Metrization on M-spaces

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Introduction

We shall prove what a space is an M-space, and what an M-space is metrizable. We begin by defining an M-space.

Main Theorems

Definition 1 A space X is an M-space iff there exists a sequence $G_1, G_2 \cdots$ of open covers of X such that

(1) For each n, G_{n+1} is a point-star refinement of G_n,

(2) if $x_n \in st(x, G_n)$, n = 1,2, ⋯ , then the sequence $x_1, x_2, \cdots$ has a cluster point.

It follows from Definition 1 that if instead of (2) we had x as a cluster point of $x_1, x_2, \cdots$, then $\{ st(x, G_n): n = 1,2, \cdots\}$ would be a base at x, and hence X would be metrizable if X is a To-space,and every M-space is a W^*-space.

Theorem 1 Every countably compact metric space is an M-space

Proof Let X be a countably compact metric space with a metric d. Let $B_\epsilon(x) = \{ y \in X : d(x, y) < \epsilon, \epsilon > 0 \}$. Then $\{ B_\epsilon(x) : x \in X, \epsilon > 0 \}$ is a base for the topology. For each n = 1,2,3, ⋯ , let $G_n = \{ B_{\frac{1}{n}}(x) : x \in X \}$ then each G_n is an open cover of X. So $\{G_n\}_{n \in N}$ is a sequence of open covers of X. Clearly, for each $n \in N$ G_{n+1} is a point-star refinement of Gn. If $x_n \in st(x, G_n)$, n = 1,2, ⋯ , then $x_1, x_2, \cdots$ has a cluster point since X is countably compact. Therefore X is an M-space.

Theorem 2 A paracompact T_2, W^*-space is an M-space.

Proof Let X be a paracompact T_2, W^*-space. Then we have a nested sequence $\{ G_n \}$ of open covers of X such that whenever $x \in X$ and $x_n \in st(x, G_n)$, $x_1, x_2, \cdots$ has a cluster point. We have known that a T_1 -space is paracompact iff each open cover has an open point-star refinement [2]. So each G_n has a sequence $\{ G_{n\,k} \}_{k=1}^{\infty}$ of open covers of X such that each $G_{n\,k+1}$ is a point-star refinement of $G_{n\,k}$. Let $H_1 = G_{11}$, $H_2 = G_{11} \cap G_{22}$ and for each $n > 2$, $H_n = G_{1\,n} \cap G_{2\,n} \cap \ldots \cap G_{n\,n}$. If $x \in X$, then $st(x, H_{n+1}) \subset st(x, G_{1\,n}) \cap \ldots \cap st(x, G_{n+1\,n}) \subset G_{1\,n} \cap \ldots \cap G_{n\,n} \in H_n$ for some $G_{1\,n} \in G_n$, ... So each H_{n+1} is a point-star refinement of H_n. Clearly, if $x_n \in st(x, H_n)$, then $x_n \in st(x, G_n)$, and $x_1, x_2, \cdots$ has a culster point.

Lemma 1 Let $G_1, G_2, \cdots$ be a sequence of opne covers of a space X satisfying conditions (1) and (2) in Definition 1. For each $x \in X$, let $C_x = \bigcap_{n=1}^{\infty} st(x, G_n)$, then

(a) each C_x is a closed countably compact subset.

(b) $\{C_x : x \in X\}$ is a partition of X.

Proof (a) Pick $x \in X$. Let $w \in \bar{C}_x$.
If $n \in N$, there exists $G \in G_{n+2}$ such that $w \in G$: G must meet C_x and so G meets st(x, G_{n+2}). So $w \in st(st(x, G_{n+1}), G_{n+2}) \subset st(x, G_n)$. Hence $w \in C_x$ and $\bar{C}_x = C_x$. Therefore C_x is closed. If $x_1, x_2, \cdots$, is a sequence in C_x, then for each $n \in N$, $x_n \in st(x, G_n)$, so $x_1, x_2, \cdots$ has a cluster point. So C_x is countably compact. Therefore each C_x is a closed countably compact subset.

(b) Suppose $C_x \cap C_y = \phi$.
Then for each n, $st(x, G_n) \cap st(y, G_n) = \phi$. Let $z \in C_x$, then for each n, $z \in st(x, G_{n+4})$ which meets $st(y, G_{n+4})$ and so $st(st(z, G_{n+4}), G_{n+4})$ meets $st(y, G_{n+4})$. Since $st(st(z, G_{n+4}), G_{n+4}) \subset st(z, G_{n+2})$ then $st(z, G_{n+2})$ meets $st(y, G_{n+2})$ and $z \in st(st((y, G_{n+2}), G_{n+2})) \subset st(y, G_n)$. So $z \in C_x$. Therefore, $C_x \subset C_y$. Similarly as before we have $C_y \subset C_x$. Hence $C_x = C_y$. Therefore, $\{C_x : x \in X\}$ is a partition of X.

Lemma 2 A continuous f: $X \to Y$ is closed iff whenever $y \in Y$ and U is an open set containing $f^{-1}(y)$, then there exists an open set V containing y such that $f^{-1}(V) \subset U$.

Proof Suppose a continuous map f: $X \to Y$ is closed. Let $y \in Y$ and U an open set contaning $f^{-1}(y)$. Let $V = Y - f(X-U)$, then V is open. Observing that $f^{-1}(V) = X - f^{-1}(f(X-U)) \subset X - (X-U) = U$ completes "only if" part. For the converse, let F be closed in X, and suppose that f(F) is not closed. Let $y \in Y - f(F)$ be a limit point of f(F). Then $f^{-1}(y) \in X - F$. So there exists an open set V containing y such that $f^{-1}(V) \subset X-F$. Let $p \in V \cap f(F)$, then there exists $x \in F$ such that $f^{-1}(x) = p$. Now, $f^{-1}(x) \in f^{-1}(V) \subset X-F \Rightarrow x \notin F$. We have a contradiction. Therefore f(F) is closed and f is closed.

Theorem 3 A space X is an M-space iff there exists a metric space Y and a closed continuous map f:$X \to Y$ from X onto Y such that $f^{-1}(y)$ is countably compact for each $y \in Y$.

Proof. Suppose X is an M-space. There exists a sequence $\{G_n\}$ of open covers of X satisfying Definition 1. For each $x \in X$, let $C_x = \bigcap_{n=1}^{\infty} st(x, G_n)$, then $\bar{C}_x = C_x$ by Lemma 1. We first show that if $p \in X$ and $U \supset C_p$ is open in X, there exists an $n \in N$ such that $st(p, G_n) \subset U$. Suppose that for each $n \in N$, $st(p, G_n) \not\subset U$. For each $n \in N$, let $p_n \in st(p, G_n) - U$, then $p_1, p_2 \cdots$ has a cluster point q. Let $n \in N$. For each $m > n$, let $H_{p_m} \in G_m$ such that $st(p_m, G_{n+1}) \subset H_{p_m}$. Let $m > n$ such that $p_m \in st(q, G_{n+1})$. Then $p \in st(p_m, G_m) \subset st(p_m, G_{n+1})$ and hence $p, q \in H_{p_m}$. Thus $q \in st(p, G_n)$ and $q \in C_p$. We have a contradiction. Therefore if $p \in X$ and $U \supset C_p$ is open in X there exists an $n \in N$ such that $st(p, G_n) \subset U$. Let $Y = \{C_x : x \in X\}$. Define f: $X \to Y$ by for each $x \in X$, $f(x) = C_x$. Then f is onto and $f^{-1}(C_x) = C_x$ for each $x \in X$. By Lemmal, each $f^{-1}(C_x)$ is countably compact for each $C_x \in Y$. Define the topdogy on Y as an identification topology determined by f. Clearly, f is continuous. Therefore, f is continuous, closed and whenever $C_p \in Y$ and U is an open set containing $f^{-1}(C_p)$ then there exists an open set V containing C_p such that $f^{-1}(V) \subset U$. Next we want to prove that Y is metrizable. We have known that a To space Y is metrizable iff there exists a sequence $\{H_n\}$ of open covers of Y with the property: for each $y \in Y$ and nbd W of y there exists a nbd V of y and an $n \in N$ such that $st(V, H_n) \subset W$ [2]. We first show that Y is To. Let C_y, $C_z \in Y$ and $C_y \neq C_z$. Then $C_y \cap C_z = \phi$ by Lemma 1. Now, $C_y \subset X - C_z$ and $X - C_z$ is open by Lemma 1. By Lemma 2, there exists a nbd V of C_y such that $f^{-1}(V) \subset X - C_z$. So Y is To. For each $n \in N$, let $H_n = \{U \subset Y : U$ is open and $f^{-1}(U)$ is contained in some set of $G_n\}$. Clearly, $\{H_n\}$ is a sequence of open covers of Y. Let $n \in N$ and $C_y \in Y$. Since $C_y = \bigcap_{n=1}^{\infty} st(y, G_n)$, then $C_y \subset st(y, G_{n+1}) \subset g_n$ for some $g_n \in G_n$. Since f is closed, there exists a nbd V of C_y such that $f^{-1}(V) \subset g_n$. So $V \in H_n$. Therefore each H_n is an open cover of Y. And $\{H_n\}_{n=1}^{\infty}$ is a sequence of open covers of Y. Let $C_y \in Y$ and W a nbd of C_y.

Then $C_y \subset f^{-1}(W)$ and there exists an $m \in N$ such that $st(y, G_m) \subset f^{-1}(W)$.

Let $C_z \in st(V, H_m)$. By Lemma 2, there exists an open set V containing Cy such that $f^{-1}(V) \subset st(y, G_m)$. Let $C_z \in st(V, H_m)$ and choose $H \in H_m$ such that $C_t \in V$ and H and $C_z \in H$. But $C_t, C_z \subset f^{-1}(H) \subset g_m \in G_m$ Since $C_t \subset f^{-1}(V)$, then $C_t \subset st(y, G_m)$ and hence $C_z \subset st(y, G_m)$. So $C_z \in W$. Therefore $st(V, H_m) \subset W$. Therefore X is metrizable. For the converse, let $\{G_n\}$ be a sequence of open covers of Y such that

(1) each G_{n+1} is a point-star refinement of G_n, and

(2) if $y \in Y$ and for each $n \in N$, $y_n \in st(y, G_n)$ then $y_1, y_2, \cdots$ has a cluster point y.

For each $n \in N$, let $H_n = \{f^{-1}(g_n) : g_n \in G_n\}$. Then $\{H_n\}$ is a sequence of open covers of X. We claim H_{n+1} to be a point-star refinement of H_n. Let $x \in X$ and $y \in Y$ such that $y = f(x)$. Let $g_n \in G_n$ such that $st(y, G_{n+1}) \subset g_n$. To show $st(x, H_{n+1}) \subset f^{-1}(g_n)$, let $p \in st(x, H_{n+1})$. Let $h_{n+1} \in H_{n+1}$ such that p, $x \in h_{n+1}$. Let $g_{n+1} \in G_{n+1}$ such that $h_{n+1} = f^{-1}(g_{n+1})$. Then f(p), $f(x) \in g_{n+1}$. Thus $f(p) \in st(y, G_{n+1})$ and so $f(p) \in g_n$. Hence $p \in f^{-1}(g_n)$. So H_{n+1} is a point-star refinement of H_n. Next suppose $x_n \in st(x, H_n)$, $n = 1,2,\cdots$. For each $n \in N$, let $g_n \in G_n$ such that $x_n, x \in f^{-1}(g_n)$. Then $y = f(x)$, $f(x_n) \in g_n$ and $f(x_n) \in st(y, G_n)$, $n = 1,2,\cdots$. So $f(x_1)$, $f(x_2)$, $\cdots$ has a cluster point y. Suppose no point of $f^{-1}(y)$ is a cluster point of $\{x_1, x_2, \cdots\}$. For each $x \in f^{-1}(y)$, let U_x be a nbd of x and $n_x \in N$ such that if $m > n_x$, $x_m \notin U_x$. For each $n \in N$, let $U_n = \cup \{U_x : x \in X$ and $n_x = n\}$. Then $U_1, U_2, \cdots U_{n_m}$ be a finite subcover of $f^{-1}(C_y)$. Let V be a nhd of y such that $f^{-1}(V) \subset \bigcup_{k=1}^{m} U_{n_k}$. Let $n \in N$ such that if $m > n$ then $f(x_n) \in V$. Choose $\ell \in N$ such that $\ell > \max(n_1, \cdots, n_m)$. Then $f(x_\ell) \in V \Rightarrow x_\ell \in f^{-1}(V) \subset \bigcup_{k=1}^{m} U_{n_k}$. Let $k \leq m$ such that $x_\ell \in U_{n_k}$. Let $x \in X$ such that $n_x = n_k$ and $x_\ell \in U_x$. Since $\ell > n_k$, then $x_\ell \in U_x$ contradition. So $x_1, x_2, \cdots$ has a cluster point in $f^{-1}(y)$. Therefore, X is an M-space.

Definition 2 A continuous map f: $X \overset{onto}{\to} Y$ is quasi-perfect iff f is closed and $f^{-1}(y)$ is countably compact for each $y \in Y$. It follows from Theorem 3 and Definition 2 that an M-space is a quasi-perfect preimage of a metric space. Note that a perfect map is quasi-perfect.

Lemma 3 Suppose X and Y are T_2 spaces. If f: $X \overset{onto}{\to} Y$ is perfect, then X is paracompact iff Y is paracompact.

Proof It follows from [2] that X is paracompact iff Y is paracompact.

Theorem 4 For T_2 space, the following are equivalent.

(1) X is a perfect preimage of a metric space.

(2) X is a paracompact M-space.

(3) X is subparacompact or metacompact M-space.

(4) X is a paracompact W^{Δ}-space.

Proof (1) ⟹ (2):

It follows from [2] that every metric space is paracompact. So X is paracompact by Lemma 3 and an M-space by the notice of Definition 3.

(2) ⟹ (3) :

It follows from [2] that X is metacompact We have known that every paracompact space is subparacompact.

(3) ⟹ (4) :

It follows from Definition 1 that X is a W^{Δ}-space Let f: X → Y be quasi-perfect and Y a metric space. Then for each $y \in Y$, $f^{-1}(y)$ is countably compact. Since X is metacompact or subparacompact, then $f^{-1}(y)$ is compact. So f is perfect.

Note that Y is paracompact.

It follows from Lemma 3 that X is paracompact. Therefore X is a paracompact W^{Δ}-space

(4) ⟹ (1) :

It follows from Theorem 2 that M is a paracompact M-space.

Let f: X → Y be quasi-perfect and Y a metric space. For each $y \in Y$, $f^{-1}(y)$ is countably compact and also paracompact, hence metacompact.

It follows from [2] that $f^{-1}(y)$ is compact. So f is perfect.

Therefore X is a perfect preimage of a metric space.

Theorem 5. An M-space with a G_δ^*-diagonal is metrizable.

Proof Let X be an M-space with a G_δ^*-diagonal. Let f: X → Y be quasi-perfect and Y a metric space. Then for each y ∈ Y, $f^{-1}(y)$ is countably compact. It follows from [1] that X has a G_δ-diagonal. We have known that if X has a G_δ-diagonal, then $f^{-1}(y)$ has a G_δ^*-diagonal and hence $f^{-1}(y)$ has a G_δ-diagonal. Let $\{G_n\}$ be a sequence of open covers of f(y) such that whenever p, q ∈ $f^{-1}(y)$ with p≠q, there exists an n ∈ N and nbds Up, Vq of p and q, respectively, such that no member of G_n meets both U_p and V_q. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $f^{-1}(y)$, and for a fixed n, let $H_\alpha = \{U_\alpha \cap G : \alpha \in \Lambda, G \in G_n\}$, then $\{H_\alpha\}$ is an open refinement of $\{U_\alpha\}$.

For p ∈ $f^{-1}(y)$ with p≠y there exist nbds, Up and U_y of p and y, respectively, such that no member of Gn meets both Up and Uy. So $\{H_\alpha\}$ is locally finite.

Therefore $f^{-1}(y)$ is paracompact and hence metacompact. It follows from [2] that $f^{-1}(y)$ is compact.

So f is perfect.

Therefore X is a perfect preimage of metric space and hence a paracompact W^Δ-space. It follows from [1] and [2] that X is metrizable.

Conclusion In our paper we have proved an exact condition to be an M-space, and also we generalized theorem 5, [1], that is, X is metrizable iff X is paracompact T_2, W^Δ-space has a G_δ-diagonal.

Literature Cited

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國文抄錄

이 논문에서는 M－공간이 될 필요 충분 조건을 증명하고 그의 거리화 문제를 증명하였다.
또한 [1]에서 보인 정리 5를 일반화 하였음을 보였다.