A Note on the Idempotent in a Ring

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槪 要

環에서 멱등원 (idempotent)의 성질을 이용하여 그 元으로 生成된 環 eRe의 根基(radical)가 R의 根基와 같음을 이용하여 R이 J-semisimple일때 eRe도 또한 J-semisimple이 됨을 보였고 또 다른 멱등원 u에 의하여 生成된 ideal Ru와 Re의 同形관계와 일치할 조건 및 멱등원이 環R에서 원시원 (primitive)이 될 조건과 環 R이 正則일때 멱등원이 원시원이 되려면 eRe가 體 (division ring)가 됨을 보임

An element e of a ring R is said to be an IDEMPOTENT if $e^2=e$.

LEMMA-1. If e is an idempotent of the ring R, then eRe is a ring with unity e.

LEMMA-2. If I is an ideal and e is an idempotent element of the ring R, then the subring $eIe = eRe \cap I$.

PROOF. Assume that $r = ere \in (eRe) \cap I$, then $r = ere \in eIe$. Thus $(eRe) \cap I \subset eIe$. Next, assume that $r \in eIe \subset I$, then $r = ere \in eRe$ since $I \subset R$. Thus $r = ere \in (eRe) \cap I$. Hence, $eIe = (eRe) \cap I$.

LEMMA-3. Let R be a ring, if e is an idempotent in R, then $R = eR \oplus (1-e)R = Re \oplus R(1-e)$.

PROOF. If $r \in R$, then $r = er + (r - er)$. Hence we have $R = eR + (1-e)R$, where $(1-e)R = \{r - er \mid r \in R\}$. But $eb = b$ for all b in eR and $eb = o$ for all b in $(1-e)R$, so that $eR \cap (1-e)R = (o)$ and thus $R = eR \oplus (1-e)R$.

Moreover, $eRe = eR \cap Re$, $eR(1-e) = eR \cap R(1-e)$,

$(1-e)Re = (1-e)R \cap Re$, $(1-e)R(1-e) = (1-e)R \cap R(1-e)$.

And we can write

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e).$$

This representation is called a two sided Peirce decomposition of R relative to e.

The prime radical of a ring R, denoted by Rad R, is the set

$$\text{Rad } R = \cap \{P \mid P \text{ is a prime ideal of } R\}.$$

Remark: Rad R is a nilpotent. (#1)

An element a of the ring R is quasi-regular iff there exists some b in R such that $a + b - ab = o$. The element b is called a quasi-inverse of a. The J-radical J(R) of a ring R, with or without an identity, is the set

$$J(R) = \{a \in R \mid ar \text{ is quasi-regular for all } r \in R\}.$$

If $J(R) = (o)$, then R is said to be a J-semisimple ring.

THEOREM-4. If e and e^* are two idempotent of the ring R such that $e - e^* \in \text{Rad } R$, then $e = e^*$.

PROOF. Consider the product $(e - e^*)(1 - (e + e^*)) = o$. Now, one may write

$$1 - (e + e^*) = (1 - 2e) + (e - e^*)$$

where $(1-2e)^2 = 1 - 4e + 4e^2 = 1 - 4e + 4e = 1$.

Hence, $1-(e+e^*)$ is the sum of an invertible element and a nilpotent element, (since Rad R is a nilpotent). It is necessarily invertible in R(#1). So that $e=e^*$.

THEOREM-5. If Ri s a J-semisimple, then eRe also a J-semisimple.

PROOF. We claim that $J(eRe)=(o)$ if $J(R)=(o)$. Then, first we will show that $J(eRe)=eJ(R)e=J(R)\cap eRe$. It is clear that $J(R)\cap eRe=eJ(R)e$ (by LEMMA-2) and that this is a quasi-regular ideal in eRe. Hence $eJ(R)e\subset J(eRe)$.

Suppose $z\in J(eRe)$. By the two-sided peirce decomposition of R, we can write x in R as $x=x_{11}+x_{10}+x_{01}+x_{00}$ where $x_{11}\in eRe$, $x_{10}\in eR(1-e)$, $x_{01}\in(1-e)Re$, and $x_{00}\in(1-e)R(1-e)$. Then $zx=zx_{11}+zx_{10}$, since $zx_{01}=zex_{01}=zx_{00}=zex_{00}=o$. Now zx_{11} has a quasi-inverse, ie. z^* in eRe. Since $zx_{10}z^*=o$, we have $zx+z^*+zxz^*=zx_{10}$. Moreover, $(zx_{10})^2=o$ and hence zx_{10} is quasi-regular, since $zx_{10}+(-zx_{10})+(-zx_{10})zx_{10}=o$. Therefore zx is quasi-regular for every $x\in R$, since the quasi-regular elements of R form a group under the circle composition. Thus $zR\subset J(R)$. Hence bza is quasi-regular for every a, b in R. But then $z\in J(R)$ and $z\in eRe\cap J(R)=eJ(R)e$. Hence $J(eRe)\subset eJ(R)e$. Thus $J(eRe)=eJ(R)e$. Hence if $J(R)=(o)$, then $eJ(R)e=J(eRe)=(o)$. Proved.

Now, consider the ideals generated by a nonzero idempotent e of the ring R, say eR or Re.

LEMMA-6. If e and u are the two nonzero idempotent of the ring R. then $eR=uR$ if and only if $eu=u$ and $ue=e$.

PROOF. Suppose that $eR=uR$, let u in R, then $eu=u\cdot u=u^2=u$. Similarly, $e=e\cdot e=ue$ ie, $ue=e$.

Conversely, if $eu=u$, $ue=e$ and $a\in eR$, then $a=er$ for some $r\in R$. Since $a=er=uer=ur^*$ ($r^*=er\in R$), $a\in uR$. Hence $eR\subset uR$. Similarly, $uR\subset eR$.

THEOREM-7. If e and u are two idempotent of the ring R, then $eR\approx uR$ as R-modules if and only if there exist r, s in R such that $rs=u$ and $sr=e$.

PROOF. Suppose $eR\approx uR$ and let $ex\to ur^*ex$, $x\in R$, be the isomorphism. Let $uy\to es^*uy$, $y\in R$, be its inverse. Then $ees^*uur^*e=e$ and $uur^*ees^*u=u$. Let $r=ur^*e$ and $s=es^*u$. Thus $rs=u$ and $sr=e$.

Conversely, suppose that $sr=e$ and $rs=u$. Then the homomorphism $ex\to rex=rsrx=rs\cdot rx=urx\in uR$ for $ex\in eR$ has the mapping $uy\to suy$ for $uy\in uR$ as inverse. Hence $eR\approx uR$.

COROLLARY. $Re\approx Ru$ as R-modules if and only if there exist r, s in R such that $rs=u$ and $sr=e$.

Now, we can prove the following theorem.

THEOREM-8. The ideals eR and uR are isomorphic as R-modules if and only if the ideals Re and Ru are isomorphic as R-modules.

PROOF. If $eR\approx uR$ if and only if there exist r,s in R such that $rs=u$, $sr=e$ if and only if $Re\approx Ru$.

LEMMA-9. Let e and u be idempotents in the ring R with 1, and let J be the radical of R. Suppose $r^*s^*=e(\mathrm{mod}\ J)$ and $s^*r^*=u(\mathrm{mod}\ J)$. Then there exist r and s in R such that $rs=e$ and $sr=u$.

PROOF. $us^*\equiv s^*r^*s^*\equiv s^*e$ and $er^*\equiv r^*s^*r^*\equiv r^*u$ (mod J) imply $er^*us^*\equiv r^*us^*\equiv r^*s^*e\equiv e^2\equiv e(\mathrm{mod}\ J)$. Therefore $x=e-er^*us^*\in J$ and since $x=ex$, $er^*us^*=e(1-x)$. Now $x\in J$ and hence there exists $y\in J$ such that $(1-x)(1-y)=1$. Let $s=us^*(1-y)$ and $r=er^*u$. Then $rs=er^*us^*(1-y)=e(1-x)(1-y)=e$ and hence $(sr)^2=srsr=ser=sr$. This implies $(u-sr)^2=u-usr-sru+sr=u-sr$. But since $sr=us^*(1-y)er^*u\equiv us^*er^*u\equiv u^3\equiv u(\mathrm{mod}\ J)$, u−

$sr \in J$. Hence $u = sr$.

THEOREM-10. Let $N \subset$ radical of R, $f: R \to R/N$ canonically and e and u are idempotent of R, then $eR \approx uR$ as R-modules if and only if $feR \approx fuR$ as R-modules.

PROOF. By THEOREM-7, LEMMA-9.

THEOREM-11. If a ring R with 1 has no nilpotent element, then every idempotent of R is in the center of R.

PROOF. Consider the element $ex(1-e)$, then $(ex(1-e))^2 = ex(1-e)ex(1-e) = (ex-exe)(ex-exe) = exex - exexe - exeex + exeexe = exex - exexe - exex + exexe = o$. Thus $ex(1-e) = o$ since R has no nilpotent element. Hence $ex = exe$.

Similarly, $((1-e)ex)^2 = ((1-e)xe)((1-e)xe) = (xe-exe)(xe-exe) = xexe - xeexe - exexe + exeexe = xexe - xexe - exexe + exexe = o$. Thus $(1-e)xe = o$, ie, $xe = exe$. Hence $ex = xe$. So that e is in the center of R.

Let $e_1, \dots, e_n$ be nonzero idempotents in a ring R. They are mutually orthogonal if $e_i e_j = o$ whenever $i \neq j$. In this case $e = e_1 + e_2 + \dots + e_n$ is also an idempotent. An idempotent is PRIMITIVE if it cannot be written as the sum of two orthogonal idempotents. Remark: It is well known that e is primitive iff Re is minimal ideal generated by e.

THEOREM-12. An idempotent $e \neq o$ of the ring R is primitive if and only if R contains no idempotent $g \neq e$ such that $eg = ge = g$.

PROOF. Suppose e is not primitive, then $e = g + h$, with $g \neq o$, $h \neq o$ orthogonal idempotents. Thus $ge = g^2 + gh = g$ and $eg = g^2 + hg = g$. Therefore $ge = eg = g$ but $e \neq g$. Hence contradiction for R.

Conversely, if there exist g in R such that $o \neq g^2 = g \neq e$ and $g = ge = eg$, then g and $e - g$ are nonzero orthogonal idempotents whose sum is e. Hence e is not primitive.

THEOREM-13. Any idempotent e in a nil-semisimple left Artinian ring R is the sum of a finite number of orthogonal primitive idempotents.

PROOF. Let $I = Re$, where $e \neq o$ is an idempotent. If I is minimal, then e is primitive and theorem proved. If I is not minimal, there exists a minimal left ideal J_1 of R such that $J_1 \subset I$.

Then, there exists an ideal J_1^* such that $J_1^* \neq (o)$ and $I = J_1 \oplus J_1^*$ and there exist orthogonal idempotents e_1, e_1^* such that $J_1 = Re_1$, $J_1^* = Re_1^*$, and $e = e_1 + e_1^*$. Since J_1 is minimal, e_1 is primitive. If J_1^* is minimal, then e_1^* is primitive and we are finished. If J_1^* is not minimal, we decompose it as $J_1^* = J_2 \oplus J_2^*$ as above, where e_2 and e_2^* are orthogonal idempotent generators of J_2 and J_2^*. Since J_2 is minimal, e_2 is primitive and $e = e_1 + e_2 + e_2^*$. Now e_1 and e_2 are orthogonal since $e_1 e_1^* = o$ and thus $e_1 e_2 + e_1 e_2^* = o$ while $e_2^* e_2 = o$, giving us $o = (e_1 e_2 + e_1 e_2^*) e_2 = e_1 e_2 + e_1 e_2^* e_2 = e_1 e_2$ and similary $e_2 e_1 = o$.

After n steps we obtain

$I = J_1 \oplus J_2 \oplus \cdots \oplus J_n \oplus J_n^*$, $J_i = Re_i, (i = 1, 2, \cdots n)$

$J_n^* = Re_n^*$, $e_1, \cdots, e_n$ mutually orthogonal and primitive and $e = e_1 + \cdots + e_n + e_n^*$.

THEOOREM-14. An idempotent $e \neq o$ of R is primitive if and only if eRe contains no idempotent other then o and e.

PROOF. Assume that ere is an idempotent for some r in R, and let e be primitive. Then, since

$(e-ere)^2=e^2-e^2re-ere^2+(ere)(ere)=e-ere-ere+ere=e-ere$, $e-ere$ is also idempotent. At the same time, $ere(e-ere)=(e-ere)ere=o$.
Hence, $e=ere+(e-ere)$, where ere and $e-ere$ are idempotent and orthogonal. From the primitivity of e, either $ere=o$ or $ere=e$.

Conversely, if e is not primitive, then we may write we may write $e=u+v$, where u and v are nonzero orthogonal idempotents. Hence, $u\neq e$ and $eu=ue=u$, which implies that the element $u=eue$ is in eRe.

THEOREM-15. If R is a regular ring and e an idempotent, then e is primitive if and only if eRe is a division ring.

PROOF. Suppose e is primitive and a in eRe, $a\neq o$. Then Re is minimal and $a\in Re$ and so $Ra\subset Re$. Hence $Ra=Re$ or $Ra=(o)$. But $a=ea\in Ra$, so that $Ra\neq(o)$. Therefore $Ra=Re$. Thus $e\in Ra$, ie, there is an $x\in R$ such that $e=xa$. Then exe is a left inverse in eRe for a. Hence eRe is a division ring.

Coversely, if eRe is a division ring and that I is a left idel of R with $I\subset Re$. Then eI is a left ideal in eRe. Hence either $eI=(o)$ or $eI=eRe$. If $eI=(o)$, then $I^2\subset ReI=(o)$ and $I=(o)$ since R is regular, R has no nonzero nilpotent ideal. (#6) Now suppose that $eI=eRe$. Then there is an $x\in I$ such that $ex\in eRe$ and $ex\neq o$. Also, $exe=ex$ since e is the identity for eRe.

Moreover, ex has an inverse in eRe, say eye. Then $(eye)(exe)=e$ and $e\in Rexe=Rex\subset I$. Then $Re\subset I$ and $I=Re$, so that Re is a minimal left ideal of R. Hence e is a primitive.

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