Stability Theorems of Volterra Integro-Differential Equations

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Volterra 적분미분방정식에서 안정성 정리

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1. Introduction

The purpose of this paper is to study the aysmptotic behavior of solutions of a Volterra integro-differential system of the form $x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)ds$-(A) where $A(t)$ is a continuous $n \times n$ matrix on $[0, \infty)$ and C is an $n \times n$ matrix continuous for $0 \le s \le t < \infty$.

Equations of this type were studied by several authours in [1, 3]. Most of them considered the equations of the form $x'(t) = Ax(t) + \int_0^t C(t,s)x(s)ds$-(B) or $x'(t) = Ax(t) + \int_0^t D(t-s)x(s)ds$-(C) where A is a constant $n \times n$ matrix, $C(t,s)$ is an $n \times n$ matrix continuous for $0 \le s \le t < \infty$ and $D(t)$ is an $n \times n$ matrix continuous for $t \ge 0$. In case A is a stable matrix, there exists a symmetric positive definite matrix B such that $A^T B + BA = -I$, and we can use the function $V = X^T BX$ as a Liapunov function to investigate asymptotic behavior of solutions of (B)(c.f. 3). For the equation (C) there is another method, that is, we can use a nice resalvent $Z(t)$ for (C) (c.f. 1). Burtan [3] contructed a number of Liapunov functionals to study the asymptotic behavior of the solutions of form (B) or (C).

In this paper, we try to find the stability criteria by using the variation of parameter formula and the integral inequality, and the Gronwall's inequality.

2. Definitions and preliminaries

Let R^n denote the Euclidean n-space. For $x \in R^n$, let $|x|$ be a suitable norm of x. For an

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$n\times n$ matrix A, define the norm $|A|$ of A by $|A|=\sup_{|x|\le 1}|Ax|$. Let R^+ be the half line $0\le t\le\infty$. For $\phi\in C(R^+)$ and $t\in R^+$, define $\|\phi\|_t=\max\{|\phi(t)|:\ 0\le s\le t\}$.

Consider the equation (A). The solution of (A) with initial values (t_0, ϕ) will be denoted by $x(t;\ t_0,\ \phi)$ where $t_0\ge 0$ and $\phi:\ [0,\ t_0]\to R^n$ is a continuous function. We give the definitions of various kinds of stability.

Definition 2.1 The zero solution of (A) is stable, if for every $\epsilon>0$ and any $t_0\ge 0$, there exists $\delta>0$ such that $\|\phi\|_{t_0}<\delta$ and $t\ge t_0$ imply $|x(t;\ t_0,\ \phi)|<\epsilon$

Definition 2.2 The zero solution of (A) is uniformly stable (US), if it stable and the above δ is independent of t_0.

Definition 2.3 The zero solution of (A) is attractive, if for any $t_0\ge 0$ there exists $\delta_0\ge 0$ such that $\|\phi\|_{t_0}<\delta_0$ implies $|x(t;\ t_0,\ \phi)|\longrightarrow 0$ as $t\to\infty$. The zero solution of (A) is asymptotically stable (AS), if it is stable and attractive. If, in addition, all solutions tend to zero, then the zero solution of (A) is globally asymptotically stable.

Definition 2.4 The zero solution of (A) is uniformly asymptoticlly stable (UAS), if it is US, the above δ_0 in Definition 2.3 is independent of t_0, and for every $\epsilon>0$ there exists $T>0$ such that $\|\phi\|_{t_0}<\delta_0$ and $t\ge t_0+T$ imply $|x(t, t_0, \phi)|<\epsilon$.

Definition 2.5 The zero solution of (A) is exponentially asymptotically stable (Ex AS), if there exists $\lambda>0$ and for every $\epsilon>0$ there exists $\delta>0$ such that $t_0\ge 0$, $\|\phi\|_{t_0}<\delta$ and $t\ge t_0$ imply $|x(t;\ t_0,\ \phi)|<\epsilon\ e^{-\lambda(t-t_0)}$.

At the end of this section, we present a lemma for integral inequalities and the Grownwall's inequality.

Lemma 2.1 [2, p.315] Let the following condition (1) or (2) hold for functions $f(t)$, $g(t)\in C([t_0,\infty), R^+]$ and $F(t,u)\in C[(t_0,\infty)\times R^+, R^+]$:

(1) $f(t)-\int_{t_0}^{t} F(s, f(s))ds\le g(t)-\int_{t_0}^{t} F(s, g(s))ds$, $t\ge t_0$ and $F(s,u)$ is strictly increasing in u for each fixed $s\ge 0$,

(1) $f(t)-\int_{t_0}^{t} F(s, f(s))ds< g(t)-\int_{t_0}^{t} F(s, g(s))ds$, $t\ge t_0$ and $F(s,u)$ is monotone nondecreasing in U for each fixed $s\ge 0$. If $f(t_0)<g(t_0)$, then $f(t)<g(t)$, $t\ge t_0$.

Lemma 2.2 (Gronwall's Inequality) Let $f, g:\ [0,\alpha)\to[0,\infty)$ be continuous and let c be a nonnegative number. If $f(t)\le c+\int_0^t g(s)f(s)ds$, $0\le t<\alpha$, then $f(t)\le \exp\int_0^t g(s)ds$, $0\le t<\alpha$.

3. Main Theorems

Theorem 3.1 Let the following conditions hold for the differential equations $x'(t)=A(t)x(t)+\int_0^t C(t,s)x(s)ds$ –(1) and $x'(t)=s(t)x(t)$ –(2):

(i) the zero solution of (2) is US, that is, there exists a constant $K\ge 1$ such that $|Y(t)Y^{-1}(s)|\le K$, $t\ge s\ge 0$, where $Y(t)$ is a fundamental matrix of (2),

(ii) there exists $F\in C(R^+\times R^+, R^+)$ such that $F(t,0)\equiv 0$ and $F(t,u)$ is monotone nondecreasing with respect to u for each fixed $t\ge 0$, and $|\int_0^s C(s,u)x(u)du|\le F(s, |x(s)|)$,

(iii) the zero solution of $x'(t)=KF(t,x(t))$ is US. Then the zero solution of (1) is US.

Proof. Let ϕ be the initial solution ϕ of (1) on $[0,t_0]$. Then the solution of (1) is given by $x(t,t_0,\phi)=Y(t)Y^{-1}(t_0)\phi(t_0)+\int_{t_0}^{t} Y(t)Y^{-1}(s)\int_0^s C$

$(s,u)x(u)duds$ by the variation of parameters formula. Thus we obtain from conditions (i) and (ii) that $|x(t)|\le K|\phi(t_0)|+\int_{t0}^{t} KF(s,|x(s)|)ds$, that is, $|x(t)|-\int_{t0}^{t}KF(s,|x(s)|)ds\le K|\phi(t_0)|$. Next let $y(t)=y(t;\ t_0,\ y_0)$ be the solution of $x'(t)=KF(t,x(t))$ passing through (t_0, y_0) and let $K|\phi(t_0)|<y_0$. Then $|x(t)|-\int_{t0}^{t}KF(s,|x(s)|)ds<y(t)-\int_{t0}^{t}KF(s,y(s))ds$. Therefore applying lemma 1, we obtain that $|x(t)|<y(t)$, $t\ge t_0$. Since the zero solution of $x'(t)=KF(t,x(t))$ is US, for any $\epsilon>0$ there exists a $\delta(\epsilon)>0$ such that if $|y_0|<\delta(\epsilon)$, then $|y(t)|<\epsilon$ for all $t\ge t_0$. Thus set $\delta(\epsilon)=\delta_0(\epsilon)K$. If $|x(t_1)|<\delta(\epsilon)$, then take a $y_0>0$ such that $K|\phi(t_0)|<y_0<\delta_0(\epsilon)$. Therefore we have that $|x(t)|<\epsilon$ for all $t\ge t_0$, which completes the proof of the theorem.

Theorem 3.2 Let the following conditions hold for the differential equations $x'(t)=A(t)x(t)+\int_0^t C(t,s)x(s)ds-(1)$ and $x'(t)=A(t)x(t)-(2)$:

(i) $|\int_0^s C(s,u(x(u)du|\le a(t)|x(t)|$,

(ii) $A(t)\in L'(R^+,R^+)\cap C[R^+,R^+]$. If the zero solution of (2) is UAS, then the zero solution of (1) is also UAS.

Proof. Since the equation (2) can be considered a perturbed linear ordinary differential equations, the proof is similar to that of ordinary differential equation (cf. 6, Thm 3.)

Example 3.1

Consider the equation $x'(t)=\begin{pmatrix}-a(t) & -b(t)\\ b(t) & -b(t)\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}+\int_0^t e^{-(t+s)}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}ds-(E)$, where $a\in C[R^+, R]$ and $a(t)\ge\lambda$ for all $t\ge 0$ and some $\lambda>0$, and $b\in C[R^+,R]$. If $|\int_0^s e^{-u}\,x_1(u)du|\le x_1(s)|$ and $[\int_0^s e^{-u}x_2(u)du|\le|x_2(s)|$. Then the zero solution of (E) is UAS.

Proof. Consider $x'(t)=\begin{pmatrix}-a(t) & -b(t)\\ b(t) & -b(t)\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}+\int_0^t e^{-(t+s)}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}-(A)$. Then a fun- damental matrix of (A) is given by $Y(t)=r(t)\begin{pmatrix}-\cos\theta(t) & -\sin\theta(t)\\ \sin\theta(t) & \cos\theta(t)\end{pmatrix}$, where $r(t)=\exp(-\int_0^t a(s)ds)$ and $\theta(t)=\int_0^t b(s)ds$. Therefore, letting $|x|$ be the Euclidean norm of x R^2, we have $|Y(t)Y^{-1}(s)|\le\exp(-\int_s^t a(r)dr)$ for $t\ge s\ge 0$. Since $a(t)\ge\lambda$ for all $t\ge 0$, $|Y(t)Y^{-1}(s)|\le\exp(-\int_s^t a(u)du)\le e^{-\lambda(t-s)}$, that is, the zero solution of (A) is UAS.

Now $|\int_0^s e^{-(s+u)}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(u)\\ x_2(u)\end{pmatrix}du|\le e^{-s}|x(s)|$. This implies $|\int_0^s C(s,u)x(u)du|\le a(s)|x(s)|$, where $a(s)=e^{-s}$. Since $a(s)=e^{-s}\in L^1(R^+,R^+)\cap C[R^+, R^+]$, the zero solution of (1) is UAS by Thm 3.2.

Theorem 3.3 Let the following conditions hold for the differential equation $x'(t)=A(t)x(t)+\int_0^t C(t,s)x(s)ds-(1)$ and $x'(t)=A(t)x(t)-(2)$

(i) the zero solution of (2) is UAS, that is, $|Y(t)Y^{-1}(s)|\le Ke^{-\alpha(t-s)}$ for some $K\ge 1$ and $\alpha>0$ and $t\ge s\ge 0$,

(ii) $|\int_0^s C(s,u)x(u)du|\le M|x(u)|$ with $MK<\alpha$. Then the zero solution of (1) is exponentially stable.

Proof. For any $\epsilon > 0$ let $\delta(\epsilon) < \epsilon/K$ and $\max_{0 \le t \le t_0} |\phi(t)| < \delta(\epsilon)$.

By the variation of parameters formula, we have $|x(t)| \le |Y(t)Y^{-1}(t_0)| |\phi(t_0)| + \int_{t_0}^{t} |Y(t)Y^{-1}(s)| \le Ke^{-\alpha(t-t_0)}\delta + KM\int_{t_0}^{t} e^{-\alpha(t-s)}|x(s)|ds$. Thus

$$e^{\alpha t}|x(t)| \le Ke^{\alpha t_0}\delta + KM\int_{t_0}^{t} e^{\alpha s}|x(s)|\ ds.$$

By the Gronwall's inequality, we have $e^{\alpha t}|x(t)| \le K\delta e^{\alpha t_0} e^{\alpha t_0} e^{KM(t-t_0)}$ and $|x(t)| \le K\delta e^{-\alpha(t-t_0)} e^{KM(t-t_0)}$. Therefore, $|x(t)| < \epsilon\ e^{-(\alpha-KM)(t-t_0)}$, which completes the proof.

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요 약

Volterra 적분미분방정식의 점근적 성질을 조사하여 적분부등식, Gronwall의 부등식과 매개변수 변화법을 이용해 Volterra 적분미분방정식의 해의 안전성의 근거를 찾았다.