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# A Quasi-Sure Flow Property and the Equivalence of Capacities for Differential Equations on the Wiener Space

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We consider a differential equation on the Wiener space. We show that the solutions for the differential equation satisfy the flow property quasi everywhere and we obtain the equivalence of capacities under the transformations of the Wiener space induced by the solutions by using the quasi flow property. 

## 1. INTRODUCTION

Let $(X, H, \mu)$ be an abstract Wiener space and $A$ be a vector field on $X$ which is, by definition, a mapping from $X$ into $H$ smooth in the sense of Malliavin. We consider the following differential equation (1.1) on $X$:

$$\begin{cases} \dfrac{d}{dt}(U_t(x)) = A(U_t(x)), \\ U_0(x) = x. \end{cases} \tag{1.1}$$

The problem was first treated by Cruzeiro [2, 3] and she established for a class of vector fields the existence and the uniqueness of solutions $U_t(x)$ for $\mu$-almost every ($\mu$-a.e.) $x$ which satisfy the almost everywhere flow property: for every $t \in \mathbb{R}$, $(U_t)_*\mu$ is absolutely continuous with respect to $\mu$ and satisfies

$$U_t \circ U_s(x) = U_{t+s}(x), \tag{1.2}$$

for $\mu$-a.e. $x$, for every $t$ and $s$.

In the previous paper [14], the author constructed a solution $U_t(x)$ which satisfies (1.1) quasi everywhere (q.e.), i.e., for all $x$ except in a slim set, that is, a set of $(r,p)$-capacity 0, for all $r \geq 0$ and $p > 1$. Here the capacities are associated with the Ornstein-Uhlenbeck operator on the Wiener space (cf. [9], [5], [10]). By the way of its construction, we see that this $U_t(x)$ is a quasi continuous modification of the unique solution of Cruzeiro [2], [3] in the sense of almost everywhere.

In the present paper, we obtain further refinements of this solution. We prove that we can choose its quasi continuous modification $U_t(x)$ so that the mapping $x \to U_t(x)$ preserves the class of slim sets for every $t$ and the flow property (1.2) holds q.e. for every $t$ and $s$.

The organization of this paper is as follows. In Section 2, we modify the solution $U_t(x)$ constructed in [14] so that it is defined for every $t$ and $x$ and show that this modification satisfies the flow property (1.2) q.e. in $x$ for every $t$ and $s$. In Section 3, we prove an equivalence of capacities in the sense of Theorem 3.5 under the transformations of the Wiener space induced by the above modification so that we can conclude that the class of slim sets is preserved under the transformations.

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## 2. A QUASI SURE FLOW PROPERTY

Let $(X, H, \mu)$ be an abstract Wiener space introduced by Gross [6] where

(i) $X$ is a real, separable Banach space with the norm $\|\cdot\|$,

(ii) $H$ is a real, separable Hilbert space densely and continuously imbedded in $X$ with the inner product $\langle x, y\rangle_H$,

(iii) $\mu$ is the standard Gaussian measure, i.e., the Borel probability measure on $X$ such that

$$\int_X \exp\{i(h, x)\}\mu(dx) = \exp(-\frac{1}{2}\langle h, h\rangle_{H^*})$$

where $h \in X^* \subset H^*$ and $(\ ,\ )$ is a natural pairing of $X^*$ and $X$.

We denote the Borel $\sigma$-field on $X$ by $\mathcal{B}(X)$ and define a system of transition probabilities $P_t(x, \cdot)$, $t \geq 0, x \in X$, by

$$P_t(x, G) = \int_X 1_G(e^{-t}x + \sqrt{1-e^{-2t}}y)\mu(dy), \quad G \in \mathcal{B}(X).$$

Then the Ornstein-Uhlenbeck semigroup $\{T_t\}$ is defined by

$$T_t u(x) = \int_X u(y)P_t(x, dy), \quad u \in \mathcal{B}_b(X),$$

where $\mathcal{B}_b(X)$ is the space of bounded $\mathcal{B}(X)$-measurable real functions on $X$.

Let $E$ be a real separable Banach space. We denote by $L^p(X,\mu;E)$ the space of all $E$-valued $\mu$-measurable functions $u$ satisfying

$$||u||_p := \left\{\int_X ||u(x)||_E^p \mu(dx)\right\}^{1/p} < \infty.$$

As usual, we identify two functions which are equal to each other $\mu$-almost everywhere ($\mu$-a.e.). For $u \in L^p(X,\mu;E)$, we set

$$T_t u(x) = \int_X u(y) P_t(x,dy).$$

Here the integral in the right hand side is a Bochner integral. Then $T_t u(x)$ can be defined for $\mu$-a.e. $x$ and $\{T_t\}$ is a strongly continuous contraction semigroup on $L^p(X,\mu;E)$.

Let us denote the generator by $L$. For $r \geq 0$, we define $(1-L)^{-r/2}$ by the following gamma transformation:

$$(1-L)^{-r/2} = \frac{1}{\Gamma(r/2)}\int_0^\infty t^{r/2-1}e^{-t}T_t dt.$$

$(1-L)^{-r/2}$ is a contraction operator on $L^p(X,\mu;E)$ since $\{T_t\}$ is a contraction semigroup. We set

$$W_r^p(X;E) := (1-L)^{-r/2}(L^p(X,\mu;E)).$$

Then $W_r^p(X;E)$ becomes a Banach space with a norm

$$||f||_{r,p} := ||u||_{L^p} \quad \text{for} \quad f = (1-L)^{-r/2}u, \quad u \in L^p(X,\mu;E).$$

We call $W_r^p(X;E)$ the Sobolev space with the differentiability index $r$ and the integrability index $p$. We denote the space $\cap_r W_r^p(X;E)$ by $W_\infty^p(X;E)$ for $p \in [1,\infty)$ and $W_\infty^\infty(X;E) = \cap_p W_\infty^p(X;E)$. If $E$ is a separable Hilbert space, we can define the gradient operator $\nabla : W_{r+1}^p(X;E) \to W_r^p(X;E \otimes H)$ and its dual, the divergence operator, $\delta : W_{r+1}^p(X;E\otimes H) \to W_r^p(E)$ as usual.

Next let us recall the notion of $(r,p)$-capacity ([5] and [8] for details). The $(r,p)$-capacity $C_{r,p}$ is defined as follows: for an open set $O \subset X$,

$$C_{r,p}(O) = \inf\{||f||_{r,p}^p \ ; \ f \in W_r^p(X;\mathbb{R}), \ f \geq 1 \text{ a.e. on } O\}$$

and for an arbitrary set $B \subset X$,

$$C_{r,p}(B) = \inf\{C_{r,p}(G) \ ; \ G \text{ is open and } G \supset B\}.$$

Now we turn to the differential equation (1.1) on the Wiener space. We set $U_t(x) = V_t(x) + x$. Then (1.1) is equivalent to the following differential equation:

$$\begin{cases} \frac{d}{dt}(V_t(x)) = A(V_t(x) + x), \\ V_0(x) = 0. \end{cases} \tag{2.1}$$

**Theorem 2.1.** (Yun[14], Theorem 5.5) Assume that

(*i*) $A \in W_\infty^\infty(X; H)$ and $\int_X \exp(\lambda \|A(x)\|)d\mu(x) < \infty$, for every $\lambda > 0$,

(*ii*) $\int_X \exp(\lambda \|\nabla^m A(x)\|)d\mu(x) < \infty$, for every $\lambda > 0$ and $m = 1, 2, \cdots$,

(*iii*) $\int_X \exp(\lambda |\delta A(x)|)d\mu(x) < \infty$, for every $\lambda > 0$.

Then, we can choose a quasi continuous modification $\tilde{A}(x)$ of $A(x)$ defined everywhere on $X$ and we can construct $U_t(x)$, $t \in \mathbb{R}$, $x \in X$, such that $U_t(x) \in W_\infty^\infty(X; C([-T,T] \to X))$, $X \ni x \mapsto U_t(x) \in C([-T,T] \to X)$ is quasi continuous for every $T > 0$ and satisfies

$$U_t(x) = x + \int_0^t \tilde{A}(U_s(x))ds \quad \text{for quasi every (q.e.) } x \in X, \tag{1.1'}$$

for all $t \in \mathbb{R}$.

Here, a property is said to hold q.e. $x$ if it holds for every $x$ except in a slim set, i.e., a set of $(r,p)$-capacity 0 for every $r \geq 0$ and $p > 1$.

The solution $U_t(x)$ constructed in Theorem 2.1 is a quasi continuous modification of the unique solution established by Cruzeiro [2], [3] in the sense of almost everywhere with respect to $\mu$. This fact can be seen by the way of our construction in [14]: to be precise, we took a sequence $\{A_n\}$ converging to $A$ such that $A_n$ depends only on finite number of coordinates and takes values in a finite dimensional subspace of $H$ for using the results in the finite dimensional case. Denoting the solution for $A_n$ by $V_t^{(n)}$, we showed that for some subsequence $\{n_i\}$, $\{V_t^{(n_i)}\}$ converges quasi everywhere to $V_t(x)$ and $U_t(x) = x + V_t(x)$ satisfies $(1.1)'$ in which $\tilde{A}(x)$ is defined by

$$\tilde{A}(x) = \begin{cases} \lim_{n\to\infty} A_n(x), & \text{if it converges,} \\ 0, & \text{otherwise.} \end{cases}$$

Here we have to notice that $U_t^{(n)}(x) = x + V_t^{(n)}(x)$ is a homeomorphism on $X$ in the finite dimensional case.

Now we show that we can modify the solution $U_t(x)$ so that it is defined for every $t \in \mathbb{R}$ and $x \in X$, satisfies $(1.1)'$ for q.e. $x \in X$ for all $t \in \mathbb{R}$ and also has the quasi sure flow property, i.e., satisfies (1.2) for q.e. $x \in X$, for all $t, s \in \mathbb{R}$.

**Lemma 2.2.** For all $r \geq 0$ and $p > 1$,

$$||V_t^{(n)} \circ U_s - V_t^{(m)} \circ U_s||_{r,p} \longrightarrow 0 \quad \text{as} \quad n, m \to \infty.$$

*Proof.* We write $V_t^{(n)}(V_s(x) + x)$ instead of $V_t^{(n)}(U_s(x))$. Since $V_t^{(n)}$ depends on finite number of coordinates, we have

$$\begin{aligned} &L(V_t^{(n)}(V_s(x) + x)) \\ &\quad = LV_t^{(n)}(V_s(x) + x) + \nabla V_t^{(n)}(V_s(x) + x) \cdot LV_s(x) \\ &\qquad + \sum_i \partial_{jk}^2 V_t^{(n)}(V_s(x) + x) \cdot \partial_i V_s^j \cdot \partial_i V_s^k(x) \\ &\qquad + 2\sum_i \partial_{ij}^2 V_t^{(n)}(V_s(x) + x) \cdot \partial_i V_s^j(x). \end{aligned} \tag{2.2}$$

Thus we have

$$\begin{aligned} &||V_t^{(n)}(V_s(x) + x) - V_t^{(m)}(V_s(x) + x)||_{2,p} \\ &\quad = E[|L(V_t^{(n)}(V_s(x) + x)) - L(V_t^{(m)}(V_s(x) + x))|^p]^{1/p} \\ &\quad \leq E[|LV_t^{(n)}(V_s(x) + x) - LV_t^{(m)}(V_s(x) + x)|^p]^{1/p} \\ &\qquad + E[|LV_s(x)|^p \cdot |\nabla V_t^{(n)}(V_s(x) + x) - \nabla V_t^{(m)}(V_s(x) + x)|^p]^{1/p} \\ &\qquad + E[(|\nabla V_s(x)|^2 + 2|\nabla V_s(x)|)^p \cdot |\nabla^2 V_t^{(n)}(V_s(x) + x) \\ &\qquad\quad - \nabla^2 V_t^{(m)}(V_s(x) + x)|^p]^{1/p} \\ &\quad \equiv E_1^{1/p} + E_2^{1/p} + E_3^{1/p}. \end{aligned}$$

We set $(d(U_s)_*\mu/d\mu)(x) = k_s(x)$. Then $\sup_{0 \leq s \leq t} ||k_s||_2 < \infty$ (see [2]). By Hölder's inequality, we have

$$\begin{aligned} E_1 &= E[|L(V_t^{(n)}(x)) - L(V_t^{(m)}(x))|^p \cdot k_s(x)] \\ &\leq ||k_s||_2 \cdot ||LV_t^{(n)} - LV_t^{(m)}||_{2p}^p, \\ E_2 &= E[|LV_s(x)|^p \cdot |\nabla V_t^{(n)}(x) - \nabla V_t^{(m)}(x)|^p \cdot k_s(x)] \\ &\leq ||k_s||_2 \cdot ||LV_s||_{4p}^p \cdot ||\nabla V_t^{(n)} - \nabla V_t^{(m)}||_{4p}^p, \end{aligned}$$

and

$$\begin{aligned} E_3 &= E[(|\nabla V_s(x)|^2 + 2|\nabla V_s(x)|)^p \cdot |\nabla^2 V_t^{(n)}(x) - \nabla^2 V_t^{(m)}(x)|^p \cdot k_s(x)] \\ &\leq ||k_s||_2 \cdot E[(|\nabla V_s(x)|^2 + 2|\nabla V_s(x)|)^{4p}]^{1/4} \cdot ||\nabla^2 V_t^{(n)} - \nabla^2 V_t^{(m)}||_{4p}^p \\ &\leq ||k_s||_2 \cdot (||(\nabla V_s)^2||_{4p} + ||2\nabla V_s||_{4p})^p \cdot ||\nabla^2 V_t^{(n)} - \nabla^2 V_t^{(m)}||_{4p}^p. \end{aligned}$$

Here we denote the $L^p$-norm by $\|\cdot\|_p$. We have proved in the proof of Theorem 2.1 in [14] that the right hand sides of the above estimates for $E_i$, $i$=1, 2, 3 tend to 0 as $n, m \to \infty$ and $V_t \in W^\infty_\infty(X; H)$. The convergence of $V_t^{(n)} \circ U_s$ in $(2,p)$-norm is proved.

To prove the convergence of $V_t^{(n)} \circ U_s$ in $(2k,p)$-norm, we apply the operator $L$ successively to (2.2). We can rewrite the formula (2.2) as follows:

$$L(V_t^{(n)}(V_s(x)+x)) = \nabla V_t^{(n)}(V_s(x)+x)\cdot LV_s(x) + F^1(LV_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_s)$$

for some polynomial $F^1$. Note in general the following chain rule

$$\begin{aligned} LF(u_1,\cdots,u_m) &= \sum_{j=1}^m (\partial_j F)(u_1,\cdots,u_m)\cdot Lu_j \\ &\quad + \sum_{j,k}^m (\partial_j\partial_k F)(u_1,\cdots,u_m)\cdot(\sum_{i=1}^n \partial_i u_j\cdot\partial_i u_k) \\ &= \sum_{j=1}^m (\partial_j F)(u_1,\cdots,u_m)\cdot Lu_j \\ &\quad + \sum_{j,k}^m (\partial_j\partial_k F)(u_1,\cdots,u_m)\cdot(\nabla u_j, \nabla u_k), \end{aligned}$$

for a polynomial $F$ and $u_j = u_j(x_1,\cdots,x_n)$, $j=1,\cdots,m$. Then we have

$$\begin{aligned} L^2 V_t^{(n)} &= \nabla V_t^{(n)}\cdot L^2 V_s \\ &\quad + F^2(L^2V_t^{(n)}, L(\nabla^2 V_t^{(n)}), L(\nabla V_t^{(n)}), LV_t^{(n)}, \\ &\qquad \nabla^4 V_t^{(n)}, \nabla^3 V_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_t^{(n)}, \\ &\qquad L(\nabla V_s), LV_s, \nabla^2 V_s, \nabla V_s), \end{aligned} \tag{2.3}$$

for some polynomial $F^2$. In fact, since

$$\begin{aligned} L^2 V_t^{(n)} &= L(\nabla V_t^{(n)}\cdot LV_s) + LF^1(LV_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_s) \\ &= L(\nabla V_t^{(n)})\cdot LV_s + \nabla V_t^{(n)}\cdot L^2 V_s \\ &\quad + \sum_{j=1}^3 (\partial_j F^1)(u_1,u_2,u_3)\cdot(Lu_j) \\ &\quad + \sum_{j,k=1}^3 (\partial_j\partial_k F^1)(u_1,u_2,u_3)\cdot(\sum_{i=1}^n \partial_i u_j\cdot\partial_i u_k) \\ &\quad (\text{where}\quad u_1 = LV_t^{(n)}, u_2 = \nabla^2 V_t^{(n)}, u_3 = \nabla V_s), \end{aligned}$$

$$
\begin{aligned}
L(\nabla V_t^{(n)}) \cdot LV_s = & [\sum_i (\nabla^3 V_t^{(n)}(V_s(x)+x) \cdot (\partial_i V_s(x)+e_i)^2 \\
& + \nabla^2 V_t^{(n)}(V_s(x)+x) \cdot (\partial_i \partial_i V_s(x))) \\
& + \sum_i (\nabla^2 V_t^{(n)}(V_s(x)+x) \cdot (\partial_i V_s(x)+e_i)) \cdot x_i] \cdot LV_s(x) \\
& \text{(where } \{e_i\} \text{ is ONB in } H \text{ chosen in} \\
& \text{the finite dimensional approximation)} \\
= & [\sum_i \nabla^3 V_t^{(n)}(V_s(x)+x) \cdot (\partial_i V_s(x)+e_i)^2 \\
& + \nabla^2 V_t^{(n)}(V_s(x)+x) \cdot \nabla^2 V_s(x) \\
& + \sum_i \nabla^2 V_t^{(n)}(V_s(x)+x) \cdot e_i \cdot x_i] \cdot LV_s(x) \\
= & F_1^2(L(\nabla V_t^{(n)}), \nabla^3 V_t^{(n)}, \nabla^2 V_t^{(n)}, LV_s, \nabla^2 V_s, \nabla V_s),
\end{aligned}
$$

$$
\begin{aligned}
& (\partial_1 F^1)(u_1,u_2,u_3) \cdot (L(LV_t^{(n)})) \\
& \qquad = F_2^2(L^2 V_t^{(n)}, \nabla^2(LV_t^{(n)}), \nabla(LV_t^{(n)}), LV_s, \nabla^2 V_s, \nabla V_s), \\
& (\partial_2 F^1)(u_1,u_2,u_3) \cdot (L(\nabla^2 V_t^{(n)})) \\
& \qquad = F_3^2(L(\nabla^2 V_t^{(n)}), \nabla^4 V_t^{(n)}, \nabla^3 V_t^{(n)}, LV_s, \nabla^2 V_s, \nabla V_s), \\
& (\partial_3 F^1)(u_1,u_2,u_3) \cdot (L(\nabla V_s)) \\
& \qquad = F_4^2(L(\nabla V_s)),
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{j,k=1}^{3} (\partial_j \partial_k F^1)(u_1,u_2,u_3) \cdot (\nabla u_j, \nabla u_k) \\
& \qquad = F_5^2(L(\nabla V_t^{(n)}), LV_t^{(n)}, \nabla^3 V_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla^2 V_s, \nabla V_s),
\end{aligned}
$$

for some polynomials $F_1^2, \cdots, F_5^2$, the formula (2.3) holds by setting

$$
F^2 = \sum_{i=1}^{5} F_i^2.
$$

Note that

$$
\begin{aligned}
L(\nabla V_t^{(n)}(V_s(x)+x)) &= L(\nabla V_t^{(n)}(V_s(x)+x))\cdot \nabla V_s(x) + L(\nabla V_t^{(n)}(V_s(x)+x)) \\
&\quad + \nabla V_t^{(n)}(V_s(x)+x)\cdot L(\nabla V_s(x)) \\
&= \nabla V_t^{(n)}(V_s(x)+x)\cdot L(\nabla V_s(x)) \\
&\quad + E^{1,1}(L(\nabla V_t^{(n)}), \nabla^3 V_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_s),
\end{aligned}
$$

where $E^{1,1}$ is some polynomial which can be calculated by the same method as $F^2$. Since it is of the same type as in the previous case for $LV_t^{(n)}$, we can prove the $L^p$ convergence of $L(\nabla V_t^{(n)}(V_s(x)+x))$. And noting the fomula (2.3), the $L^p$-convergence of $L^2 V_t^{(n)}(V_s(x)+x)$ can be also proved by the same reason. For the general case $L^l \nabla^m V_s(x)$ ($l=0,1,\cdots,k,\ m=0,1,\cdots,2k,\ 2l+m\le 2k,\ k=2,3,\cdots$), the equation can be calculated by

$$
\begin{aligned}
&L^l\nabla^m(V_t^{(n)}(V_s(x)+x)) \qquad (2.4)\\
&\quad = \nabla V_t^{(n)}(V_s(x)+x)\cdot L^l\nabla^m V_s(x) \\
&\qquad + E^{l,m}(L^i\nabla^j V_t^{(n)}, L^{i'}\nabla^{j'} V_s; \\
&\qquad\qquad i=0,1,\cdots,m, \quad j=0,1,\cdots, \\
&\qquad\qquad i'=0,1,\cdots,m-1, \quad j'=0,1,\cdots, \\
&\qquad\qquad 2i+j\le 2m+n, \quad 2i'+j'\le 2(m-1)+n), \\
&l=0,1,\cdots,k, \quad m=0,1,\cdots,2k, \quad 2l+m\le 2k, \\
&k=2,3,\cdots,
\end{aligned}
$$

for some polynomial $E^{l,m}$.

Repeating the same argument as above, we can prove the convergence of $V_t^{(n)}\circ U_s$ in $(2k,p)$-norm. ∎

**Theorem 2.3.** We can redefine $U_t(x)$ as $\tilde{U}_t(x)$ so that it is defined for every $t\in\mathbb{R}$ and $x\in X$ and satisfies (1.1)′ and (1.2) q.e. $x$ for all $t,s\in\mathbb{R}$.

*Proof.* The solution $U_t(x)$ in Theorem 2.1 was constructed as a quasi everywhere limit of $U_t^{(n_i)}(x)$ for some subsequence $\{n_i\}$. We define $\tilde{V}_t(x)$ for every $t\in\mathbb{R}$ and $x\in X$ by

$$
\tilde{V}_t(x) = \begin{cases} \lim_{n\to\infty} V_t^{(n_i)}(x), & \text{if } V_t^{(n_i)}(x) \text{ converges,} \\ 0, & \text{if } V_t^{(n_i)}(x) \text{ does not converge.} \end{cases}
$$

Then, $\tilde{U}_t(x)=\tilde{V}_t(x)+x$ is quasi continuous and $U_t(x)=\tilde{U}_t(x)$ q.e. By the almost sure flow property of $U_t(x)$, we have $\tilde{U}_t\circ\tilde{U}_s(x)=U_t\circ U_s(x)=U_{t+s}(x)=\tilde{U}_{t+s}(x)$ a.a. $x(\mu)$. But $\tilde{U}_{t+s}(x)$ is quasi continuous and hence, if we can show that $\tilde{U}_t\circ\tilde{U}_s(x)$ has a quasi continuous modification, we have

$\tilde{U}_t \circ \tilde{U}_s(x) = \tilde{U}_{t+s}(x)$ q.e. $x$.
By Lemma 2.2, we showed that $V_t^{(n)} \circ \tilde{U}_s$ is Cauchy in $W_r^p(X;H)$ and therefore, we can take a subsequence $\{\tilde{n}_j\}$ from $\{n_i\}$ such that for any $\varepsilon > 0$, there exists a closed set $Y$ with $C_{r,p}(X \backslash Y) < \varepsilon$ and $V_t^{(\tilde{n}_j)} \circ \tilde{U}_s(x)$ converges to $\tilde{V}_t \circ \tilde{U}_s(x)$ uniformly in $x \in Y$. Hence we can deduce that $\tilde{U}_t \circ \tilde{U}_s(x)$ is quasi continuous. ∎

## 3. An equivalences of capacity

Let $U_t(x)$ be the solution constructed in Theorem 2.3. In this section, we prove the equivalence of capacities between a set $B$ in $X$ and $U_t(B)$ in the sense given in Theorem 3.5 which refines to the property of mutual absolute continuity for $(U_t)_*\mu$ and $\mu$.

**Lemma 3.1.** For any $1 < p < p_1$ and $r \geq 0$, if $g \in W_r^{p_1}(X;\mathbb{R})$, then the family $\{g \circ U_t^{(n)}\}$ forms a bounded set in $W_r^p(X;\mathbb{R})$. To be more precise, there exists a constant $C$ such that $||g \circ U_t^{(n)}||_{r,p}^p \leq C \cdot ||g||_{r,p_1}^p$ for all $n$ and $g \in W_r^{p_1}(X;\mathbb{R})$.

*Proof.* We prove only the case $r = 2$. The general case can be proved similarly. Note that

$$\begin{aligned}
||L(g(V_t^{(n)}(x)+x))||_p \leq\ & ||(Lg)(V_t^{(n)}(x)+x)||_p \\
& + ||(\nabla g)(V_t^{(n)}(x)+x) \cdot LV_t^{(n)}(x)||_p \\
& + ||\sum_i (\nabla^2 g)(V_t^{(n)}(x)+x) \cdot \partial_i V_t^{(n)} \cdot \partial_i V_t^{(n)}||_p \\
& + ||2\sum_i (\partial_i \nabla g)(V_t^{(n)}(x)+x) \cdot \partial_i V_t^{(n)}||_p \\
\equiv\ & E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

If we put $k_t^{(n)}(x) = (d(U_t^{(n)})_*\mu/d\mu)(x)$, then for all $p > 1$, $\sup_n\{||k_t^{(n)}||_p + ||LV_t^{(n)}||\} < \infty$, and $\sup_n ||\nabla V_t^{(n)}||_p < \infty$ ([14]). By Hölder's inequality, the above $L^p$ norms can be estimated as follows:

$$\begin{aligned}
E_1 &= ||(Lg)(x) \cdot k_t^{(n)}(x)||_p \leq ||k_t^{(n)}||_{q'}^{1/p} \cdot ||Lg||_{pp'}, \\
E_2 &= ||LV_t^{(n)}(x) \cdot (\nabla g)(x) \cdot k_t^{(n)}(x)||_p \leq ||k_t^{(n)}||_{2q'}^{1/p} \cdot ||LV_t^{(n)}||_{2pq'} \cdot ||\nabla g||_{pp'}, \\
E_3 &= ||\sum_i (\nabla^2 g)(x) \cdot \partial_i V_t^{(n)}(x) \cdot \partial_i V_t^{(n)}(x) \cdot k_t^{(n)}(x)||_p \\
&\leq ||k_t^{(n)}||_{2q'} \cdot ||(\nabla V_t^{(n)})^2||_{2pq'} \cdot ||\nabla^2 g||_{pp'},
\end{aligned}$$

$$E_4 = ||2\sum_i(\partial_i\nabla g)(x)\cdot\partial_i V_t^{(n)}(x)\cdot k_t^{(n)}(x)||_p$$
$$\leq ||k_t^{(n)}||_{2q'}\cdot||2\nabla V_t^{(n)}||_{2pq'}\cdot||\nabla^2 g||_{pp'},$$

for all $p' > 1$ and $q' > 1$ with $1/p' + 1/q' = 1$. By Meyer's equivalence,

$$\sum_i E_i \leq C\cdot||Lg||_{p_1},$$

for some constant $C$. Thus we have $||g\circ U_t^{(n)}||_{2,p}^p \leq C\cdot||g||_{2,p_1}^p$. ∎

To show the equivalence, we need the tightness of capacity.

**Definition 3.2.** An increasing sequence $\{F_n\}$ of closed sets is called a *nest* with respect to the capacity if

$$\lim_{n\to\infty} C_{r,p}(X\backslash F_n) = 0, \quad \text{for all} \quad r\geq 0 \quad \text{and} \quad p > 1.$$

Further we say that the capacity is *tight* ([11]) if a nest $\{F_n\}$ exists such that $F_n$ is compact for all $n$.

It is known ([13]) that the capacity is tight in our case of an abstract Wiener space. Using the tightness, we have

**Lemma 3.3.** There exists a compact nest $\{F_n\}$ such that $U_t|_{F_n}$, the restriction of $U_t$ to $F_n$, is a homeomorphism.

*Proof.* Since $U_t$ is quasi continuous, for all $\varepsilon > 0$, there exists a closed set $F$ such that $C_{r,p}(F^c) < \varepsilon$ and $U_t$ is continuous on $F$ where $F^c$ denotes the complement of $F$. Since $\{U_t\}$ is a quasi flow by Theorem 2.3, we see that $U_{-t}\circ U_t$ is the identity mapping on $F_1$ for some closed set $F_1$ with $C_{r,p}(F_1^c) < \varepsilon$. Let $\{F_n'\}$ be a compact nest. Then $F_n = F\cap F_1\cap F_n'$ becomes also a compact nest. For $x, y\in F_n$, if $U_t(x) = U_t(y)$, then $x = U_{-t}(U_t(x)) = U_{-t}(U_t(y)) = y$ since $U_{-t}\circ U_t$ is the identity on $F_n$. Thus $U_t$ is continuous and one-to-one on $F_n$.
Therefore, $U_t$ is a homeomorphism from $F_n$ to $U_t(F_n)$. ∎

**Lemma 3.4.** For any $1 < p < p_1$ and $r\geq 0$, if $g\in W_r^{p_1}(X;\mathbb{R})$, then $g\circ U_t\in W_r^p(X;\mathbb{R})$; more precisely, there exists a constant $C$ such that $||g\circ U_t||_{r,p}^p \leq C\cdot||g||_{r,p_1}^p$ for all $g\in W_r^{p_1}(X;\mathbb{R})$.

*Proof.* By Lemma 3.1, $\|g \circ U_t^{(n)}\|_{r,p}^p \leq C \cdot \|g\|_{r,p_1}^p$ for some constant $C$. Thus the proof is complete if we prove that $g \circ U_t^{(n)}$ converges to $g \circ U_t$ in $W_r^p(X;\mathbb{R})$.

Since

$$\begin{aligned} L(g(V_t^{(n)}(x)+x)) = &\ Lg(V_t^{(n)}(x)+x) + \nabla g(V_t^{(n)}(x)+x)\cdot LV_t^{(n)}(x) \\ &+ \sum_i \nabla^2 g(V_t^{(n)}(x)+x)\cdot \partial_i V_t^{(n)}(x)\cdot \partial_i V_t^{(n)}(x) \\ &+ 2\sum_i \partial_i \nabla g(V_t^{(n)}(x)+x)\cdot \partial_i V_t^{(n)}(x), \end{aligned}$$

we can prove that

$$\|g \circ U_t^{(n)} - g \circ U_t\|_{r,p} \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

by the same method as in the proof of Lemma 2.2. ∎

**Theorem 3.5.** For $1 < p_2 < p < p_1$ and $r \geq 0$, there exist constants $C_1, C_2 > 0$ such that

$$C_2 \cdot (C_{r,p_2}(B))^{p/p_2} \leq C_{r,p}(U_t(B)) \leq C_1 \cdot (C_{r,p_1}(B))^{p/p_1}, \quad \forall B \subset X. \tag{3.1}$$

*Proof.* By Lemma 3.3, there exists a compact nest $\{F_n\}$ such that $U_t|_{F_n}$ is a homeomorphism. Let $O$ be an open set in $X$. Then $O \cap F_n$ is open in $F_n$ and $U_t(O \cap F_n)$ is open in $U_t(F_n)$. Thus there exists an open set $O'$ in $X$ such that $U_t(O \cap F_n) = O' \cap U_t(F_n)$. Since $U_t(O) \subset [O' \cap U_t(F_n)] \cup U_t(F_n^c)$, we have

$$\begin{aligned} &C_{r,p}(U_t(O)) \\ &\quad \leq C_{r,p}([O' \cap U_t(F_n)] \cup U_t(F_n^c)) \\ &\quad = \inf\{\|f\|_{r,p}^p \,;\ f \in W_r^p(X;\mathbb{R}),\ f \geq 1 \text{ a.e. on } [O' \cap U_t(F_n)] \cup U_t(F_n^c)\} \\ &\quad \leq \inf\{\|f\|_{r,p}^p \,;\ f \in W_r^p(X;\mathbb{R}),\ f \circ U_t \geq 1 \text{ a.e. on } O \cup F_n^c\} \\ &\quad = \inf\{\|g \circ U_{-t}\|_{r,p}^p \,;\ g \circ U_{-t} \in W_r^p(X;\mathbb{R}),\ g \geq 1 \text{ a.e. on } O \cup F_n^c\}. \end{aligned}$$

By Lemma 3.4, we have

$$\begin{aligned} C_{r,p}(U_t(O)) &\leq C_1 \cdot \inf\{\|g\|_{r,p_1}^p \,;\ g \in W_r^{p_1}(X;\mathbb{R}),\ g \geq 1 \text{ a.e. on } O \cup F_n^c\} \\ &= C_1 \cdot (C_{r,p_1}(O \cup F_n^c))^{p/p_1} \\ &\leq C_1 \cdot (C_{r,p_1}(O)^{p/p_1} + C_{r,p_1}(F_n^c)^{p/p_1}) \\ &\longrightarrow C_1 \cdot (C_{r,p_1}(O))^{p/p_1}, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, for an open set $O$ in $X$, $C_{r,p}(U_t(O)) \leq C_1 \cdot (C_{r,p_1}(O))^{p/p_1}$ for some constant $C_1$.

For an arbitrary set $B \subset X$, we take an open set $G \supset B$. Then

$$C_{r,p}(U_t(B)) \leq C_{r,p}(U_t(G)) \leq C_1 \cdot (C_{r,p_1}(G))^{p/p_1}.$$

Taking infimum for all open sets $G \supset B$, we have

$$C_{r,p}(U_t(B)) \leq C_1 \cdot (C_{r,p_1}(B))^{p/p_1}.$$

For the first inequality of (3.1), we have to note that for an arbitrary set $B$, $C_{r,p_2}(B) = C_{r,p_2}(U_{-t} \circ U_t(B))$ by Theorem 2.3. Thus we have

$$C_{r,p_2}(B) = C_{r,p_2}(U_{-t} \circ U_t(B)) \leq C_1 \cdot (C_{r,p}(U_t(B)))^{p_2/p},$$

by the second inequality of (3.1). Hence

$$C_2 \cdot (C_{r,p_2}(B))^{p/p_2} \leq C_{r,p}(U_t(B)),$$

for some constant $C_2$. The proof is complete. ∎

**Corollary 3.6.** The flow $U_t(x)$ constructed in Theorem 2.3 preserves the class of slim sets, that is, if $B \subset X$ is slim in the sense that $C_{r,p}(B) = 0$ for every $r > 0$ and $1 < p < \infty$, then $U_t(B)$ is also slim for every $t \in \mathbb{R}$.

Finally, we give a typical application. Let $F : W \to \mathbb{R}^d$ be a $d$-dimensional Wiener mapping satisfying $F \in W_\infty^\infty(X;\mathbb{R}^d)$ and $(\det(\nabla F|\nabla F))^{-1} \in \cap_{1<p<\infty} L^p$. By taking a suitable modification, if necessary, we may assume that $F$ is quasi continuous. Let $A \in W_\infty^\infty(X;H)$ be a vector field which satisfies the conditions (i), (ii) and (iii) of Theorem 2.1. Assume further that

$$< A(x), \nabla F(x) >_H = 0, \quad \text{a.a. } x(\mu).$$

For $a \in \mathbb{R}^d$, let $S_a = \{x | F(x) = a\}$ and $\nu_a = \det < \nabla F|\nabla F >^{-1/2} (x)\delta_a(F)$ be the area measure on $S_a$ induced from $\mu$ (cf. Airault-Malliavin [1], Sugita [13]). Assume that $\nu_a \neq 0$. Since every slim set is a $\nu$-null set (cf. [1], [13]), the flow $U_t(x)$ is a well-defined random element on the measure space $(S_a, \nu_a)$. Now we can conclude that, for a.a. $x$ $(\nu_a)$, $U_t(x)$ has the following property:

(i) $U_t(x) \in S_a$ for all $t \in \mathbb{R}$,

(ii) $U_{t+s}(x) = U_t \circ U_s(x)$ for all $t, s \in \mathbb{R}$, that is, $U_t(x)$ defines a flow on $S_a$. For the proof, we first note that $F$ is quasi continuous and by Theorem 3.5 we can deduce easily that $F \circ U_t(x)$ is quasi continuous. Since

$$\frac{d}{dt}(F \circ U_t(x)) = < \nabla F \circ U_t(x), AU_t(x) >_H = 0, \quad \mu - \text{a.e. } x,$$

we can conclude that $F \circ U_t(x) = F(x)$, a.a $x(\mu)$. Hence by the quasi continuity, we deduce that

$$F \circ U_t(x) = F(x), \quad \text{quasi everywhere.}$$

In particular, $F \circ U_t(x) = F(x)$, $\nu_a$-a.e. Hence (i) is proved. Also (ii) holds $\nu_a$-a.e. because it holds quasi everywhere by Theorem 2.3.

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