On the w^Δ-spaces

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w^Δ-공간에 관한 연구

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Summary

Given a definition of the w^Δ-space, we have proved what a w^Δ-space is, and a metrization of the w-space with some conditions.

Furthermore, we will have an open problem:

Is a collectionwise normal w-space with a G_δ-diagonal metrizable?

Introduction

In this paper, we will show the following.

(a) What space is a w^Δ-space?

(b) What is the property of a w^Δ-space with somespace?

(c) What w^Δ-space is metrizable?

To do this we will introduce the following propositions.

propositions:

(1) Every collectionwise normal Moore space is metrizable.

(2) A T_2-space X is paracompact iff X is subparacompact and collectionwise normal.

(3) A compact T_2-space with a G_δ-diagonal is metrizable.

(4) If X is a T_2-space, then X is compact iff X is countably compact and subparacompact.

(5) Every developable space is subparacompact.

(6) A regular subparacompact space with a G_δ-diagonal has a G_δ*-diagonl.

Main Theorems

Definition

A space X is a w^Δ-space iff there exists a sequence $G_1, G_2, \ldots$ of open covers of X such that if $x \in X$ and $x_n \in \mathrm{st}(x, G_n)$, $n=1,2,\ldots$, then the sequence $x_1, x_2, \ldots$ has a cluster point.

It follows from Definition that developable spaces and countably compact spaces are w^Δ-spaces.

Theorem 1

A locally compact subparacompact space is a w^Δ-space.

Proof: Let X be a locally compact subparacompact space. Then for each $x \in X$, there exists an open set O_x such that $x \in O_x$ and $\bar{O}_x$ is compact. So, $\{O_x : x \in X\}$ is an open cover of X.

Let $A_1, A_2, \ldots$ be open covers of X such that $A_1 = \{O_x : x \in X\}$ and for each n, A_{n+1} refines A_n and given n and x, there exists m such that $\mathrm{st}(x, A_m) \subset V_n \in A_n$.

Let $x \in X$ and $O_y \in A_1$ such that $O_y \supset \mathrm{st}(x, A_m) \supset \mathrm{st}(x, A_{m+1}) \cdots$. Let $x_n \in \mathrm{st}(x, A_n)$.

Then $x_m, x_{m+1}, \rightarrow O_y$ and hence $x_1, x_2, \ldots$ has a cluster point. Therefore, X is a w^Δ-space.

Theorem 2

Every w^Δ-space with a G_δ*-diagonal is developable.

Proof: Let X be a w^4-space with a G_δ^*-diagonal. Then there exists $G_1, G_2, \ldots$, open covers of X such if $x \in X$ and for each n, $x_n \in \mathrm{st}(x, G_n)$ then the sequence $x_1, x_2, \ldots$ has a cluster point. Furthermore, there exists a sequence $H_1, H_2, \ldots$ open covers of X such that $x \neq y \in X$, then there exists n such that

$y \notin \overline{\mathrm{st}(x, H_n)}$.

Let $K_1 = \{g_1 \cap h_1 : g_1 \in G_1,\ h_1 \in H_1\}$ and for each $n \geq 2$, let $K_{n+1} = \{g_{n+1} \cap h_{n+1} \cap k_n : g_{n+1} \in G_{n+1} \in H_{n+1},\ k_n \in K_n\}$.

Then $K_1, K_2, \ldots$ is a sequence of open covers of X.

Let $x \in X$ and U an open set of x.

Suppose for each n, st $(x, K_n) - U \neq \phi$.

Let $x_n \in \mathrm{st}(x, K_n) - U$for each n. Then $x_n \in \mathrm{st}(x, G_n)$ for each n.

Let y be a cluster point of the sequence $x_1, x_2, \ldots$ and $y \neq x$.

Then there exists $n \in N$ such that $y \in \mathrm{st}(x, H_n)$.

So, $y \notin \overline{\mathrm{st}(x, K_n)}$. On the other hand, $y \in X$-$\overline{\mathrm{st}(x, K_n)}$ and$X$-$\overline{\mathrm{st}(x, K_n)}$ is open.

Since st $(x, K_n) \cap (X - \mathrm{st}(x, K_n)) = \phi$, we have a contradiction.

Hence, $\mathrm{st}(x, K_n) \subset U$. Therefore, X is developable.

Theorem 3

A regular countably compact space with a G_δ^*-diagonal is metrizable.

Proof: Let X be a regular countably compact space with a G_δ^*-diagonal.

Since X is countably compact, then X is a w^4-space and so by Theorem 2, X is developable. From proposition (5), X is subparacompact.

From proposition (5), X is compact. Since X has a G_δ^*-diagonal, then X has a G_δ-diagonal. From proposition (3), X is metrizable.

Theorem 4

The following are equivalent for a regular space X.

(1) X is a Moore space.

(2) X is a w^4-space with a G_δ^*-diagonal.

(3) X is a subparacompact w^4-space with a G_δ-diagonal.

Proof: (1)⇒(2)

Let $G_1, G_2, \ldots,$ be a sequence of open covers of X such that if $x \in U$ open, then there exists n such $\mathrm{st}(x, G_n) \subset U$.

Suppposse $x \neq y$.

Since X is regular, there exists an open U such that $x \in U$ $\bar{U} \subset X - \{y\}$ and there exists n such that $\mathrm{st}(x, G_n) \subset U$

So, $y \notin \overline{\mathrm{st}(x, G_n)}$.

Hence X has a G_δ^*-diagonal.

For each n, let $x_n \in \mathrm{st}(x, G_n)$.

Since $\mathrm{st}(x, G_n)$, $n = 1, 2, \ldots$ is a base at x, then $x_1, x_2, \ldots$ has x as a cluster point.

Therefore, X is a w^4-space.

(2)⇒(3)

From Theorem (2), X is developable and so X is subparacopact by proposition (5).

Since X has a G_δ^*-diagonal, then X has a G_δ-diagonal.

Therefore, X is a subparacompact w^4-space with a G_δ-diagonal.

(3)⇒(1)

From Proposition (6), X has a G_δ^*-diagonal.

Hence, X is developable from Theorfm 2.

Therefore, X is a Moore space.

Theorem 5

A T_2-space X is metrizable iff X is a paraco-mpact w^4-space with a G_δ-diagonal.

Proof : Suppose X is metrizable.

Then X is paracompact and so by Proposition (2), X is subparacompact and collectionwise normal. Furtherrmore, X is normal and regular.

Since X is metrizable, then X is a Moore space.

From Theorem 4, X is a subparacompact w^4 -space with a G_δ -diagonal.

But then X is paracompact.

Therefore X is a paracompact w^4 -space with a G_δ -diagonal.

For the converse, suppose X is a paracompact w^4 -space with a G_δ-diagonal.

From Proposition (2) and Theorem 4, X is subparacompact and collectionwisenormal, and X is a Moore space.

From proposition (1), X is metrizable.

Conclusion.

From Definition, if we required x to be a cluster poiut of the sequence $x_1, x_2, \ldots$, then st (x, G_n) would be a base at x and we would have defined a developable space.

This paper ends by giving an open problem.

Is a collectionwise normal w^4 -space with a G_δ -diagonal metrizable?

Literatures Cited

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〈초 록〉

w^4－공간에 관한 연구

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도입과정에서 주어진 명제의 w^4-space의 정의를 가지고 w^4-space가 될 수 있는 공간, 그리고 어떤조건을 주어서 이의 거리화 문제를 증명하였다. 그러나 어떤 조건을 가진 w^4-space는 거리화가 될 수 있는지의 미해결문제를 결론에서 제시함으로 이 논문을 끝내고자 한다.