On the General Weighted Orlicz—Sobolev Spaces

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一般加重 올리쯔 · 소볼레프 空間에 관하여

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Summary

The space $W^m L_{\Phi,\omega}(\ell_\phi)$ is a Banach space.
For $\Phi_1=\{\phi_{1n}\}$ and $\Phi_2=\{\phi_{2n}\}$, if ϕ_{1n} dominate ϕ_{2n} then $W^m L_{\Phi,\omega}(\ell_\phi) \subset W^m L_{\Phi_2}(\ell_\phi)$ and ϕ_1 dominates ϕ_2 then $W^m L_{\Phi,\omega}(\ell_{\phi_1}) \subset W^m L_{\Phi,\omega}(\ell_{\phi_2})$.

Introduction

For a given open set Ω in R^n and a given N-function ϕ, $W^m L\phi(\Omega)$ denotes an Orlicz—Sobolev space Which consists of those (equivalence class of) functions u in $L\phi(\Omega)$ whose distributional derivatives D^α also belong to $L\phi(\Omega)$ for all α with $|\alpha| \le m$. $W^m L\phi(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{m,p} = \|u\|_{m,\phi,\Omega} = \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_{\phi,\Omega}$$

General weighted Orlicz—Sobolev spaces

Let A be a Banach space with norm $\|\cdot\|_A$

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and le Ω be open in R^n. If ω is a Lebesgue measurable function on Ω such that $0 < \omega(x) < \infty$ for $x \in \Omega$. We define $L_{p,\omega}(A)$, the space of all functions $u : \Omega \to A$ such that

$$\| u \| L_{p,\omega}(A) = \{\int_\Omega (\| u(x) \|_A \omega(x))^p dx\}^{1/p} < \infty$$

Then $L_{p,\omega}(A)$ is a Banach space (Triebel, 1978; Turpin, 1978). For an N−function ϕ, the Orlicz−sequence ℓ_ϕ is a Banach space with respect to the norm $\| \cdot \|_{L_{p,\omega}(\ell_\phi)}$.

For a sequence $\Phi = \{\phi_n\}$ of N−functions, define the class

$$k_{\Phi,\omega}(A) = \{u \in A^\Omega : \sum_n \int_\Omega \phi_n (\| u(x) \|_A \omega(x)) dx < \infty\}$$

Clearly $K_{\Phi,\omega}(A)$ is a convex set since all ϕ_n are convex but it may not be a vector space.

A sequence $\Phi = \{\phi_n\}$ of N−function is said to satisfy a Δ_2−uniform condition globally if there exists a positive constant C such that for every $t \geq 0$ and all $n \in N$

$$\phi_n(2t) \leq C\phi_n(t)$$

This is equivalent that for every $r > 1$ there exists a positive constant $C = C(r)$ such that for all $t \geq 0$ and all $n \in N$,

$$\phi_n(rt) \leq C\phi_n(t)$$

LEMMA 1. $K_{\Phi,\omega}(A)$ is a vector space if Φ satisfies a Δ_2−uniform condition globally.

Proof. Since all ϕ_n are convex we have;

(i) $\lambda u \in K_{\Phi,\omega}(A)$ provided, $u \in K_{\Phi,\omega}(A)$ and $|\lambda| \leq 1$ and

(ii) if $u \in K_{\Phi,\omega}(A)$ implies $\lambda u \in K_{\Phi,\omega}(A)$ for every complex λ, then $u, v \in K_{\Phi,\omega}(A)$ implies $u + v \in K_{\Phi,\omega}(A)$.

It follows that $K_{\Phi,\omega}(A)$ is a vector space if and only if $u \in K_{\Phi,\omega}(A)$ and $|\lambda| > 1$ implies $\lambda u \in K_{\Phi,\omega}(A)$.

If Φ satisfies a global Δ_2−uniform condition and $|\lambda| > 1$, then there is a constant C such that $\phi_n(|\lambda| t) \leq C|\lambda|\phi_n(t)$ for all $t > 0$ and all $n \in N$

Thus, for $u \in K_{\Phi,\omega}(A)$.

$$\sum_n \int_\Omega \phi_n (\| \lambda u(x) \|_A \omega(x)) dx \leq c|\lambda| \sum_n \int_\Omega \phi_n (\| u(x) \|_A \omega(x)) dx. ///$$

The general weighted Orlicz space $L_{\Phi,\omega}(A)$ is defined to be the linear hull of the class $K_{\Phi,\omega}(A)$. Thus $K_{\Phi,\omega}(A) \subset L_{\Phi,\omega}(A)$ and these sets are equal if Φ satisfies Δ_2−uniform condition globally.

We define the functional $\| \cdot \|_{L_{\Phi,\omega}(A)}$ on $L_{\Phi,\omega}(A)$

$$\| \cdot \|_{L_{\Phi,\omega}(A)} = \inf\{k > 0 : \sum_n \int_\Omega \phi_n \left(\frac{\| u(x) \|_A \omega(x)}{k} \right) dx \leq 1\}$$

Clearyly, $\| \cdot \|_{L_{\Phi,\omega}(A)}$ is a norm. We have the following.

THEOREM 2. $L_{\Phi,\omega}(A)$ is a Banach space with respect to the norm $\| \cdot \|_{L_{\Phi,\omega}(A)}$.

COROLIARY 3. $L_{\Phi,\omega}(\ell_\phi)$ is a Banach space with respect to the norm

$\| \cdot \|_{L_{\Phi,\omega}(\ell_\phi)}$

The general weighted Orlicz−Sobolev space $W^m L_{\Phi,\omega}(\ell_\phi)$ is defined by

$$W^m L_{\Phi,\omega}(\ell_\phi) = \{u : D^\alpha u \in L_{\Phi,\omega}(\ell_\phi),\ |\alpha| \leq m\}$$

with the norm

$$\| u \|_{W^m L_{\phi,\omega}(\ell_\phi)} = \sum_n \| D^\alpha u \|_{L_{\phi,\omega}(\ell_\phi)}$$

THEOREM 4. $W^m L_{\Phi,\omega}(\ell\phi)$ is a Banach space with respect to $\| \cdot \|_{W^m L_{\phi,\omega}(\ell_\phi)}$.

Proof. Since $L_{\Phi,\omega}(\ell_\phi)$ is a Banach space, the vector space $W^m L_{\Phi,\omega}(\ell\phi)$ is complete with respect to the norm $\| \cdot \|_{W^m L_{\phi,\omega}(\ell_\phi)}$ by Theorem 7.13 in Kufner (1977).

REMARK 1. Triebel (1980) considered the space $L_{p,\omega}(\ell_q)$. This is the case $\phi(t)=|t|^q$ and $\phi_n(t)=|t|^p$ for each n in our spaces (Trjebel, 1980)

we consider imbedding between general weighted Orlicz−Sobolev spaces.

THEOREM 5. If ϕ_{1n} dominate ϕ_{2n} and ϕ_1 dominates ϕ_2 then we have the following imbedding diagram

$$W^m L_{\Phi,\omega}(\ell_{\phi_1}) \hookrightarrow W^m L_{\Phi,\omega}(\ell_{\phi_2}) \hookrightarrow$$
$$\hookrightarrow W^m L_{\Phi_1,\omega}(\ell_{\phi_1}) \hookrightarrow W^m L_{\Phi_2,\omega}(\ell_{\phi_2})$$

References

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〈摘要〉

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$W^m L_{\Phi,\omega}(\ell_\phi)$는 바나크공간이다. $\Phi_1=\{\phi_{1n}\}$와 $\Phi_2=\{\phi_{2n}\}$에 대해서, 만일 ϕ_{1n}이 ϕ_{2n}를 지배하면 $W^m L_{\Phi,\omega}(\ell_\phi) \subset W^m L_{\Phi_2\omega}(\ell_\phi)$이다. 또한 만일 ϕ_1이 ϕ_2를 지배하면 $W^m L_{\Phi,\omega}(\ell_{\phi_1}) \subset W^m L_{\Phi,\omega}(\ell_{\phi_2})$이다.