ of $\{1,\dots,n\}$ such that $T(E_{i,i}) = E_{\gamma(i)\gamma(i)}$ for each $i = 1,\dots,n$. Define L: $\mathcal{M}_n(\mathbb{Z}^+) \to \mathcal{M}_n(\mathbb{Z}^+)$ by $\mathrm{L}(X) = P\mathrm{T}(X)P^t$, where P is the permutation matrix corresponding to γ so that $\mathrm{L}(E_{i,i}) = E_{i,i}$ for each $i = 1,\dots,n$. Then we can easily show that L is a bijective linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$ which preserves pairs of commuting matrices. By Lemma 3.1, L maps a cell onto a cell. Therefore, there exists $(p,q) \in \Delta_n$ such that $\mathrm{L}(E_{r,s}) = E_{p,q}$ for any $(r,s) \in \Delta_n$.

Suppose that $r \neq s$. Since L is bijective, we have $p \neq q$ because $\mathrm{L}(E_{i,i}) = E_{i,i}$ for each $i = 1,\dots,n$. Assume that $p \neq r$ and $p \neq s$. Then

$$E_{r,s}(E_{r,r} + E_{s,s} + E_{p,p}) = (E_{r,r} + E_{s,s} + E_{p,p})E_{r,s}$$

so that

$$\mathrm{L}(E_{r,s})\mathrm{L}(E_{r,r} + E_{s,s} + E_{p,p}) = \mathrm{L}(E_{r,r} + E_{s,s} + E_{p,p})\mathrm{L}(E_{r,s}),$$

equivalently,

$$E_{p,q}(E_{r,r} + E_{s,s} + E_{p,p}) = (E_{r,r} + E_{s,s} + E_{p,p})E_{p,q}.$$

It follows that $q = r$ or $q = s$. Since $E_{r,s}(E_{r,r} + E_{s,s}) = (E_{r,r} + E_{s,s})E_{r,s}$, we have

$$\mathrm{L}(E_{r,s})\mathrm{L}(E_{r,r} + E_{s,s}) = \mathrm{L}(E_{r,r} + E_{s,s})\mathrm{L}(E_{r,s}),$$

equivalently,

$$E_{p,q}(E_{r,r} + E_{s,s}) = (E_{r,r} + E_{s,s})E_{p,q}.$$

Since $q = r$ or $q = s$, we have $E_{p,q}(E_{r,r} + E_{s,s}) = E_{p,r}$ or $E_{p,s}$, but $(E_{r,r} + E_{s,s})E_{p,q} = 0$, a contradiction. Hence we have $p = r$ or $p = s$. Similarly we obtain $q = r$ or $q = s$. Therefore we have $\mathrm{L}(E_{r,s}) = E_{r,s}$ or $\mathrm{L}(E_{r,s}) = E_{s,r}$ for each $(r,s) \in \Delta_n$. Suppose that $\mathrm{L}(E_{r,s}) = E_{r,s}$ with $r \neq s$ and $\mathrm{L}(E_{r,t}) = E_{t,r}$ for some $t \neq r, s$. Then we have $\mathrm{L}(E_{s,t} + E_{t,s}) = E_{s,t} + E_{t,s}$. Let $A =$

$E_{r,r} + E_{s,t} + E_{t,s}$ so that $\mathrm{L}(A) = E_{r,r} + E_{s,t} + E_{t,s}$. Then $(E_{r,s} + E_{r,t})A = A(E_{r,s} + E_{r,t})$, and hence

$$\mathrm{L}(E_{r,s} + E_{r,t})\mathrm{L}(A) = \mathrm{L}(A)\mathrm{L}(E_{r,s} + E_{r,t}).$$

But

$$\mathrm{L}(E_{r,s} + E_{r,t})\mathrm{L}(A) = E_{r,t} + E_{t,r},$$

while

$$\mathrm{L}(A)\mathrm{L}(E_{r,s} + E_{r,t}) = E_{r,s} + E_{s,r}.$$

Thus we have $t = s$, a contradiction. It follows that if $\mathrm{L}(E_{i,j}) = E_{i,j}$ for some pair $(i,j) \in \Delta_n$ with $i \neq j$, then $\mathrm{L}(E_{r,s}) = E_{r,s}$ for all pairs $(r,s) \in \Delta_n$. Similarly, if $\mathrm{L}(E_{i,j}) = E_{j,i}$ for some pair $(i.j) \in \Delta_n$ with $i \neq j$, then $\mathrm{L}(E_{r,s}) = E_{s,r}$ for all pairs $(r,s) \in \Delta_n$. We have established that either $\mathrm{L}(X) = X$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$ or $\mathrm{L}(X) = X^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$. Therefore $\mathrm{T}(X) = P^tXP$ or $\mathrm{T}(X) = P^tX^tP$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$. If $U = P^t$, then we have $\mathrm{T}(X) = UXU^t$ or $\mathrm{T}(X) = UX^tU^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$.

The converse is immediate. ■

Thus we have characterized the linear operators that preserve commuting pairs of matrices over nonnegative integers.

References

[1] D. J. Richman and H. Schneider, Primes in the semigroup of nonnegative matrices, *Linear and Multilinear Algebra* **2** (1974) 135-140.

[2] D. de Caen and D. A. Gregory, Primes in the semigroup of Boolean matrices, *Linear Algebra Appl.* **37** (1981) 227-241.

[3] G. H. Chan and M. H. Lim, Linear transformations on symmetric matrices that preserve commutativity, *Linear Algebra Appl* **47** (1982) 11-22.

[4] H. Radjavi, Commutativity-preserving operators on symmetric matrices, *Linear Algebra Appl.* **61** (1984) 219.

[5] K. Rao and P. Rao, On generalized inverses of Boolean matrices, *Linear Algebra Appl.* **11** (1975) 135-153.

[6] K. H. Kim, *Boolean Matrix Theory and Applications, Pure Appl.* Math. 70, Marcel Dekker, New York, (1982).

[7] K. Rao and P. Rao, On generalized inverses of Boolean matrices. II, *Linear Algebra Appl.* **42** (1982) 133-153.

[8] L. B. Beasley, Linear transformations on matrices: The invariance of commuting pairs of matrices, *Linear and Multilinear Algebra* **6**(1978), 179-183.

[9] L. B. Beasley and N. J. Pullman, Boolean rank preserving operators and Boolean rank-1 spaces, *Linear Algebra Appl.* **65**(1984), 55-77.

[10] L. B. Beasley and N. J. Pullman, Linear operators that strongly preserve commuting pairs of Boolean matrices, *Linear Algebra Appl.* **132** (1990), 137-143.

[11] M. D. Choi, A. A. Jafarian and H. radjavi, Linear maps preserving commutativity, *Linear Algebra Appl.* **87**(1987) 227-241.

[12] S. Z. Song and L. B. Beasley, *Linear operators that preserve commuting pairs of nonnegaitve real matrices, Linear and Multilinear Algebra.* **51(3)**(2003) 279-283.

[13] S. Z. Song and K. T. Kang, Column ranks and their preservers of matrices over max algebra, *Linear and Multilinear Algebra* **51** (2003) 311-318

[14] S. Z. Song and K. T. Kang, *Linear maps that preserve commuting pairs of matrices over general Boolean algebras, to appear in J. Korean Math. Soc.* (2006)

[15] W. Watkins, Linear maps that preserve commuting pairs of matrices, *Linear Algebra Appl.* **14** (1976) 29-35.