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전위에 의한 매개 삼차곡면의 부피

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The Volume of a Parametrized 3-Surface under Inversion

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Abstract

A mapping $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ which sends a point P into a point P' is called an inversion in an Euclidean space E^3 with respect to a given circle or sphere with center O and radius R, if $OP \cdot OP' = R^2$ and if the points P, P' are on the same side of O and O, P, P' are collinear.

This thesis shows that, for a parametrized 3-surface in E^3 is given by $X(u_1, u_2, u_3) = \Big(x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3)\Big)$, the volume of $f(X)$ is equal to $R^6 \int_U \frac{1}{|X|^6} \sqrt{g}\, du_1\, du_2\, du_3$,where $\sqrt{g}$ is the absolute value of Jacobian matrix of x, y, z with respect to u_1, u_2, u_3.

Introduction

In this paper, we study the volume of the parametrized 3-surface in Euclidean space E^3.

In section 1, we present the basic concepts of a parametrized 3-surface in E^3 and a natural instrument to treat the volume of a parametrized 3-surface in E^3. And we also show how to find the volume of a parametrized 3-surface.

In section 2, we introduce the definition and some properties of inversion in E^3 and show that $f(X) : U \longrightarrow E^3$ is a parametrized 3-surface, and $\sqrt{\hat{g}} = \frac{R^6}{|X|^6} \sqrt{g}$.

Finally, in section 3, we show the volume of $f(X)$ under inversion is

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equal to $R^6 \int_U \frac{1}{|X|^6}\sqrt{g}\, du_1\, du_2\, du_3$, and give the example for the above theorem.

1. The volume of a parametrized 3-surface

In this section, we introduce the basic concepts of a parametrized 3-surface in E^3. And we define the volume of a parametrized 3-surface.

Definition 1.1 A parametrized 3-surface is a smooth map $X : U \longrightarrow E^3$ which is regular, where $U \subset E^3$ is open.

If we write $X(u_1, u_2, u_3) = (x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3))$ for any $(u_1, u_2, u_3) \in U \subset E^3$, then X is smooth if and only if $x(u_1, u_2, u_3)$, $y(u_1, u_2, u_3)$ and $z(u_1, u_2, u_3)$ have continuous partial derivatives of all orders in U. Regular condition means that dX_q is non-singular (has rank 3) for each $q \in U$. Let us compute the matrix of the linear map dX_q with respect to the canonical bases $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ of E^3 with coordinates (u_1, u_2, u_3) and $i_1 = (1,0,0), i_2 = (0,1,0)$ and $i_3 = (0,0,1)$ of E^3 with coordinates (x,y,z).
By the definition of the differential, we have

$$dX_q(e_1) = \left(\frac{\partial x}{\partial u_1}, \frac{\partial y}{\partial u_1}, \frac{\partial z}{\partial u_1}\right) = \frac{\partial X}{\partial u_1} = X_{u_1}, \tag{1.1}$$

$$dX_q(e_2) = \left(\frac{\partial x}{\partial u_2}, \frac{\partial y}{\partial u_2}, \frac{\partial z}{\partial u_2}\right) = \frac{\partial X}{\partial u_2} = X_{u_2}, \tag{1.2}$$

$$dX_q(e_3) = \left(\frac{\partial x}{\partial u_3}, \frac{\partial y}{\partial u_3}, \frac{\partial z}{\partial u_3}\right) = \frac{\partial X}{\partial u_3} = X_{u_3}. \tag{1.3}$$

Regular condition implies that for each $q \in U$,

$$\frac{\partial X}{\partial u_1} \cdot \frac{\partial X}{\partial u_2} \times \frac{\partial X}{\partial u_3} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} \neq 0. \tag{1.4}$$

The mapping X is called a parametrization or a system of local coordinates in a neighborhood of $p \in U$.

Example 1.2 Let $X : U \longrightarrow E^3$ be defined by
$X(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$,
where $U = \{(r,\theta,\phi) \,|\, 0 < r < a, 0 < \theta < 2\pi, 0 < \phi < \pi\}$.
Then $X : U \longrightarrow E^3$ is a parametrized 3-surface
and the image $X(U) = B\backslash\{(x_1,x_2,x_3) \in B \,|\, x_1 \geq 0, x_2 = 0\}$,
where $B = \{(x_1,x_2,x_3) \,|\, x_1^2 + x_2^2 + x_3^2 < a^2\}$.
Since $x(r,\theta,\phi) = r\cos\theta\sin\phi$, $y(r,\theta,\phi) = r\sin\theta\sin\phi$ and $z(r,\theta,\phi) = r\cos\phi$ have continuous partial derivatives of all orders in U, X is smooth.
Moreover, since

$$
\begin{aligned}
E_1(p) &= \frac{\partial X}{\partial r} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi),\\
E_2(p) &= \frac{\partial X}{\partial \theta} = (-r\sin\theta\sin\phi, r\cos\theta\sin\phi, 0) \text{ and}\\
E_3(p) &= \frac{\partial X}{\partial \phi} = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, -r\sin\phi), p = (r,\theta,\phi) \in U,
\end{aligned}
$$

we have $\left|\frac{\partial X}{\partial r} \cdot \frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial \phi}\right| \neq 0$. Hence the regular condition is satisfied.

Definition 1.3 Let $X : U \longrightarrow E^3$ be a parametrized 3-surface where $U \in E^3$ is open.
Then the volume of a parametrized 3-surface X denoted by $V(X)$ is defined by

$$
V(X) = \int_U \left|\frac{\partial X}{\partial u_1} \cdot \frac{\partial X}{\partial u_2} \times \frac{\partial X}{\partial u_3}\right| du_1\, du_2\, du_3, \tag{1.5}
$$

where (u_1, u_2, u_3) is a local coordinate system on U.

The function $|X_{u_1} \cdot X_{u_2} \times X_{u_3}|$ defined in U, measures the volume of a parallelepiped generated by the vectors $X_{u_1}, X_{u_2}, X_{u_3}$.

Proposition 1.4 Let $X : U \longrightarrow E^3$ be a parametrized 3-surface and let $g_{ij} = X_{u_i} \cdot X_{u_j} = \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + \frac{\partial y}{\partial u_i} \cdot \frac{\partial y}{\partial u_j} + \frac{\partial z}{\partial u_i} \cdot \frac{\partial z}{\partial u_j}$. Then

$$
V(X) = \int_U \sqrt{g}\, du_1 du_2 du_3, \tag{1.6}
$$

where $\quad g = |\det(g_{ij})|$.

Corollary 1.5 The parametrization X has the regularity condition iff $\sqrt{g}$ is never zero, that is $\sqrt{g} > 0$.

Example 1.6 Let $X : U \longrightarrow E^3$ be the parametrization in the example 1.2. Then

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2 \phi & 0 \\ 0 & 0 & r^2 \end{vmatrix} = r^4 \sin^2 \phi.$$

Hence we get

$$\begin{aligned} V(X) &= \int_0^{\pi} \int_0^{2\pi} \int_0^{a} \sqrt{g}\, dr\, d\theta\, d\phi \\ &= \int_0^{\pi} \int_o^{2\pi} \int_0^{a} r^2 \sin\phi\, dr\, d\theta\, d\phi \\ &= \frac{4}{3}\pi a^3. \end{aligned}$$

2. Definition and some properties of an inversion

In this section, we define an inversion in E^3 and study some properties of an inversion.

Let the symbol $(O)_R$ denote the sphere with center O and radius R

Definition 2.1 Two points P and P' of E^3 are said to be inverse with respect to a given sphere $(O)_R$ if

$$OP \cdot OP' = R^2. \tag{2.1}$$

and if P, P' are on the same side of O and the points O, P, P' are collinear.

A $(O)_R$ is called the sphere of inversion, and the transformation which sends a point P into P' is called an inversion.
Note that the center O of the sphere of inversion has no inverse point.
From now on, we take the center O as an origin in E^3, and denote the distance from O to a point X by $|X|$.

Then we have the following properties.

Proposition 2.2 An inversion in a space E^3 is a mapping $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ such that

$$f(X) = \frac{R^2 X}{< X, X >} = \frac{R^2 X}{|X|^2}, \tag{2.2}$$

where $< X, X >= X \cdot X$ is the dot product.

Proof. Since f is an inversion and O, X and $f(X)$ are collinear.
Hence $f(X) = kX$ for some positive real number k.
Since $f(X)$ is inverse point of X, by means of (2.1),

$$|X||f(X)| = R^2.$$

$$k|X|^2 = R^2.$$

Since $|X| \neq 0$, we have

$$k = \frac{R^2}{|X|^2}.$$

The inverse point $f(X) = \dfrac{R^2 X}{|X|^2}$ is the vector of length $R^2|X|^{-1}$ on the ray of X.

Theorem 2.3
(1) A plane through O inverts into a plane through O.
(2) A plane not through O inverts into a sphere through O.
(3) A sphere through O inverts into a plane not through O.
(4) A sphere not through O inverts into a sphere not through O.

Proof. Let B be any nonzero constant vector in E^3, and consider the equation

$$a|X|^2 + < B, X > + c = 0, \tag{2.3}$$

where $a, c \in R$.
Then the equation (2.3) represents a sphere for $a \neq 0$ and a plane for $a = 0$.
For $|X| \neq 0$, multiplying both sides of (2.3) by $\dfrac{R^2}{|X|^2}$, we have

$$R^2 a + \frac{R^2 < B, X >}{|X|^2} + \frac{R^2 c}{|X|^2} = 0. \tag{2.4}$$

Let $Y = \frac{R^2 X}{|X|^2}$. Then we have

$$\frac{c}{R^2}|Y|^2 + < B, Y > + R^2 a = 0. \tag{2.5}$$

Thus (2.3) is transformed into (2.5) under inversion. Hence we get:

(1) When $a = 0, c = 0$, (2.3) and (2.5) represent a plane through O.

(2) When $a = 0, c \neq 0$, (2.3) represents a plane not through O and (2.5) represents a sphere through O.

(3) When $a \neq 0, c = 0$, (2.3) represents a sphere through O and (2.5) represents a plane not through O.

(4) When $a \neq 0, c \neq 0$, (2.3) and (2.5) represents a sphere not through O.

Define $f \circ X : U \longrightarrow E^3$ by $(f \circ X)(u_1, u_2, u_3) = \frac{R^2 X}{|X|^2}$, where $(u_1, u_2, u_3) \in U$ and $X = \big(x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3)\big)$.

Theorem 2.4 Let $X : U \to E^3 - \{(0,0,0)\}$ be a parametrized 3-surface for $U \subset E^3$ and $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion. Then $f(X) = f \circ X$ is a parametrized 3-surface.

Proof. Since X is a parametrized 3-surface and $f \circ X = f(X) = \frac{R^2 X}{|X|^2}$, $f(X)$ is smooth and regular.
Hence $f(X)$ is a parametrized 3-surface.

Theorem 2.5 Let $f(X) : U \to E^3$ be an inversion of a parametrized 3-surface X. Then

$$\widehat{g_{ij}} = < \frac{\partial f(X)}{\partial u_i}, \frac{\partial f(X)}{\partial u_j} > = \frac{R^4}{|X|^4} g_{ij}, \tag{2.6}$$

where $g_{ij} = < \frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_j} >$.

Proof. Since f is an inversion, we have $f(X) = \frac{R^2}{|X|^2} X$ from (2.2).

By the equation $\frac{\partial f(X)}{\partial u_i} = \frac{R^2}{|X|^2}\frac{\partial X}{\partial u_i} - \frac{2R^2}{|X|^3}\frac{\partial |X|}{\partial u_i} X$,

$$\widehat{g_{ij}} = < \frac{\partial f(X)}{\partial u_i}, \frac{\partial f(X)}{\partial u_j} >$$
$$= < \frac{R^2}{|X|^2}\frac{\partial X}{\partial u_i} - \frac{2R^2}{|X|^3}\frac{\partial |X|}{\partial u_i}X, \frac{R^2}{|X|^2}\frac{\partial X}{\partial u_j} - \frac{2R^2}{|X|^3}\frac{\partial |X|}{\partial u_j}X >$$
$$= < \frac{R^2}{|X|^2}\frac{\partial X}{\partial u_i}, \frac{R^2}{|X|^2}\frac{\partial X}{\partial u_j} >$$
$$= \frac{R^4}{|X|^4} < \frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_j} >$$
$$= \frac{R^4}{|X|^4} g_{ij}.$$

Corollary 2.6 Let $X : U \longrightarrow E^3 - \{(0,0,0)\}$ be a parametrized 3-surface and $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion.
Then the $\sqrt{\hat{g}}$ of $f(X)$ is equal to $\frac{R^6}{|X|^6}\sqrt{g}$ where $\hat{g} = \det(\hat{g}_{ij})$.
Proof.

$$\hat{g} = \begin{vmatrix} \widehat{g}_{11} & \widehat{g}_{12} & \widehat{g}_{13} \\ \widehat{g}_{21} & \widehat{g}_{22} & \widehat{g}_{23} \\ \widehat{g}_{31} & \widehat{g}_{32} & \widehat{g}_{33} \end{vmatrix}$$
$$= \begin{vmatrix} \frac{R^4}{|X|^4}g_{11} & \frac{R^4}{|X|^4}g_{12} & \frac{R^4}{|X|^4}g_{13} \\ \frac{R^4}{|X|^4}g_{21} & \frac{R^4}{|X|^4}g_{22} & \frac{R^4}{|X|^4}g_{23} \\ \frac{R^4}{|X|^4}g_{31} & \frac{R^4}{|X|^4}g_{32} & \frac{R^4}{|X|^4}g_{33} \end{vmatrix}$$
$$= \left(\frac{R^4}{|X|^4}\right)^3 \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$
$$= \frac{R^{12}}{|X|^{12}} g.$$

Hence $\sqrt{\hat{g}} = \frac{R^6}{|X|^6}\sqrt{g}$.

3. The volume of a parametrized 3-surface under inversion

In this section, we show that the volume of $V(f(X))$ under inversion is equal to $R^6 \int_U \frac{1}{|X|^6}\sqrt{g}du_1du_2du_3$ and give example for the following theorem.

Theorem 3.1 Let $X : U \longrightarrow E^3 - \{(0,0,0)\}$ be a parametrized 3-surface defined by
$X(u_1,u_2,u_3) = \big(x(u_1,u_2,u_3), y(u_1,u_2,u_3), z(u_1,u_2,u_3)\big)$ for $(u_1,u_2,u_3) \in U \subset E^3$, and let $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion of X, then the volume of $f(X)$ under inversion is equal to

$$V(f(X)) = R^6 \int_U \frac{1}{|X|^6}\sqrt{g}\, du_1\, du_2\, du_3. \tag{3.1}$$

Proof. By corollary 2.6 and proposition 1.4, we have

$$\begin{aligned} V(f(X)) &= \int_U \sqrt{\hat{g}}\, du_1\, du_2\, du_3 \\ &= \int_U \frac{R^6}{|X|^6}\sqrt{g}\, du_1\, du_2\, du_3 \\ &= R^6 \int_U \frac{1}{|X|^6}\sqrt{g}\, du_1\, du_2\, du_3. \end{aligned}$$

Example 3.2 Let $X : U \longrightarrow E^3 - \{(0,0,0)\}$ be a mapping defined by
$X(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$,
where $U = \{(r,\theta,\phi) \,|\, 1 < r < 2, 0 < \theta < 2\pi, 0 < \phi < \pi\}$.
Then, by example (1.2), X is a parametrized 3-surface, and

$$\begin{aligned} |X|^2 &= r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\phi \\ &= r^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + r^2\cos^2\phi \\ &= r^2(\sin^2\phi + \cos^2\phi) \\ &= r^2. \end{aligned}$$

Thus $|X|^6 = r^6$ and $\sqrt{g} = r^2 \sin\phi$. Hence

$$\begin{aligned} V(f(X)) &= R^6 \int_0^{\pi} \int_0^{2\pi} \int_1^2 \frac{1}{r^4} \sin\phi \, dr \, d\theta \, d\phi \\ &= \frac{7}{6}\pi R^6. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(X) &= \frac{R^2}{r^2}(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) \\ &= \frac{R^2}{r}(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi). \end{aligned}$$

Thus

$$\begin{aligned} \widehat{E_1(p)} &= \frac{\partial f(X)}{\partial r} = -\frac{R^2}{r^2}(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi), \\ \widehat{E_2(p)} &= \frac{\partial f(X)}{\partial \theta} = \frac{R^2}{r}(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) \text{ and} \\ \widehat{E_3(p)} &= \frac{\partial f(X)}{\partial \phi} = \frac{R^2}{r}(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi). \end{aligned}$$

Hence

$$\begin{aligned} \hat{g} &= \begin{vmatrix} \widehat{g}_{11} & \widehat{g}_{12} & \widehat{g}_{13} \\ \widehat{g}_{21} & \widehat{g}_{22} & \widehat{g}_{23} \\ \widehat{g}_{31} & \widehat{g}_{32} & \widehat{g}_{33} \end{vmatrix} \\ &= \begin{vmatrix} \frac{R^4}{r^4} & 0 & 0 \\ 0 & \frac{R^4}{r^2}\sin^2\phi & 0 \\ 0 & 0 & \frac{R^4}{r^2} \end{vmatrix} \\ &= \frac{R^4}{r^4} \cdot \frac{R^8}{r^4} \sin^2\phi. \end{aligned}$$

Thus

$$\begin{aligned} V(f(X)) &= R^6 \int_0^{\pi} \int_0^{2\pi} \int_1^2 \frac{1}{r^4} \sin\phi \, dr \, d\theta \, d\phi \\ &= \frac{7}{6}\pi R^6. \end{aligned}$$

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<국 문 초 록>

전위에 의한 매개 삼차곡면의 부피

중심이 O 이고 반지름의 길이가 R 인 주워진 원 또는 구에서 Euclid 공간 E^3 의 두 점 P 와 P' 이 중심 O 와 같은 쪽에 있고 점 P 에서 P' 으로 $OP \cdot OP' = R^2$ 가 되도록 보내는 변환 $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ 를 전위라 한다. 이 논문은 Euclid 공간 E^3 에서 주어진 매개 3 차 곡면, X는

$$X(u_1, u_2, u_3) = \Big(x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3)\Big)$$

에 대하여 전위에 의한 매개 3 차 곡면의 부피 $V(f(X))$ 는 x, y, z 가 u_1, u_2, u_3 에 대한 Jacobian 행렬의 고유치하에서 $R^6 \int_U \frac{1}{|X|^6} \sqrt{g}\, du_1\, du_2\, du_3$ 와 같다는 것을 보인다.