A Note on Reimann-Stieltjes Integral

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Riemann—Stieltjes 積分에 관한 小考

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要 約

本論文에서는 복소 Riemann-stieltjes 적분에서 f가 연속이고 유계변동이며 α가 유계변동일 때 $\int_a^b fd_\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha d_f$가 성립하는데 실함수 f의 Riemann−stieltjes 적분에서는 f가 유계변동이 아니고 有界만 되어도 本定理가 成立되며 本定理로부터 f가 有界하고 α가 有界變動이며 연속일 때 Riemann−stieltjes 적분 가능함을 알 수 있다.

그리고 복소함수의 Riemann−stieltjes적분을 다른 방법으로 정의하여 보았다.

(Ⅰ) Introduction

Definition(1)

Let $\alpha(x)$ be Monotonic increas-ing function and continuous function on[a. b].

Corresponding to each partition; p $(a=x_0 < x_1 < \cdot\cdot\cdot\cdot < X_n = b)$ of [a. b], We form the sum $S(p, f, \alpha) = \sum_{i=1}^{n} f(t_i)[\alpha(X_i) - \alpha(X_{i-1})]$ $(t_i \epsilon [X_{i-1}\ X_i])$, If the sum $S(p, f, \alpha)$ tends to a limit as $\mu(p) \to 0$, then this limit is Reimann stieltjes Integral of f with respect to $\alpha(x)$ on [a, b].

In this case, we denote by $\int_a^b f(x) d(x)$ or $\int_a^b fd_\alpha$ and we say that $f(x)$ is Reimann stieltjes Integrable with respect to $\alpha(x)$ on [a, b] and write $f \epsilon R(\alpha)$.

Definition(2)

Let us, f [a, b] into R^k.

if p= $\{X_0, X_1, \ldots\ldots X_n\}$ is a partion of [a, b] and $\Delta f_i = f(X_i) - f(X_{i-1})$, we define V (f, a, b) = lub $\sum_{i=1}^{n} |\Delta f_i|$ the lub being taken over all partition of [a, b], and call V(f, a, b)the total variation of f on [a, b].

The function is said to be bounded variation on [a, b] if and only if V (f, a, b) $< \infty$.

The class of function of bounded variation is closed with respect to operation of addition and Multiplication.

Theorem 1)

A function f; I $\to$R is of bounded variation if and only if it is the difference of two non decreasing function.

Proof) The proof of sufficient condition is trivial.

Let us prove the necessity condition

Define two function g, h: I$\longrightarrow$R by taking

$g(x) = V_a^x f$

$h(x) = V_a^x f - f(x)$ for every $x \epsilon I = [a, b]$. Then $f = g - h$.

The function g is clearly non decreasing.

On the other hand, for only two real number X and y in I with $x \leq y$, We have $h(y) - h(x) = [V_a^y(f) - f(x)] - [V_a^x(f) - f(x)] = V_x^y(f) - [f(y) - f(x)] \geq V_x^y(f) - V_x^y(f) = 0$

Hence h is also non decreasing function.

(Lemma)

Let us put $U(p, f, \alpha) = \sum_{i=1}^{n}$ lub $f(x)$ $[\alpha(X_i) - \alpha(X_{i-1})]$ $X \epsilon [X_i X_{i-1}]$

$L(p, f\alpha)=\sum_{i=1}^{n} glbf(x) \, [\alpha(X_i)-\alpha(X_{i-1})]$

$f \epsilon R(\alpha)$ on [a, b] if and only if for every $\epsilon<0$.

There exist a partitition p, such that $U(p, f, \alpha)-L(p, f, \alpha)<\epsilon$

Theorem 2)

If f is continuous, and α is monotonic on [a, b], then $f \epsilon R(\alpha)$.

Theorem 3)

If f is monotonic on [a, b] and α is Continuous on [a, b], then $f \epsilon R(\alpha)$.

Theorem 4)

If f is bounded variation on [a, b] and α is continuous on[a, b], then $f \epsilon R(\alpha)$.

Definition 5)

If $f=f_1+if_2$, $\alpha=\alpha_1+i\alpha_2$ and the one of he following condition; (a) f is continuous and α is of bounded variation (b) f, α is of bounded variation, and is satisfied, we define

$\int f d\alpha=\int f_1 d\alpha_1-\int f_2 d\alpha_2+i\int f_1 d\alpha_2+i\int f_2 d\alpha_1$

as for Reimann stieltjes Integral of the complex function.

In this paper, we consider the following facts; If f is continous real function and bounded variation and α is only bounded real functions, the following hold

$\int_a^b f d_\alpha=f(b)\alpha(b)-f(a)\alpha(a)-\int_a^b \alpha \, df.$

We also intend to define the complex Reimann- stieltjes Integral for another method.

Theorem 6)

If f is continuous and bounded real function on[a, b] and α is bounded real function, then $\int_a^b f d_\alpha= f(b)\alpha(b)- f(a)\alpha(a)- \int_a^b \alpha \, df.$

Proof) If f is continuous, $\int_a^b f d_\alpha=\lim_{\mu(p)\to 0} S(p, f, \alpha)=\lim_{\mu(p)\to 0} \Sigma f(t_i) \, \alpha(X_i)$

Choose a partition $p=\{X_0 \, X_1. . . . Xn\}$ of [a, b], Choose $t_1 \, t_2 . . . . t_n$ such that put $t_0=a$ $t_{n+1}=b$ and let Q be the partition $\{t_0 t_1. . . t_{n+1}\}$ of [a, b]

$\int_a^b f d_\alpha=\lim_{\mu(p)\to 0}\mu \, \Sigma f(t_i)\Delta\alpha_i=\lim_{\mu(p)\to 0}\sum_{i=1}^{n}\alpha(t_{i-1})$ $[f(t_i)-f(t_{i-1})]$

$=f(b)\alpha(b)-f(a)\alpha(a)-\lim_{\mu(p)\to 0}\sum_{i=1}^{n+1}\alpha(X_{i-1})$ $[(t_i)-f(t_{i-1})]=f(b)\alpha(b)-f(a)\alpha(a)-$ $\lim_{\mu(Q)\to 0} S(Q, \alpha, f)=f(b)\alpha(b)-f(a)\alpha(a)-\int_a^b \alpha df,$

$if \; \mu(p)->0 \quad \mu(Q)->0.$

In the end, this theorem is able to define another extension of Reiman -Stieltjes Integral; namely if f is bounded and α is continuous bounded variation, then f is Reimann-Stieltjes Integrable.

Defintion 7)

Let us $f=f_1+if_2=\alpha=\alpha_1+i\alpha_2$

We define $\int f d\alpha=\int f d\|\alpha\|+i\int f_2 d\|\alpha\|$

where $\|\alpha\|=\sqrt{\alpha_1^2+\alpha_2^2}$

When $f_1 f_2$ is Reimann Stieltjes Integrable with respect to $\|\alpha\|$, f is Reimann Stieltjes Integrable.

Theorem 8)

a) If $f \epsilon R(\alpha)$ and $g \epsilon R(\alpha)$, then $f+g \epsilon R(\alpha)$ and $\int f+g d_\alpha=\int f d_\alpha+\int g d_\alpha$.

b) If $f \epsilon R(\alpha)$ and C is constant, then c $f \epsilon R(\alpha)$ and $\int c f d_\alpha=c\int f d_\alpha$

Proof) $\int(f+g)d_\alpha=\int(f_1+if_2+g_1+ig_2)d_\alpha=\int(f_1+g_1)d\|\alpha\|+i\int_a^b(f_2+g_2)d\|\alpha\|=\int f_1 d\|\alpha\|+i\int f_2 d\|\alpha\|+\int g_1 d\|\alpha\|+i\int g_2 d\|\alpha\|=\int f d_\alpha+\int g d_\alpha$ Since, $f \epsilon R(\alpha)$, $g \epsilon R(\alpha)$. $f_1, f_2 \epsilon R(\|\alpha\|)$ $g_1 g_2 \epsilon R(\|\alpha\|)$, $f_1+f_2 \epsilon R(\|\alpha\|)$, $g_1+g_2 \epsilon R(\|\alpha\|)$, $f+g \epsilon R(\|\alpha\|)$

Perhaps we will easily check the other properties of complex Reimann-Stieltjes Integral

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