J. of Basic Sciences, Cheju Nat. Univ.
10(2), 1-13, 1997

기초과학연구 제주대학교
10(2), 1-13, 1997

FUZZY STRONG LAW OF LARGE NUMBERS

JI SOON LEE AND YONG SIK YUN

Department of Mathematics, Clieju National University, Korea

ABSTRACT. **We investigate properties of fuzzy variables, fuzzy random variables, and its membership functions. Furthermore, we study the relationship between expectation of fuzzy random variable, its membership function and convex fuzzy variable. We define the convergence of sequences of fuzzy variables and fuzzy random variables, and then prove the convergence of sequences of fuzzy random variables and its membership function.**

1. INTRODUCTION

Tlie notion of **a** fuzzy set wits introduced by Zadeh[6] and the theory of fuzzy sets have been used to model situations where knowledge is imprecise. This imprecision is presumed to arise when dealing with concepts that are ill-defined. We shall base our theory on the work of Stein and Talati [10]. In that paper, they defined the concept of afuzzy ***variable X*** **as** a real-valued function defined on an arbitrary set Γ and proved some properties of fuzzy variables and their membership functions. They defined also a fuzzy ***random variable*** $Z : \Omega \to \mathbb{R}^\Gamma$ **as** a fuzzy variable valued function defined on a probability space $(\Omega, \mathcal{B}, P)$. The set Ω is a sample space and we assume that **a** probability measure P is defined on a a-algebra $\mathcal{B}$ of subsets of Ω while $\mathbb{R}^\Gamma$ represents all real-valued functions defined on Γ. For some fuzzy random variables $Z_1, \cdots, Z_n$, we proved the convergence of sequence $\{\bar{Z}_n = \frac{1}{n}(Z_1 + \cdots + Z_n)\}$ and find the sufficient condition for the convergence of sequence of their membership functions.

Tlie organization of this paper is as follows. In Section 2, we review certain properties of fuzzy variables and relationship for their membership functions. We define the convex fuzzy variable and observe properties of tlie convex fuzzy variables. In Section **3,** we define the fuzzy random variable and its expectation as **a** fuzzy variable and find the membership function of the expectation of fuzzy random variable. We investigate properties of the expectation of fuzzy random variable and its membership function. Furthermore, we study the relationship between expectation of fuzzy random variable, its membership function, and convex fuzzy variable. We carry over usual linear properties of probabilitic expectation to fuzzy random variable and view the linear properties of expectation of fuzzy random variables. In Section **4,** we define the convergence of sequences of fuzzy variables and fuzzy randon variables and tlieii prove tlie convergence of sequences of a fuzzy random variable and its membership function.

2. Fuzzy variables

We define a *scale* σ on the class of all subsets of a set Γ as a function satisfying
(i) $\sigma(\phi) = 0$ and $\sigma(\Gamma) = 1$,
(ii) for any arbitrary collection of subsets $\{A_\alpha\}$ of Γ,

$$\sigma(\bigcup_\alpha A_\alpha) = \sup_\alpha \sigma(A_\alpha).$$

Definition 2.1. *A fuzzy variable X is a real valued function defined on Γ.*

The scale σ is analogous to a probability measure P. The distribution of a random variable is obtained from P and the definition of the random variable, while the membership function of a fuzzy variable is determined from σ and the definition of the fuzzy variable.
The *membership function* $\mu_X : \mathbb{R} \to [0,1]$ of the fuzzy variable X is defined by

$$\mu_X(x) = \sigma\{\gamma \in \Gamma : X(\gamma) = x\}, \quad x \in \mathbb{R}.$$

To obtain the membership function of $g(X)$ where X is a fuzzy variable and $g : \mathbb{R} \to \mathbb{R}$ is any function, Nahmias [4] proved that

$$\mu_{g(X)}(t) = \sup_{u:g(u)=t} \mu_X(u).$$

Example 2.2.
(i) $\mu_{\alpha X}(t) = \mu_X(\frac{t}{\alpha})$, for $\alpha \neq 0$ and t.
(ii) $\mu_{X^2}(t) = \mu_X(\sqrt{t}) \vee \mu_X(-\sqrt{t})$, for $t \geq 0$.
(iii) If $\mu_X(x) = e^{-(x-a)^2/b^2}$, then $\mu_{X^2}(t) = e^{-(\sqrt{t}-|a|)^2/b^2}$, for $t \geq 0$.

Proof. (i) By definition,

$$\begin{aligned}\mu_{\alpha X}(t) &= \sigma\{\gamma \in \Gamma | (\alpha X)(\gamma) = t\} \\ &= \sigma\{\gamma \in \Gamma | X(\gamma) = \frac{t}{\alpha}\} \\ &= \mu_X(\frac{t}{\alpha}).\end{aligned}$$

(ii) By definition,

$$\begin{aligned}\mu_{X^2}(t) &= \sigma\{\gamma \in \Gamma | X^2(\gamma) = t\} \\ &= \sigma\{\gamma \in \Gamma | X(\gamma) = \pm\sqrt{t}\} \\ &= \mu_X(\sqrt{t}) \vee \mu_X(-\sqrt{t}).\end{aligned}$$

(iii) By (ii), if $\mu_X(x) = e^{-(x-a)^2/b^2}$, then $\mu_{X^2} = e^{-(\sqrt{t}-|a|)^2/b^2}$ for $t \geq 0$. □

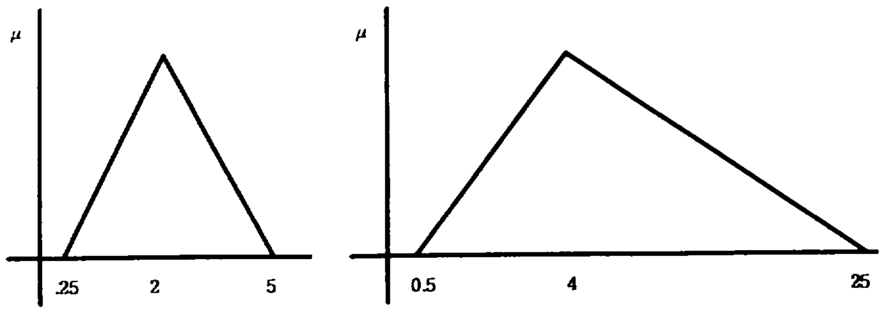

Fig. 1. The transformation $g(x) = x^2$ applied to a triangular membership function.

In Fig. 1, the effect of the transformation $g(X) = X^2$ is shown applied to a triangular membership function.

Nahmias [4] calls two fuzzy variables X, Y are *unrelated* (or *noninteractive*) if

$$\sigma(X = x \cap Y = y) = \sigma(X = x) \wedge \sigma(Y = y) \quad \text{for all} \quad x, y \in \mathbb{R}.$$

This is analogous to the concept of independent random variables. A collection of fuzzy variables is called *mutually unrelated* if every finite subcollection has the property that the scale of the intersection can be computed by the minimum of the scale of each term (see [4]).

Using the concept of unrelated fuzzy variables, Nahmias [4] derived Zadeh's extension principle for the sum of two fuzzy variables.

Theorem 2.3 (Nahmias[4]). *If X,Y are fuzzy variables, then*
(i) $\mu_{X+Y}(t) = \sup_x \sigma(\{X = x\} \cap \{Y = t - x\})$.
Furthermore, if X, Y are unrelated, then
(ii) $\mu_{X+Y}(t) = \sup_x [\mu_X(x) \wedge \mu_Y(t - x)]$.

Proof. (i) We have that

$$\begin{aligned}
\mu_{X+Y}(t) &= \sigma\{\gamma \in \Gamma | (X+Y)(\gamma) = t\} \\
&= \sigma\{\gamma \in \Gamma | ((X+Y)(\gamma) = t) \cap \Gamma\} \\
&= \sigma\{\gamma \in \Gamma | ((X+Y)(\gamma) = t) \bigcap (\textstyle\bigcup_x (X(\gamma) = x))\} \\
&= \sigma\{\gamma \in \Gamma | \bigcup_x ((X(\gamma) = x) \cap ((X+Y)(\gamma) = t))\} \\
&= \sup_x \sigma\{\gamma \in \Gamma | (X(\gamma) = x) \cap ((X+Y)(\gamma) = t)\} \\
&= \sup_x \sigma\{\gamma \in \Gamma | (X(\gamma) = x) \cap (Y(\gamma) = t - x)\}.
\end{aligned}$$

(ii) Since X, Y are unrelated, $\mu_{X+Y}(t) = \sup_x[\mu_X(x) \wedge \mu_Y(t-x)]$. □

Note that in the above theorem, $X+Y$ refers to the fuzzy variable defined by

$$(X+Y)(\gamma) = X(\gamma) + Y(\gamma), \quad \text{for all} \quad \gamma \in \Gamma.$$

The set of all fuzzy variables, $\mathbb{R}^\Gamma$, is a vector space over $\mathbb{R}$. Scalar multiplication is also defined in the usual manner.

$$(\alpha X)(\gamma) = \alpha(X(\gamma)) \quad \text{and} \quad (\alpha+\beta)X = \alpha X + \beta X.$$

This can be extended to define products and ratios of fuzzy variables. Let $*$ be any binary operation defined between pairs of real numbers. Then we can define the fuzzy variable $X * Y$ by $(X*Y)(\gamma) = X(\gamma) * Y(\gamma)$. The following theorem is a definition in Chang [1] and Nguyen [5] by the extension principle.

Theorem 2.4 (Stein and Talati [10]). *If X, Y are unrelated fuzzy variables, then*

$$\mu_{X*Y}(t) = \sup_{x*y=t}[\mu_X(x) \wedge \mu_Y(y)].$$

Proof. Since

$$\begin{aligned}
\mu_{X*Y}(t) &= \sigma\{\gamma \in \Gamma | (X*Y)(\gamma) = t\} \\
&= \sigma\{\gamma \in \Gamma | (X(\gamma) * Y(\gamma) = t) \cup \Gamma\} \\
&= \sigma\{\gamma \in \Gamma | (X(\gamma) * Y(\gamma) = t) \bigcap (\textstyle\bigcup_{x,y}((X(\gamma) = x) \cap (Y(\gamma) = y)))\} \\
&= \sigma\{\gamma \in \Gamma | \bigcup_{x,y}((X(\gamma) * Y(\gamma) = t) \cap ((X(\gamma) = x) \cap (Y(\gamma) = y)))\} \\
&= \begin{cases} \sigma(\phi) & \text{if} \quad x*y \neq t, \\ \sup_{x*y=t} \sigma\{\gamma \in \Gamma | (X(\gamma) = x) \cap (Y(\gamma) = y)\} & \text{if} \quad x*y = t, \end{cases}
\end{aligned}$$

we have

$$\mu_{X*Y}(t) = \sup_{x*y=t}[\mu_X(x) \wedge \mu_Y(y)].$$ □

Definition 2.5. *A fuzzy variable X is convex if its membership function is quasi-concave. That is, $\mu_X(\lambda a + (1-\lambda)b) \geq \mu_X(a) \wedge \mu_X(b)$, for all $a, b \in \mathbb{R}$ and $0 \leq \lambda \leq 1$. We call μ_X is convex if X is convex.*

A convex membership function is called a *fuzzy number* by Dubois and Prade [2]. The class of convex membership function includes
(i) functions that only assume the values 0 or 1,
(ii) monotone functions and
(iii) $N(a, b) = e^{-(x-a)^2/b^2}$, for $a \in \mathbb{R}$ and $b > 0$.
In fact, (i) is clear. Since monotone functions are increasing or decreasing, (ii) holds. We know that all $N(a, b)$ are continuous and concave, thus (iii) holds.

We note the following result, that is, sufficient condition for convexity of composite function without proof.

Theorem 2.6 (Chang [1]). *If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $X_1, \cdots, X_n$ are unrelated convex fuzzy variables, then $f(X_1, \cdots, X_n)$ is also convex fuzzy variable.*

The above theorem implies that $X + Y$, $X - Y$, $X \cdot Y$ and X^2 are all convex if X and Y are unrelated and convex. It is quite easy to find counter examples to show that 'unrelated' is a necessary condition in the above theorem. For example, if X is $N(0,1)$ and

$$Y = \begin{cases} X & \text{if} \quad |X| \leq 1, \\ 0 & \text{if} \quad |X| > 1, \end{cases}$$

then

$$X - Y = \begin{cases} 0 & \text{if} \quad |X| \leq 1, \\ X & \text{if} \quad |X| > 1, \end{cases}$$

so that $X - Y$ has a nonconvex membership function.

Theorem 2.6 can be extended to the case when f is defined on a convex subset of $\mathbf{R}^n$. For example, if X is convex and $X > 0$, then X^2 is also convex (see Fig. 1).

The concept of a convex fuzzy variable will play a key role in the next sections.

3. Fuzzy random variables

We will consider only fuzzy random variables that take on a finite number of values (each value is a fuzzy variable):

$$Z(\omega) = \sum_{i=1}^{n} I_{E_i}(\omega) X_i$$

where $X_1, \cdots, X_n$ are fuzzy variables and $E_1, \cdots, E_n$ are a partition of the sample space Ω and $\omega \in \Omega$. Thus Z takes the value X_i with probability $P(E_i)$. We shall call Z is convex if each X_i is convex. Note that $\sum_{i=1}^{n} p_i = 1$ if $P(E_i) = p_i$.

Definition 3.1. *With the above notation, define the expectation $E(Z) = \sum_{i=1}^{n} p_i X_i$. This defines $E(Z)$ as a fuzzy variable.*

Proposition 3.2. *If Z is a fuzzy random variable, then $E(\alpha Z + \beta) = \alpha E(Z) + \beta$ for all reals α and β.*

Proof. Since $\alpha Z + \beta = \alpha \sum_{i=1}^{n} I_{E_i} X_i + \beta = \sum_{i=1}^{n} I_{E_i}(\alpha X_i + \beta)$, we have $E(\alpha Z + \beta) = \sum_{i=1}^{n} p_i(\alpha X_i + \beta) = \alpha \sum_{i=1}^{n} p_i X_i + \beta = \alpha E(Z) + \beta$. □

The following theorem can be used to determine the membership function of $E(Z)$.

Theorem 3.3 (Stein and Talati [10]). *Let $Z = \sum_{i=1}^{n} I_{E_i} X_i$, where $\{X_i\}$ are fuzzy variables and $p_i = P(E_i)$. Then*

$$\mu_{E(Z)}(t) = \sup[\mu_{X_1}(x_1) \wedge \cdots \wedge \mu_{X_n}(x_n)],$$

where the supremum is taken over $(x_1, \cdots, x_n)$ subject to the constant $\sum p_i x_i = t$.

Proof. We have that

$$\begin{aligned}
\mu_{E(Z)} &= \sigma\{\gamma \in \Gamma | E(Z)(\gamma) = t\} \\
&= \sigma\{\gamma \in \Gamma | (\sum_{i=1}^{n} p_i X_i)(\gamma) = t\} \\
&= \sigma\{\gamma \in \Gamma | (\sum_{i=1}^{n} p_i \cdot X_i(\gamma) = t) \cap \Gamma\} \\
&= \sigma\{\gamma \in \Gamma | (\sum_{i=1}^{n} p_i \cdot X_i(\gamma) = t) \\
&\qquad \bigcap (\cup_{x_1, \cdots, x_n} (X_1(\gamma) = x_1 \cap \cdots \cap X_n(\gamma) = x_n))\} \\
&= \sigma\{\gamma \in \Gamma | \bigcup_{x_1, \cdots, x_n} ((\sum_{i=1}^{n} p_i \cdot X_i(\gamma) = t) \\
&\qquad \cap (X_1(\gamma) = x_1 \cap \cdots \cap X_n(\gamma) = x_n))\}
\end{aligned}$$

$$= \begin{cases} \sigma(\phi) & \text{if } \sum_{i=1}^{n} p_i \cdot x_i \neq t \\ \sup_{\sum p_i x_i = t} \sigma\{\gamma \in \Gamma | (X_1(\gamma) = x_1 \cap \cdots \cap X_n(\gamma) = x_n)\} & \\ & \text{if } \sum_{i=1}^{n} p_i \cdot x_i = t \end{cases}$$

$$= \sup_{\sum p_i x_i = t} [\mu_{X_1}(x_1) \wedge \cdots \wedge \mu_{X_n}(x_n)]. \qquad \square$$

Corollary 3.4. *If Z is a fuzzy random variable with*

$$Z = \begin{cases} X & \text{with probability } p, \\ Y & \text{with probability } q, \end{cases}$$

where X, Y are unrelated fuzzy variables and $p + q = 1$, then

$$\mu_{E(Z)}(t) = \sup_x [\mu_X\left(\frac{x}{p}\right) \wedge \mu_Y\left(\frac{t-x}{q}\right)].$$

The following theorem shows that convexity is required for a sensible interpretation of the expectation when all values are unrelated.

Theorem 3.5 (Stein and Talati [10]). *Assume that Z is a fuzzy random variable as in Theorem 3.3. Also assume that each X_i has the same membership function μ. Then $E(Z)$ has membership function μ (for every $p_1, \cdots, p_n$) if and only if Z is convex.*

Proof. First we consider the case $n = 2$. Then $E(Z) = p_1 X_1 + p_2 X_2$ with $p_1 + p_2 = 1$. Now for fixed p_1 and p_2, the following statements are equivalent.

(1) $p_1X_1 + p_2X_2$ has membership function μ.

(2) For all $t, \mu(t) = \sup[\mu(x_1) \wedge \mu(x_2)]$ where the supremum is taken over x_1 and x_2 such that $p_1x_1 + p_2x_2 = t$.

(3) $\mu(t) \geq \mu(x_1) \wedge \mu(x_2)$ for all t, where $p_1x_1 + p_2x_2 = t$ (equality occurs at $x_1 = x_2 = t$).

(4) $\mu(p_1x_1 + p_2x_2) \geq \mu(x_1) \wedge \mu(x_2)$ for all t, x_1 and x_2.

Note that these equivalences hold for all probabilities p_1 and p_2 with $p_1 + p_2 = 1$. We see from above that μ is required to be convex. Thus by (1) and (4), the result holds for $n = 2$.

For arbitrary $n \in \mathbb{N}$, consider

$$\begin{aligned} E(Z) &= p_1X_1 + p_2X_2 + \cdots + p_nX_n \\ &= \Big(\frac{p_1}{p_1 + \cdots + p_{n-1}}X_1 + \cdots + \frac{p_{n-1}}{p_1 + \cdots + p_{n-1}}X_{n-1}\Big) \\ &\quad \cdot (p_1 + \cdots + p_{n-1}) + p_nX_n. \end{aligned}$$

Then we can extend the result from $n = 2$ to arbitrary n by induction. □

Corollary 3.6. *Let $Z = \sum I_{E_i}X_i$ with unrelated convex fuzzy variables X_i. Then $E(Z)$ is a convex fuzzy variable.*

Proof. Since $E(Z) = \sum p_iX_i$, $E(Z)$ is a convex fuzzy variable by Theorem 2.6. □

We may extend μ to the positive reals to obtain another membership function $\tilde{\mu}$. If we consider the computation of $E(Z)$ as in Theorem 3.5, we will obtain different results if we choose $\tilde{\mu}$ rather than μ. The following theorem summarizes the extent of the differences between these results and leads to an 'optimal' extension.

Theorem 3.7 (Stein and Talati [10]). *Let $Z = \sum I_{E_i}X_i$ with unrelated fuzzy variables X_i, each with membership function μ with support contained in the nonnegative integers. Let $\tilde{\mu}$ be an extension of μ to the nonnegative reals that is convex. Let $\tilde{Z}$ be the fuzzy random variable obtained if we use $\tilde{\mu}$ instead of μ. Then*

(i) $\mu \leq \mu_{E(Z)} \leq \mu_{E(\tilde{Z})} = \tilde{\mu}$ *and*

(ii) $\mu_{E(Z)}(t) = \mu_{E(\tilde{Z})}(t)$ if $t \in \{\mu > 0\}$.

Proof. (i) Taking $x_i = t$,

$$\mu_{E(Z)}(t) = \sup_{\sum p_ix_i = t} [\mu(x_1) \wedge \cdots \wedge \mu(x_n)] \geq \mu(t).$$

Since $\tilde{\mu}$ is extension of μ, we have $\mu \leq \tilde{\mu}$. Thus from the definition, $\mu_{E(Z)} \leq \mu_{E(\tilde{Z})}$. By Theorem 3.5 and convexity of $\tilde{\mu}$, we have $\mu_{E(\tilde{Z})} = \tilde{\mu}$.

(ii) Since $\tilde{\mu}$ is an extension, $\mu = \tilde{\mu}$ on $\{\mu > 0\}$. Thus (ii) follows from (i). □

This theorem shows in (i) that the membership functions of $E(Z)$ and $E(\tilde{Z})$ are close if μ and $\tilde{\mu}$ are ; and in (ii) that the membership function of $E(Z)$ and $E(\tilde{Z})$ agree on the original points. So we see that we should choose an extension $\tilde{\mu}$ that is close to μ as is possible and is also convex.

Definition 3.8. *Let μ be as above. Define for $t \geq 0$,*

$$\bar{\mu}(t) = \begin{cases} \mu(t) & \text{if } \mu(t) > 0, \\ \mu([t]) \wedge \mu([t]+1) & \text{if } \mu(t) = 0. \end{cases}$$

If $\bar{\mu}$ is convex, we call it the minimal extension of μ.

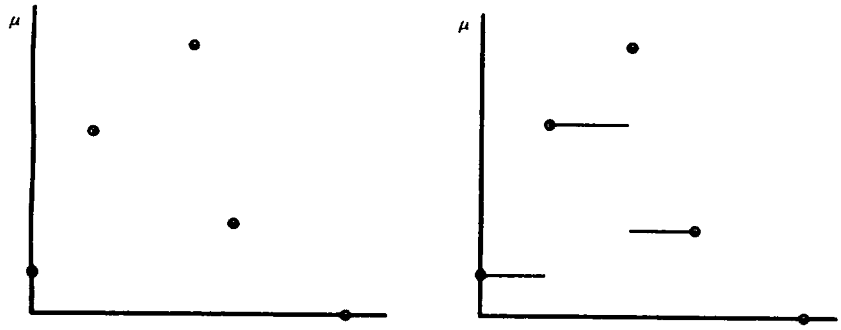

Fig. 2. A discrete membership function and its minimal extension.

In Fig. 2, the minimal extension of a typical membership function is given. It is clear that if $\bar{\mu}$ is convex and $\tilde{\mu}$ is any extension of μ that is also convex, then $\bar{\mu} \leq \tilde{\mu}$. So in this sence, $\bar{\mu}$ is the closest convex membership function to μ.

A fuzzy random variable can be considered as a generalized random variable, since it takes values in the linear space $\mathbb{R}^{\Gamma}$. This is sufficient to be able to define an expectation (Definition 3.1) as a linear operator. Extending the linearity proven in Proposition 3.2, we can show the following theorem.

Theorem 3.9 (Stein and Talati [10]). *Let $Z_1, \cdots, Z_k$ be fuzzy random variables and let $\alpha_1, \cdots, \alpha_k$ be any real numbers. Then*

$$E(\sum_{i=1}^{k} \alpha_i Z_i) = \sum_{i=1}^{k} \alpha_i E(Z_i).$$

Proof. Since

$$\begin{aligned} E(\sum_{i=1}^{k} \alpha_i Z_i) &= E(\alpha_1 Z_1 + \cdots + \alpha_k Z_k) \\ &= E(\alpha_1 Z_1) + \cdots + E(\alpha_k Z_k) \\ &= \alpha_1 E(Z_1) + \cdots + \alpha_k E(Z_k) \\ &= \sum_{i=1}^{k} \alpha_i E(Z_i), \end{aligned}$$

the theorem holds by Proposition 3.2. □

In probability theory, we know that the sample mean $\bar{X}$ is an unbiased estimator of the population mean when all the summand show the same expectation.

Corollary 3.10. *Let $Z_1, \cdots, Z_n$ be fuzzy random variables with all the $E(Z_i)$ unrelated and each having the same convex membership function μ. If $\bar{Z} = \frac{1}{n}(Z_1 + \cdots + Z_n)$, then $E(\bar{Z})$ has the membership function μ.*

Proof. By Theorem 3.9,

$$E(\bar{Z}) = \frac{1}{n}\{E(Z_1) + \cdots + E(Z_n)\}.$$

Thus $E(\bar{Z})$ has the membership function μ by Theorem 3.5. □

Proposition 3.11(Stein and Talati [10]). *Let Z, W be independent fuzzy random variables. Then $E(ZW) = E(Z)E(W)$.*

Proof. Let $Z = \sum_{i=1}^{n} I_{E_i} X_i$ and $W = \sum_{j=1}^{n} I_{F_j} Y_j$, where $\{E_i\}$ and $\{F_j\}$ are both partitions of Ω. Since Z and W are independent, $P(E_i \cap F_j) = P(E_i)P(F_j)$. Thus

$$\begin{aligned} E(ZW) &= E(\sum_{i=1}^{n} I_{E_i} X_i \sum_{j=1}^{n} I_{F_j} Y_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i \cap F_j) X_i Y_j \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i)P(F_j) X_i Y_j \\ &= \sum_{i=1}^{n} P(E_i) X_i \sum_{j=1}^{n} P(F_j) Y_j \\ &= E(Z)E(W). \end{aligned}$$

□

4. Convergence of fuzzy random variables

Definition 4.1. *The sequence of fuzzy variables $\{X_n\}$ is said to converge to the fuzzy variable X if $X_n(\gamma) \to X(\gamma)$ for all $\gamma \in \Gamma$.*

Definition 4.2. *The sequence of fuzzy random varibles $\{Z_n\}$ is said to converge almost surely* (a.s.) to the fuzzy random variable Z if there exist a set $F \subset \Omega$ with $P(F) = 0$ such that for every $\omega \in F^c$, $Z_n(\omega) \to Z(\omega)$ as $n \to \infty$.

Definition 4.1 is not same as pointwise convergence of the membership functions. For example, consider $\Gamma = \{\alpha, \beta\}$ with a scale σ defined on each element. Define the fuzzy variables:

$$X_n(\alpha) = 1, \qquad X(\alpha) = 1,$$
$$X_n(\beta) = 1 + 1/n, \quad X(\beta) = 1,$$

so that $X_n \to X$. Now,

$$\lim \mu_{X_n}(t) = \begin{cases} \sigma(\alpha) & \text{if t=1,} \\ 0 & \text{otherwise,} \end{cases}$$

whereas

$$\mu_{\lim X_n}(t) = \begin{cases} \sigma(\alpha \cup \beta) & \text{if t=1,} \\ 0 & \text{otherwise,} \end{cases}$$

which will not be the same if $\sigma(\beta) > \sigma(\alpha)$.

Theorem 4.3. *Let $\{Z_i\}$ be independent fuzzy random variables. Let*

$$Z_i = \begin{cases} X & \textit{with probability} \quad p, \\ Y & \textit{with probability} \quad q, \end{cases}$$

where X, Y are unrelated fuzzy variables. If $\bar{Z}_n = \frac{1}{n}(Z_1 + \cdots + Z_n)$, then $\bar{Z}_n$ converge almost surely (a.s.) to $E(Z_1)$ as $n \to \infty$ and $\mu_{E(\bar{Z}_n)} = \mu_{E(Z_1)}$.

Proof. Note that $\bar{Z}_n(\omega) = \left(\frac{k}{n}\right)X + \left(\frac{n-k}{n}\right)Y$, where k is a binomial random variable that represents the number of the successes in the first n trials. By the nonfuzzy strong law of large numbers, $\frac{k}{n}$ converge to p a.s., so that there exist $F \subset \Omega$ with $P(F) = 0$ such that for every $\omega \in F^c$, $\frac{k(\omega)}{n}$ converge to $p(\omega) = p$ as $n \to \infty$. Therefore $\bar{Z}_n$ converges to $E(Z_1) = pX + qY$ a.s..

By Theorem 3.9, $E(\bar{Z}_n(\omega)) = \frac{1}{n}(E(Z_1) + \cdots + E(Z_n))$. Since $E(Z_i) = E(Z_j)$ for all i and j,

$$\begin{aligned} \mu_{E(\bar{Z}_n(\omega))} &= \mu_{\frac{1}{n}(E(Z_1)+\cdots+E(Z_n))} = \mu_{\frac{1}{n}((pX+qY)+\cdots+(pX+qY))} \\ &= \mu_{\frac{1}{n}(npX+nqY)} = \mu_{pX+qY} \\ &= \mu_{E(Z_1)}. \end{aligned}$$ □

Theorem 4.4. *Let $\{Z_i\}$ be independent fuzzy random variables such that*

$$Z_i = \begin{cases} X_1 & \textit{with probability} \quad p_1, \\ \vdots & \\ X_m & \textit{with probability} \quad p_m, \end{cases}$$

where $\sum_{i=1}^{m} p_i = 1$, X_i's are unrelated fuzzy variables. If $\bar{Z}_n = \frac{1}{n}(Z_1 + \cdots + Z_n)$, then $\bar{Z}_n$ converge almost surely(a.s.) to $E(Z_1)$ as $n \to \infty$ and $\mu_{E(\bar{Z}_n)} = \mu_{E(Z_1)}$.

Proof. Note that $Z_n(\omega) = (\frac{k_1}{n})X_1 + \cdots + \left(\frac{n-(k_1+\cdots+k_{m-1})}{n}\right)X_m$, where k_i's are multinomial random variables. By the nonfuzzy strong law of large numbers, $\frac{k_i}{n}$ converge to p_i a.s., so that there exist $F \subset \Omega$ with $P(F) = 0$ such that for every $\omega \in F^c$, $\frac{k_i(\omega)}{n}$ converge to $p_i(\omega) = p_i$ as $n \to \infty$. Therefore $\bar{Z}_n(\omega)$ converges to $E(Z_1) = p_1X_1 + \cdots + p_mX_m$ a.s..

By Theorem 3.9, $E(\bar{Z}_n(\omega)) = \frac{1}{n}(E(Z_1) + \cdots + E(Z_n))$. Since $E(Z_i) = E(Z_j)$ for all i and j,

$$\begin{aligned}
\mu_{E(\bar{Z}_n(\omega))} &= \mu_{\frac{1}{n}(E(Z_1)+\cdots+E(Z_n))} \\
&= \mu_{\frac{1}{n}((p_1X_1+\cdots+p_mX_m)+\cdots+(p_1X_1+\cdots+p_mX_m))} \\
&= \mu_{\frac{1}{n}(np_1X_1+\cdots+np_mX_m)} = \mu_{p_1X_1+\cdots+p_mX_m} \\
&= \mu_{E(Z_1)}.
\end{aligned}$$

□

Using the same approach as in the above theorem, it is also possible to reformulate the fuzzy random variables Z_i so that the values are merely unrelated fuzzy variables. In this case, we shall consider pointwise convergence of membership functions. The following lemma will be required.

Lemma 4.5. *If $X_1, \cdots, X_n$ are unrelated fuzzy variables with X_i having an $N(a_i, b_i)$ membership function, $b_i > 0$, so that $\mu_{X_i}(x) = exp\{-(x-a_i)^2/{b_i}^2\}$, then $c_1X_1 + \cdots + c_nX_n$ is a fuzzy variable with membership function $N(\sum c_ia_i, \sum c_ib_i)$ for any positive real numbers $c_1, \cdots, c_n$.*

Proof. Consider $Z = X_1 + X_2$. Without loss of generality, we can assume that $a_2 > a_1$ and $b_2 > b_1$. Then we have $\mu_Z(z) = \sup_x[\mu_{X_1}(x) \wedge \mu_{X_2}(z-x)]$ by Theorem 2.3. Let $g_z(x) = \mu_{X_1}(x) \wedge \mu_{X_2}(z-x)$ for fixed z. As a function of x, $g_z(x)$ is unimodal and achives its maximum of $x_1(z)$ solving

$$\mu_{X_1}(x_1(z)) = \mu_{X_2}(z - x_1(z)),$$

which gives $x_1(z)$ satisfying the quadratic equation

$$[(x_1(z) - a_1)/b_1]^2 = [(z - x_1(z) - a_2)/b_2]^2.$$

The solutions obtained from the quadratic formula.

$$x_1(z) = ({b_2}^2 - {b_1}^2)^{-1}\{a_1{b_2}^2 - (z-a_2){b_1}^2 \pm b_1b_2(z - a_1 - a_2)\}.$$

Since $\mu_Z(z) = \mu_{X_1}(x_1(z)) = \exp\{-(x_1(z) - a_1)^2/{b_1}^2\}$, we can alternatively substitute the positive and negative roots for $x_1(z)$. Substituting the positive root, we obtain

$$\mu^+_{X_1}(x_1(z)) = \exp\{-((b_2 - b_1)^{-1}(z - a_1 - a_2))^2\}$$

and substituting the negative root,

$$\mu^-_{X_1}(x_1(z)) = \exp\{-((b_2 + b_1)^{-1}(z - a_1 - a_2))^2\}.$$

Since $b_1 > 0$ and $b_2 > b_1$, we have $\mu^-_{X_1}(x_1(z)) > \mu^+_{X_1}(x_1(z))$ and the supremum is achived at the negative root. Hence

$$\mu_Z(z) = \exp\{-((b_2 + b_1)^{-1}(z - (a_1 + a_2)))^2\}.$$

For any $n \in \mathbb{N}$, we consider $Z = \sum_{i=1}^n X_i$, then Z is a fuzzy variable with membership function $N(\sum a_i, \sum b_i)$ by induction.

Suppose that $Z = c_1 X_1$, $c_1 \neq 0$. Then $\mu_Z(z) = \mu_{X_1}(\frac{z}{c_1}) = \exp(-(\frac{z}{c_1} - a_1)^2/{b_1}^2) = \exp(-(z - c_1 a_1)^2/(c_1 b_1)^2)$, so that Z is a fuzzy variable with membership function $N(c_1 a_1, c_1 b_1)$. Thus, if $Z = \sum_{i=1}^n c_i X_i$ for $c_i \neq 0$, then Z is a fuzzy variable with membership function $N(\sum c_i a_i, \sum c_i b_i)$ by induction. □

Theorem 4.6. *Let $\{Z_i\}$ be independent fuzzy random variables. Let*

$$Z_i = \begin{cases} X_i & \text{with probability} \quad p, \\ Y_i & \text{with probability} \quad q. \end{cases}$$

Aassume that $\{X_i\}$ and $\{Y_i\}$ are all unrelated fuzzy variables. Suppose that each X_i has an $N(1,1)$ membership function while Y_i has an $N(0,1)$. If $\bar{Z}_n = \frac{1}{n}(Z_1 + \cdots + Z_n)$, then the membership functions of the sequence $\bar{Z}_n(\omega)$ converge almost surely (a.s.) to the membership function of $E(Z_1) = pX_1 + qY_1$ which is $N(p, 1)$.

Proof. By Lemma 4.5, we know that $X_1 + X_2$ has the same membership function as $2X_1$, so the membership function of $\bar{Z}_n(\omega)$ is the same as that of $\left(\frac{k(\omega)}{n}\right)X_1 + \left(\frac{n - k(\omega)}{n}\right)Y_1$, where $k(\omega)$ is the value of a binomial random variable.

Since X_i has a membership function $N(1,1)$ and Y_i has a membership function $N(0,1)$, $\left(\frac{k(\omega)}{n}\right)X_1 + \left(\frac{n - k(\omega)}{n}\right)Y_1$ has a membership function $N\left(\frac{k(\omega)}{n}, 1\right)$ by Lemma 4.5. Since $k(\omega)$ is the value of binomial random variable and $N(a, b)$ is continuous, $N\left(\frac{k}{n}, 1\right)$ converges a.s. to $N(p, 1)$ as $n \to \infty$. So there exist a set $F \subset \Omega$ with $P(F) = 0$ such that for every $\omega \in F^c$, $\mu_{\bar{Z}_n(\omega)}$ converges to $\mu_{E(Z_1)}$ with membership function $N(p, 1)$ as $n \to \infty$. □

Theorem 4.7. *Let $\{Z_i\}$ be independent fuzzy random variables. Let*

$$Z_i = \begin{cases} X_i^1 & \text{with probability} \quad p_1, \\ \vdots & \\ X_i^m & \text{with probability} \quad p_m, \end{cases}$$

where $\sum_{j=1}^{m} p_j = 1$.
Aassume that $\{X_i^j\}$*'s are unrelated fuzzy variables. Suppose that each* X_i^j *has an* $N(a_j, b_j)$ *membership function ,* $1 \le j \le m$ *and* $1 \le i \le n$. *If* $\bar{Z}_n = \frac{1}{n}(Z_1 + \cdots + Z_n)$, *then the membership functions of the sequence* $\bar{Z}_n(\omega)$ *converge a.s. to the membership function of* $E(Z_1) = p_1 X_1^1 + \cdots + p_m X_1^m$ *which is* $N(\sum_{j=1}^{m} p_j a_j, \sum_{j=1}^{m} p_j b_j)$.

Proof. By Lemma 4.5, we know that $X_1^1 + X_2^1$ has the same membership function as $2X_1^1$, so the membership function of $\bar{Z}_n(\omega)$ is the same as that of $\left(\frac{k_1(\omega)}{n}\right)X_1^1 + \cdots + \left(\frac{k_{m-1}(\omega)}{n}\right) \cdot X_1^{m-1} + \left(\frac{n-(k_1(\omega)+\cdots+k_{m-1}(\omega))}{n}\right)X_1^m$,where $k_j(\omega)$ is the value of a multinomial random variables.

Since X_i^j has a membership function $N(a_j, b_j)$, $\left(\frac{k_1(\omega)}{n}\right)X_1^1 + \cdots + \left(\frac{k_{m-1}(\omega)}{n}\right)X_1^{m-1} + \left(\frac{n-(k_1(\omega)+\cdots+k_{m-1}(\omega))}{n}\right)X_1^m$ has a membership function $N(\sum_{j=1}^{m}\left(\frac{k_j}{n}\right)a_j, \sum_{j=1}^{m}\left(\frac{k_j}{n}\right) b_j)$ by Lemma 4.5. Since k_j's are the multinomial random variables and $N(a, b)$ is continuous, $N(\sum_{j=1}^{m}\left(\frac{k_j}{n}\right)a_j, \sum_{j=1}^{m}\left(\frac{k_j}{n}\right)b_j)$ converges a.s. to $N(\sum_{j=1}^{m} p_j a_j, \sum_{j=1}^{m} p_j b_j)$ as $n \to \infty$. So there exist a set $F_j \subset \Omega$ with $P(F_j) = 0$ such that for every $\omega \in {F_j}^c$, $\mu_{\bar{Z}_n(\omega)}$ converges to $\mu_{E(Z_i)}$ as $n \to \infty$ with membership function $N(\sum_{j=1}^{m} p_j a_j, \sum_{j=1}^{m} p_j b_j)$. □

REFERENCES

[1] S.S.L. Chang, On risk and decision making in a fuzzy environment. in: L.A. Zadeh et al., Eds., Fuzzy Sets and Their Applications to Cognitive and Decision Processes (Academic Press. New York, 1975).

[2] D. Dubois and H. Prade, Fuzzy real algebra: some results. Fuzzy Sets and Systems 2 (1979) 327-348.

[3] S. Nahmias, Discrete fuzzy random variables, Technical Report 18. Department of Ind. Eng. Syst. Manag. Eng. and Operat. Res., University of Pittsburg. PA (1974).

[4] S. Nahmias, Fuzzy variables, Fuzzy Sets and Systems 1 (1978) 97-110.

[5] H.T. Nguyen, A note on the extension principle for fuzzy sets, J. Math. Anal. Appl. 64(1978) 369-380.

[6] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems 1 (1978) 3-28.

[7] D.A. Kappos, Probability Algebra and Stochastic Spaces (Academic Press. New York, 1969).

[8] L.A. Zadeh, The concept of a linguistic variable and its application to approximation reasoning II, Information Sci. 8 (1975) 301-357.

[9] L.A. Zadeh, The concept of a linguistic variable and its application to approximation reasoning III, Information Sci. 9 (1975) 43-80.

[10] W.E. Stein and K. Talati, Convex fuzzy random variables, Fuzzy Sets and Systems 6 (1978) 271-283.