# A Note on the Grill–Determined Space and the Cauchy Filter

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## Grill– Determined 空間과 Cauchy Filter에 관한 小考

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## INTRODUCTION

It is known (Herrlich 1974a, Herrlich 1974b) the concept of nearness structures gives rise to a single method for the investigation of various known structures. e.g. topological, uniform, proximity or contiguity structures.

Moreover it is known (Hong et al 1979) the category *T–Near*(or *C–Near, P–Near*) of topological (or contigual, proximal, resp.) nearness spaces and nearness preserving maps is contained in the category *Grill* of grill-determined spaces and nearness preserving maps.

I study some properties in the grill-determined space, which is satisfying in the nearness space. In this present note, we have the most results in the grill-determined space are analogous to that of the nearness space.

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## I. PRELIMINARY

1.1. DEFINITION. Let PX denote the power set of a set X and $P^2X=PPX$. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of PX.

(1) sec $\mathcal{A}=\{B\subset X$ : for any $A\in\mathcal{A}$, $A\cap B\neq\phi\}$.

(2) stack $\mathcal{A}=\{B\subset X$ : there is $A\in\mathcal{A}$ with $A\subset B\}$.

(3) $\mathcal{A}$ is called a *stack* in X if stack $\mathcal{A}=\mathcal{A}$.

(4) $\mathcal{A}\vee\mathcal{B}=\{A\cup B : A\in\mathcal{A}$ and $B\in\mathcal{B}\}$,

$\mathcal{A}\wedge\mathcal{B}=\{A\cap B : A\in\mathcal{A}$ and $B\in\mathcal{B}\}$.

(5) $\mathcal{A}$ is said to *corefine* $\mathcal{B}$ if for any $A\in\mathcal{A}$, there is $B\in\mathcal{B}$ with $B\subset A$. In this case we denote by $\mathcal{A}<\mathcal{B}$.

(6) $\mathcal{G}\subset PX$ is called a *grill* on X if $\phi\notin\mathcal{G}$, and for any subsets A, B of X, $A\cup B\in\mathcal{G}$ iff $A\in\mathcal{G}$ or $B\in\mathcal{G}$.

1.2. PROPOSITION. Let X be a set and $\mathcal{A}$, $\mathcal{B}\subset PX$. Then

(1) sec $\mathcal{A}=\{B\subset X : X-B\notin$ stack$\mathcal{A}\}$,

stack $\mathcal{A} = \{B \subset X : X-B \notin \sec\mathcal{A}\}$.

(2) $\mathcal{A} \subset \mathcal{B}$ implies $\sec \mathcal{B} \subset \sec \mathcal{A}$.

(3) stack $\mathcal{A} = \sec^2 \mathcal{A}$, $\sec^3 \mathcal{A} = \sec \mathcal{A}$ (i.e. $\sec \mathcal{A}$ is a stack).

(4) grills(or filters) on X are stacks in X.

(5) $\mathcal{A} < \mathcal{B}$ iff $\sec \mathcal{B} < \sec \mathcal{A}$,
$\mathcal{A} < \mathcal{B}$ implies stack $\mathcal{A}$ < stack $\mathcal{B}$.

(6) $\mathcal{A} < \mathcal{B}$ iff $\mathcal{A} \subset$ stack $\mathcal{B}$.

1.3. PROPOSITION. Let SX be the set of all stacks in X, and let $\mathcal{A}$ and $\mathcal{B}$ be elements of SX. Then

(1) $\mathcal{A} < \mathcal{B}$ iff $\mathcal{A} \subset \mathcal{B}$ iff $\sec \mathcal{B} \subset \sec \mathcal{A}$.

(2) $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.

(3) $\mathcal{A} = \sec \mathcal{B}$ iff $\mathcal{B} = \sec \mathcal{A}$.

(4) $\mathcal{A}$ is a filter iff $\sec \mathcal{A}$ is a grill.

1.4. REMARK. (1) grills are precisely the union of ultrafilters.(2) If $\mathcal{B}$ is a filter base for a filter $\mathfrak{F}$, then stack $\mathcal{B} = \mathfrak{F}$.

The following definition is due to Herrlich (1974b)

1.5. DEFINITIONS. Let X be a set and let $\xi$ be a subset of $P^2X$. Consider the following axioms:

(N1) if $\mathcal{A} < \mathcal{B}$ and $\mathcal{B} \in \xi$, then $\mathcal{A} \in \xi$.

(N2) if $\bigcap \mathcal{A} \neq \phi$, then $\mathcal{A} \in \xi$.

(N3) $\phi \neq \xi \neq P^2X$.

(N4) if $\mathcal{A} \vee \mathcal{B} \in \xi$, then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$.

(N5) if $\{Cl_\xi A : A \in \mathcal{A}\} \in \xi$, then $\mathcal{A} \in \xi$,
where $Cl_\xi A = \{x \in X : \{A, \{x\}\} \in \xi\}$.

$\xi$ satisfying (N1), (N2) and (N3) is called a *prenearness structure* on X. $\xi$ satisfying (N1), (N2), (N3) and (N4) is called a *quasinearness structure* on X.

Finally $\xi$ satisfying (N1), (N2), (N3), (N4) and (N5) is called a *nearness structure* on X. The pair (X, $\xi$) is called a *(pre—, quasi—) nearness space.* A map $f:(X, \xi) \to (Y, \eta)$ between prenearness spaces is called *nearness preserving* if $\mathcal{A} \in \xi$ implies $f(\mathcal{A}) \in \eta$

1.6. DEFINITION. For a prenearness space (X, $\xi$), $\gamma(\xi)$, or shortly $\gamma$, is defined to be the family $\gamma = \{\mathcal{A} \subset PX : \sec \mathcal{A} \in \xi\}$ and is called the *associated merotopic structure* with $\xi$.

1.7. REMARK. Let (X, $\xi$) be a prenearness space and $\gamma$ be the associated merotopic structure with $\xi$. Then $\xi$ is precisely the family $\{\mathcal{A} \subset PX : \sec \mathcal{A} \in \gamma\}$. (i.e. $\mathcal{A} \in \xi$ iff $\sec \mathcal{A} \in \gamma$, and $\mathcal{A} \in \gamma$ iff $\sec \mathcal{A} \in \xi$).

1.8. DEFINITION. A prenearness space (X, $\xi$) is called *seperated* iff $\mathcal{A} \in \xi \cap \gamma$ implies $\xi(\mathcal{A}) \in \xi$, where $\xi(\mathcal{A}) = \{B \subset X : \mathcal{A} \cup \{B\} \in \xi\}$.

1.9. NOTATION. Let (X, $\xi$) be a prenearness-space and $\mathcal{A} \subset PX$.

$\mathcal{A}(<_\xi) = \{B \subset X :$ there is $A \in \mathcal{A}$ with $\{A, X-B\} \notin \xi\}$.

1.10. REMARK. In the above notation

$\sec(\mathcal{A}(<_\xi)) = \{B \subset X :$ for any $A \in \mathcal{A}$, $\{A, B\} \in \xi\}$.

1.11. PROPOSITION. If (X, $\xi$) is a prenearness space, then the following conditions are equivalent:

(1) if $\mathcal{A}(<_\xi) \in \xi$ then $\mathcal{A} \in \xi$.

(2) if $\mathcal{A} \in \gamma$ then $\mathcal{A}(<_\xi) \in \gamma$.

(3) $\mathcal{A} \in \gamma$ iff $\sec(\mathcal{A}(<_\xi)) \in \xi$.

(4) $\mathcal{A} \in \xi$ iff $\sec(\mathcal{A}(<_\xi)) \in \gamma$.

1.12. DEFINITION. A prenearness space is called *regular* iff it satisfies one of the above conditions in 1.11.

1.13. REMARK. Every regular prenearness

space is seperated.

## Ⅱ. A GRILL—DETERMINED SPACE

2.1. DEFINITION. Let (X, $\xi$) be a prenearness space. A non-empty subset $\mathcal{A}$ of PX is called:

(1) a *$\xi$-cluster* iff $\mathcal{A}$ is a maximal element of the set $\xi$, ordered by inclusion.

(2) a *$\xi$-cocluster* iff $\mathcal{A}$ is a minimal element of the set $\{\mathcal{B} \in \gamma : \mathcal{B} = \text{stack}\,\mathcal{B}\}$, ordered by inclusion.

(3) a *$\xi$-grill* iff $\mathcal{A}$ is a grill and $\mathcal{A} \in \xi$.

(4) a *$\gamma$-filter*(or *Cauchy filter*) if $\mathcal{A}$ is a filter and $\mathcal{A} \in \gamma$.

2.2. REMARK.

(1) If $\mathcal{A}$ is a filter in X then $\mathcal{A} \subset \sec \mathcal{A}$.

(2) If $\mathcal{A}$ is a grill in X then $\sec \mathcal{A} \subset \mathcal{A}$.

(3) $\mathcal{A}$ is a $\xi$-grill iff sec $\mathcal{A}$ is a Cauchy filter, and $\mathcal{B}$ is a Cauchy filter on (X, $\xi$) iff sec $\mathcal{B}$ is a $\xi$-grill.

2.3. PROPOSITION. *Let (X, $\xi$) be a prenearness space, and let $\mathcal{A}$ be non—empty stack in X. If $\mathcal{A}$ is a $\xi$-grill, then $\mathcal{A} \in \xi \cap \gamma$.*

PROOF. Since $\mathcal{A}$ is a $\xi$-grill, sec $\mathcal{A} \subset \mathcal{A}$ by 2.2(2). Thus sec$\mathcal{A} < \mathcal{A}$, so that sec$\mathcal{A} \in \xi$. Hence $\mathcal{A} \in \gamma$. But $\mathcal{A}$ is a $\xi$-grill, which implies $\mathcal{A} \in \xi$. Therefore $\mathcal{A} \in \xi \cap \gamma$.

2.4. DEFINITION. A prenearness space (X, $\xi$) is called *grill-determined* if for any $\mathcal{A} \in \xi$ there is a $\xi$-grill $\mathcal{G}$ with $\mathcal{A} \subset \mathcal{G}$

2.5. NOTATION. The category of grill—determined spaces and nearness preserving maps will be denoted by Grill (Hong et al 1978).

2,6. PROPOSITION. Let (X, $\xi$) be a prenearness space and $\gamma$ the associated merotopic structure with it. Then (X, $\xi$) $\in$ *Grill* iff for any $\mathcal{A} \in \gamma$, there is a Cauchy filter $\mathcal{F}$ with $\mathcal{F} \subset \text{stack}\,\mathcal{A}$, i.e. $\mathcal{F} < \mathcal{A}$.

PROOF. It is immediate from 1.2(3) and 2.2(3).

2.7. PROPOSITION. *Every grill determined space is a quasinearness space.*

PROOF. Let (X, $\xi$) $\in$ Grill, and suppose $\mathcal{A} \vee \mathcal{B} \in \xi$. Then there is a $\xi$-grill $\mathcal{G}$ with $\mathcal{A} \vee \mathcal{B} \subset \mathcal{G}$. If $\mathcal{A} \not\subset \mathcal{G}$ and $\mathcal{B} \not\subset \mathcal{G}$, then there is $A \in \mathcal{A} - \mathcal{G}$ and $B \in \mathcal{B} - \mathcal{G}$. Hence $A \cup B \in (\mathcal{A} \vee \mathcal{B}) - \mathcal{G}$, which is a contradiction. Therefore $\mathcal{A} \subset \mathcal{G}$ or $\mathcal{B} \subset \mathcal{G}$, so that $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$. Hence (X, $\xi$) is a quasinearness space.

## Ⅲ. THE MAIN RESULTS

3.1. LEMMA. *Let (X, $\xi$) $\in$ Grill, and let $\mathcal{A}$ be a non-empty subset of PX. If $\mathcal{A}$ is a $\xi$-cluster, then $\mathcal{A}$ is a maximal $\xi$-grill.*

PROOF. For any A, B $\in \mathcal{A}$, it is clear that $A \cup B \in \mathcal{A}$. Since $(\mathcal{A} \cup \{A\}) \vee (\mathcal{A} \cup \{B\}) < \mathcal{A}$ and $\mathcal{A} \in \xi$, $(\mathcal{A} \cup \{A\}) \vee (\mathcal{A} \cup \{B\}) \in \xi$. But (X, $\xi$) $\in$ Grill, then by 2.7 $\mathcal{A} \cup \{A\} \in \xi$ or $\mathcal{A} \cup \{B\} \in \xi$. This implies $A \in \mathcal{A}$ or $B \in \mathcal{A}$. Thus $\mathcal{A}$ is a $\xi$-grill. Assume that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ is a $\xi$-grill. Then $\mathcal{A} = \mathcal{B}$, so that $\mathcal{A}$ is maximal.

3.2. THEOREM. *Let (X, $\xi$) be a seperated grill—determined space. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is a $\xi$-cluster.*

(2) ***$\mathcal{A}$ is a maximal $\xi$-grill.***

(3) ***sec $\mathcal{A}$ is a minimal $\gamma$-filter.***

PROOF. By the sec—operator, it is obvious (2) iff (3). It suffices to show that (2) implies (1). Suppose that $\mathcal{A}$ is a maximal $\xi$-grill. Then $\mathcal{A}$ is a stack in X, and by 2.3, $\mathcal{A} \in \xi \cap \gamma$. Since (X, $\xi$) is seperated, $\xi(\mathcal{A}) \in \xi$. Obviousely $\mathcal{A} \subseteq \xi(\mathcal{A})$. If for any $\mathcal{A} \in \xi(\mathcal{A})$, $\{A\} \cup \mathcal{A} \in \xi$. And $\{A \cup B\} \cup \mathcal{A} < \{A\} \cup \mathcal{A}$. Thus $\{A \cup B\} \cup \mathcal{A} \in \xi$, so $A \cup B \in \xi(\mathcal{A})$. Note that $(\{A\} \cup \mathcal{A}) \vee (\{B\} \cup \mathcal{A}) < \{A \cup B\} \cup \mathcal{A}$. Then $A \in \xi(\mathcal{A})$ or $B \in \xi(\mathcal{A})$. Therefore $\xi(\mathcal{A})$ is a $\xi$ -grill. Since $\mathcal{A}$ is maximal, $\mathcal{A} = \xi(\mathcal{A})$. Because (X, $\xi$) $\in$ Grill, $\xi(\mathcal{A})$ is a $\xi$ -cluster.

3.3. COROLLARY. *Let (X, $\xi$) is a seperated grill-determined space. If $\mathcal{A}$ is a $\xi$ -grill then there exists a unique $\xi$ -cluster containing $\mathcal{A}$, namely $\xi(\mathcal{A})$.*

PROOF. Let $\mathcal{A}$ be a $\xi$ -grill. Then $\xi(\mathcal{A}) \in \xi$ an also $\xi(\mathcal{A})$ is a $\xi$ -cluster. Assume that $\mathcal{B}$ be any $\xi$ -cluster with $\mathcal{A} \subset \mathcal{B}$. Then $\mathcal{B} \subset \xi(\mathcal{A})$. Hence $\mathcal{B} = \xi(\mathcal{A})$.

3.4. COROLLARY. *In a seperated grill-determined space, every Cauchy filter contains a unique minimal Cauchy filter.*

PROOF. It is immediate from the sec-operator in 3.3.•

3.5. THEOREM. *Every regular grill-determined space is a nearness space.*

PROOF. Let (X, $\xi$) be a regular grill-determined space. Then (X, $\xi$) is a regular quasinearness space. We must show that $\xi$ satisfies (N5). suppose $\mathcal{A} \subset PX$ with $\{Cl_\xi A : A \in \mathcal{A}\} \in \xi$. Assume $\mathcal{A} \notin \xi$. Then $\mathcal{A}(\langle_\xi) \notin \xi$ because of being regular. Take any $B \in \mathcal{A}(\langle_\xi)$, there exist $A \in \mathcal{A}$ such that $\{A, X-B\} \notin \xi$. If $x \in X-B$, then $\{A, \{x\}\} \notin \xi$, which implies $x \notin Cl_\xi B$ and also $Cl_\xi A \subset B$. So we have $\mathcal{A}(\langle_\xi) < \{Cl_\xi A : A \in \mathcal{A}\}$. Thus $\mathcal{A}(\langle_\xi) \in \xi$. This is a contradiction.

The following is due to Herrlich (1974b)

3.6. PROPOSITION. *For any regular grill-determined space (X, $\xi$), the underlying topological space (X, $\xi_t$) is a regular space.*

3.7. LEMMA. *If (X, $\xi$) is a regular grill-determined space and $\mathcal{A} \in \xi \cap \gamma$, then*

(1) $\sec(\mathcal{A}(\langle_\xi)) = \xi(\mathcal{A})$,

(2) $\sec(\xi(\mathcal{A})) = \mathcal{A}(\langle_\xi)$.

PROOF. (1) Since (X, $\xi$) is regular, $\mathcal{A} \in \gamma$ implies $\sec(\mathcal{A}(\langle_\xi)) \in \xi$. It is obvious that $\xi(\mathcal{A}) \subset \sec(\mathcal{A}(\langle_\xi))$. But from 1.13 and 3.3, $\xi(\mathcal{A})$ is a $\xi$ -cluster. Hence $\xi(\mathcal{A}) = \sec(\mathcal{A}(\langle_\xi))$.

(2) By (1), $\sec^2 \mathcal{A}(\langle_\xi) = \sec(\xi(\mathcal{A}))$. So that stack $\mathcal{A}(\langle_\xi) = \sec(\xi(\mathcal{A}))$. On the other hand, $\mathcal{A}(\langle_\xi)$ is stack in X. Therefore $\sec(\xi(\mathcal{A})) = \mathcal{A}(\langle_\xi)$.

3.8. PROPOSITION. *If (X, $\xi$) is a regular grill-determined space and $\mathcal{A} \in \xi \cap \gamma$, then $(\sec \mathcal{A})(\langle_\xi)$ is the unique minimal Cauchy filter contained in $\mathcal{A}$.*

PROOF. If $\mathcal{A} \in \xi \cap \gamma$, then $\sec \mathcal{A} \in \xi \cap \gamma$ by 1.7. From 3.3, $\xi(\sec \mathcal{A})$ is the unique $\xi$ -cluster containing $\sec \mathcal{A}$. Then $\sec \xi(\sec \mathcal{A})$ is the unique $\xi$ -cocluster cotained in stack $\mathcal{A}$. Hence $(\sec \mathcal{A})(\langle_\xi)$ is the unique minimal Cauchy filter contained in $\mathcal{A}$.

3.9. THEOREM. *Let (X, $\xi$) be a regular grill-determined space. If $\mathcal{A}$ is a Cauchy filter, then $\mathcal{A}(\langle_\xi)$ is the unique minimal Cauchy filter contained in $\mathcal{A}$.*

PROOF. Since $\mathcal{A}$ is a Cauchy filter, $\sec \mathcal{A} \in \xi$ and also $\mathcal{A} \subset \sec \mathcal{A}$. This implies $\mathcal{A} \in \xi$. Thus $\mathcal{A} \in \xi \cap \gamma$. But from 3.8, $(\sec \mathcal{A})(\langle_\xi)$ is the unique minimal Cauchy filter contained in $\mathcal{A}$. On the other hand, $\mathcal{A}(\langle_\xi) = \sec(\xi(\mathcal{A}))$ by 3.7(2), 3.3 and

3.2 is a minimal Cauchy filter. But $\mathcal{A}(\langle_\xi)\subset(\sec \mathcal{A})(\langle_\xi)$, so that $\mathcal{A}(\langle_\xi)=(\sec \mathcal{A})(\langle_\xi)$. Hence $\mathcal{A}(\langle_\xi)$ is the unique minimal Cauchy filter contained in $\mathcal{A}$.

3.10. THEOREM. *Let* $(X, \xi)\in$ *Grill. If* $\mathcal{A}(\langle_\xi)$ *is a Cauchy filter for any Cauchy filter* $\mathcal{A}$, *then* $(X, \xi)$ *is regular.*

PROOF. Since $(X, \xi)\in$ Grill, pick any $\mathcal{B}\in\gamma$, there is a Cauchy filter $\mathcal{F}$ with $\mathcal{F}\langle\mathcal{B}$. We will show that $\mathcal{F}(\langle_\xi)\subset\mathcal{B}(\langle_\xi)$. Take any $A\in\mathcal{F}(\langle_\xi)$, there is $F\in\mathcal{F}$ with $\{F, X-A\}\notin\xi$. Now $\mathcal{F}\langle\mathcal{B}$ implies $\{F, X-A\}\langle\{B, X-A\}$ for some $B\in\mathcal{B}$. So that $\{B, X-A\}\notin\xi$, and also $A\in\mathcal{B}(\langle_\xi)$. On the other hand, $\mathcal{B}(\langle_\xi)$ is stack in X by the definition of $\mathcal{B}(\langle_\xi)$. From 1.3(1), $\mathcal{F}(\langle_\xi)\langle\mathcal{B}(\langle_\xi)$. But $\mathcal{F}(\langle_\xi)$ is a Cauchy filter, $\mathcal{F}(\langle_\xi)\in\gamma$. Therefore $\mathcal{B}(\langle_\xi)\in\gamma$.

## Literature


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## 國 文 抄 錄

Grill-Determined 공간(주로 Regular Grill-Determined 공간)에 대해 연구 조사하였다. 결과로써,

1) Regular Grill-Determined 공간은 Nearness 공간이 되며

2) Regular Grill-Determined 공간에서는, 임의의 Canchy filter $\mathcal{A}$ 를 택하면 $\mathcal{A}(\langle_\xi)$는 $\mathcal{A}$에 포함되는 유일한 Canchy filter가 되고

3) Grill-Determined 공간은, 임의의 Canchy filter $\mathcal{A}$ 에 대해 $\mathcal{A}(\langle_\xi)$ 역시 Canchy filter 이면, Regular Grill-Determined 공간이 된다.