UMP Unbiased Tests for Multiparameter Exponential Farmilies

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I. Introduction

A simple condition that one may wish to impose on tests of the hypothesis $H{:}\theta\in\Omega_H$ against the composite class of alternatives $K{:}\ \theta\in\Omega_K$ is that for no alternatives in K the probability of rejection should be less than the size of the test. In special cases it may of course turn out that the same test maximizes the power for all alternatives in K even when there is more than one. Uniformly most powerful(UMP) test is a test defined by a uniformly most powerful critical region.

A test ϕ for which the above condition holds, that is, for which the power function $\beta_\phi(\theta) = E_\theta\phi(X)$ satisfies

$$(1.1)\qquad \begin{aligned} &\beta_\phi(\theta)\leq\alpha \ \text{ if } \theta\in\Omega_H \\ &\beta_\phi(\theta)\geq\alpha \ \text{ if } \theta\in\Omega_K \end{aligned}$$

for a given level of significance α, $0<\alpha<1$, is said to be unbiased.

Whenever a UMP test exists, it is unbiased since its power cannot fall below that of the test $\phi(x)\equiv\alpha$. In many important testing problems, the hypothesis concerns a single real-valued parameter, but the distribution of observable random variables depends in addition on certain nuisance parameters. Specially, in an exponential family, a class of hypotheses concerns a real-valued parameter, with the remaining parameters occurring as unspecified nuisance parameters. In these cases, UMP unbiased tests exist and can be constructed by means of some theories.

II. UMP Unbiased Tests for Multiparameter Exponential Families

Let X be distributed according to

$$(2.1)\qquad dP^X_{\theta,\vartheta}(x)= C(\theta,\vartheta)\exp\Big[\theta U(x) + \sum_{i=1}^{k}\vartheta_i T_i(x)\Big]\, d\mu(x),\ (\theta,\vartheta)\in\Omega$$

and let $\vartheta= (\vartheta_1, \ldots, \vartheta_k)$ and $T=(T_1, \ldots, T_k)$.

We shall consider the problems of testing the following hypotheses H_j against the alternatives K_j, $j = 1,2,3,4$:

$$\begin{aligned} &H_1 : \theta\leq\theta_0 && K_1 : \theta>\theta_0 \\ &H_2 : \theta\leq\theta_1 \text{ or } \theta\geq\theta_2 && K_2 : \theta_1<\theta<\theta_2 . \end{aligned}$$

$H_3 : \theta_1 \leq \theta \leq \theta_2 \qquad K_3 : \theta < \theta_1 \text{ or } \theta > \theta_2$
$H_4 : \theta = \theta_0 \qquad K_4 : \theta \neq \theta_0$

We shall assume that the parameter space Ω is convex, and that it has dimension k + 1, that is, that it is not contained in a linear space of dimension $<$k + 1. This is the case in particular when Ω is the natural parameter space of the exponential family. We shall also assume that there are points in Ω with θ both $<$ and $>\theta_0, \theta_1$ and θ_2 respectively. The following lemma is well known.

Lemma 1. Let $(\mathcal{T},\mathcal{B})$ and $(\mathcal{Y},\mathcal{C})$ be Euclidean spaces, and let $P_0^{T,Y}$ be a distribution over the product space $(\mathcal{X},\mathcal{A}) = (\mathcal{T}\times\mathcal{Y},\mathcal{B}\times\mathcal{C})$. Suppose that another distribution P_1 over $(\mathcal{X},\mathcal{A})$ is such that

$$dP_1(t,y) = a(y)\, b(t)\, dP_0(t,y),$$

with $a(y) > 0$ for all y. Then under P_1 the marginal distribution of T and a version of the conditional distribution of Y given t are given by

$$dP_1^T(t) = b(t)\left[\int a(y)\, dP_0^{Y|t}(y)\right] dP_0^T(t)$$

and

$$dP_1^{Y|t}(y) = \frac{a(y)\, dP_0^{Y|t}(y)}{\int_{\mathcal{Y}} a(y')\, dP_0^{Y|t}(y')} .$$

Theorem 1. Let X be distributed according to the exponential family (2.1). Then there exist measure λ_θ and probability measure ν_t such that

(1) The sufficient statistics (U,T) which have the joint distribution

$$(2.2)\quad dP_{\theta,\vartheta}^{U,T}(u,t) = C(\theta,\vartheta)\exp(\theta u + \sum_{i=1}^{k}\vartheta_i t_i)\, d\nu(u,t)$$

(2) When T = t is given, U is the only remaining variable and the conditional disbution of U given t constitutes an exponential family

$$(2.3)\quad dP_\theta^{U|t}(u) = C_t(\theta)\exp(\theta u)\, d\nu_t(u).$$

Proof. Let (θ^0, ϑ^0) be a point of the natural parameter space, and let $\mu^* = P^X_{\theta^0,\vartheta^0}$.
Then

$$dP^X_{\theta,\vartheta}(x) = \frac{C(\theta,\vartheta)}{C(\theta^0,\vartheta^0)} \exp\left[(\theta-\theta^0)\, U(x) + \sum_{j=1}^{k}(\vartheta_j-\vartheta_j^0)\, T_j(x)\right] d\mu^*(x)$$

and the result follows from Lemma 1, with

$$d\lambda_\theta(t) = \exp(\Sigma\vartheta_i^0 t_i)\left[\int \exp\left\{(\theta_i-\theta^0)u\right\} dP^{U|t}_{\theta^0,\vartheta^0}(u)\right] dP^T_{\theta^0,\vartheta^0}(t)$$

and

$$d\nu_t(u) = dP^{U|t}_{\theta^0,\vartheta^0}(u) .$$

The following lemma is well known.

Lemma 2. Let θ be a real parameter, and let X have probability density with respect to some measure μ

$$(2.4)\quad P_\theta(x) = C(\theta)\, e^{Q(x)T(x)} h(x)$$

where Q is strictly monotone. Then there exists a UMP test ϕ for testing $H:\theta\leq\theta_0$ against $K:\theta>\theta_0$. If Q is increasing,

$$\phi(x) = \begin{cases} 1 & : T(x) > C \\ \gamma & : T(x) = C \\ 0 & : T(x) < C \end{cases}$$

where C and γ are determined by $E_{\theta_0}\phi(X) = \alpha$. If Q is decreasing, the inequalities are reversed.

Theorem 2. Let U have the conditional probability density

$$dP_\theta^{U|t}(u) = C_t(\theta)\exp(\theta u)\, d\nu_t(u)$$

Then there exists a UMP test for testing H_1 with critical function ϕ_1 satisfying

$$(2.5)\quad \phi(u,t) = \begin{cases} 1 & \text{when } u > C_0(t) \\ \gamma_0(t) & \text{when } u = C_0(t) \\ 0 & \text{when } u < C_0(t) \end{cases}$$

where the function C_0 and γ_0' are determined by

$$(2.6)\quad E_{\theta_0}\,[\phi_1(U,T) \mid t] = \alpha \text{ for all } t.$$

Proof. This theorem follows from Lemma 2.

Theorem 3. For testing the hypothesis H: $\theta<\theta_1$ or $\theta\geq\theta_2$ against the alternatives K: $\theta_1<\theta<\theta_2$, there exists a UMP test given by

$$(2.7)\quad \phi(u,t) = \begin{cases} 1 & \text{when } C_1(t)<U(x) < C_2(t) \\ \gamma_i(t) & \text{when } U(x) = C_i(t),\ i = 1,2 \\ 0 & \text{when } U(x) < C_1(t) \text{ or } > C_2(t) \end{cases}$$

where the C's and γ's are determined by

$$(2.8)\quad E_{\theta_1}\,[\phi_2(U,T)|t] = E_{\theta_2}\,[\phi_2(U,T)|t] = \alpha\,.$$

Proof. One can restrict attention to the sufficient statistic U=U(X), the distribution is

$$dP_\theta(u) = C(\theta)\, e^{\theta u}\, d\nu(u),$$

where θ is assumed to be strictly increasing. Let $\theta_1<\theta'<\theta_2$, and consider first the problem of maximizing $E_{\theta'}\psi(U)$ subject to (2.8) with $\phi(x) = \psi[U(x)]$. If M denotes the set of all points $(E_{\theta_1}\psi(U),\ E_{\theta_2}\psi(U))$ as ψ ranges over the totality of critical functions, then the point (α,α) is an inner point of M. This follows from the fact that the set M contains points (α,υ_1) and (α,υ_2) with $\upsilon_1<\alpha<\upsilon_2$ and that it contains all points (υ,υ) with $0<\upsilon<1$. Hence there exist constants k_1, k_2 and a test $\psi_0(u)$ such that $\phi_0(x) = \psi_0[U(x)]$ satisfies (2.8) and that $\psi_0(u) = 1$ when

$$k_1C(\theta_1)e^{\theta u} + k_2C(\theta_2)e^{\theta u} < C(\theta')e^{\theta' u}$$

and therefore when

$$a_1e^{b_1 u} + a_2e^{b_2 u} < 1 \quad (b_1<0<b_2),$$

and $\psi_0(u) = 0$ when the left-hand side is >1. Here not both a's can be ≤ 0 since then the test would always reject. If one of the a's is ≤ 0 and the other one is >0, then the left-hand side is strictly monotone, and the test is of the one-sided type considered in Lemma 2, which has a strictly monotone power function and hence cannot satisfy (2.8). Since therefore both a's are positive, the test satisfies (2.7). It also maximizes $E_{\theta'}\psi(T)$ subject to the weaker restriction $E_{\theta_i}\psi(T)\leq\alpha$, $i = 1,2$. Second to complete the proof that this test is UMP for testing H, it is necessary to show that it satisfies $E_\theta\psi(T) \leq\alpha$ for $\theta\leq\theta_1$ and $\theta\leq\theta_2$ This follows from the fast that this test minimizes $E_\theta\phi(X)$ subject to (2.8) for all $\theta<\theta_1$ and $>\theta_2$ by comparison with the test $\psi(t)\equiv\alpha$.

III. Example; The Sign Test

Suppose that to test consumer preference between two products A and B, each subject of n samples is recorded as plus or minus as it favors product A or B.

1. The total number Y of plus signs has binomial distribution b(p,n). Consider the problem of testing the hypothesis p=1/2 of no difference against the alternatives $p \neq 1/2$. The appropriate test is the two-sided sign test, which rejects when $|Y - \frac{1}{2}n|$ is too large, this is UMP unbiased.

2. Suppose that the subjects are also given the possibility of declaring themselves as undecided. If p_+, p_- and p_0 denote the probabilities of preference A, B, and of no preference respectively, the number X, Y and Z of decisions in favor are distributed by multinomial distribution

$$(3.1)\quad \frac{n!}{x!\,y!\,z!}\ p_-^x\ p_+^y\ p_0^z\,,\ (x+y+z = n)$$

and the hypothesis to be tested is $H{:}p_+ = p_-$. The distribution (3.1) can be written as

$$(3.2)\quad \frac{n!}{x!\,y!\,z!}\left(\frac{p_+}{1-p_0-p_+}\right)^y\left(\frac{p_0}{1-p_0-p_+}\right)^z(1-p_0-p_+)^n$$

and (3.2) constitutes an exponential family $U=Y$, $T=Z$, $\theta=\log[p_+/(1-p_0-p_+)]$, $\vartheta=\log[p_0/(1-p_0-p_+)]$. Rewriting the hypothesis H as $p_+ = 1-p_0-p_+$ it is equivalent to $\theta=0$. There exists therefore a UMP unbiased test of H, which is obtained by considering z as fixed and determining the best unbiased conditional test of H given Z = z. Since the conditional distribution of Y given z is a binomial distribution b(p, n-z) with $p = p_+/(p_+ + p_-)$,the problem reduces to that of testing the hypothesis p = 1/2 in a binomial distribution with n-z trials, for which the rejection region is $|Y-\frac{1}{2}(n-z)|>C(z)$. The UMP unbiased test is obtained by disregarding the number of cases in which no preference is expressed, and applying the sign test to the remaining data.

Literature cited

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國 文 抄 錄

대부분의 檢定문제에 있어서 假說은 한개의 實數値를 가진 母數와 관련되지만, 그러나 관찰 가능한 確率變數의 分布는 그외에도 nuisance 母數에 의존하게 된다. 특히 指數族에 있어서 假說類들이 明記되지 않는 nuisance 母數들과 관련되었을때 一樣最大 不偏檢定을 유도하고 sign test를 그 例로 제시하였다.