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Comparisons of the Determinant of a Fuzzy matrix by t and s norms

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Abstract

In this article we compare the determinants of a fuzzy matrix. over the interval[0,1] when it is endowed with the algebraic structure a rising via various triangular t noms and their corresponding s norms.

1. Introduction

There are some papers on the determinant theory of fuzzy matrices algebraic structures.

A fair amount of effort has been directed towards discovering various properties of matrices when the underlying algebraic structure is a semiring.

A rich determinantal theory is available for matrices over semiring (See Kim[1])

The adjoint of a square fuzzy matrix is defined by Regab and Emam[2].

In this paper we compare the determinants of a fuzzy matrix over the interval [0,1] when it is endowed with the algebraic structure arising via various triangular t norms and s norms

2. Triangular norms and s norms

It is well known that triangular norms (t norm) and s norms are used very often in fuzzy set theory

Triangular norms (t norms) are used to define the intersection of fuzzy sets and theory. Triangular norms (t norms) are used to define the intersection of fuzzy sets and the conjunction of fuzzy statements. s norms are used to define the union of fuzzy sets and the disjunction of fuzzy statements.

Triangular norms and s norms are binary operations on the interval [0,1] that satisfy certain conditions.

An inportant review of fuzzy

connectives, aggregation operators and t norms and s norms is given in the paper by Dubois and Prade[3].

Definition 2.1 A binary operation $\otimes$: $[0,1]\times[0,1]\rightarrow[0,1]$ is called t norm if for $x, y, z\in[0,1]$

(2.1a) $x\otimes y=y\otimes x$

(2.1b) $x\otimes(y\otimes z)=(x\otimes y)\otimes z$

(2.1c) $x\otimes y\leq x\otimes z$ if $y\leq z$

(2.1d) $x\otimes 1=x$.

Definition 2.2 A binary operation $\oplus$: $[0,1]\times[0,1]\rightarrow[0,1]$ is s norm if for all $x, y, z\in[0,1]0$

(2.2a) $x\oplus y=y\oplus x$

(2.2b) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$

(2.2c) $x\oplus y\leq x\oplus z$ if $y\leq z$

(2.2d) $x\oplus 0=x$.

In the sequel $\otimes$ shall always denote a t norm and $\oplus$ shall denote a s norm.

Here are some examples of t norm and s norms.

1. $(\otimes, \oplus)=(t_0, s_0)$

$$x\otimes y=\begin{cases}\min(x,y) & \text{if } \max(x,y)=1\\ 0 & \text{otherwise}\end{cases}$$

$$x\oplus y=\begin{cases}\max(x,y) & \text{if } \min(x,y)=0\\ 1 & \text{otherwise}\end{cases}$$

2. $(\otimes, \oplus)=(t_1, s_1)$

$$x\otimes y=\max(0,x+y-1)$$
$$x\oplus y=\min(1,x+y)$$

3. $(\otimes, \oplus)=(t_2, s_2)$

$$x\otimes y=xy$$
$$x\oplus y=x+y-xy$$

4. $(\otimes, \oplus)=(t_3, s_3)$

$$x\otimes y=\frac{xy}{x+y-xy}$$
$$x\oplus y=\frac{x+y-2xy}{1-xy}$$

5. $(\otimes, \oplus)=(t_5, s_5)$

$$x\otimes y=\min(x,y)$$
$$x\oplus y=\max(x,y)$$

Proposition 2.1 Let $\otimes$ be a t norm and $\oplus$ be a s norm Then

(2.3a) $x\oplus y\geq x\vee y$ for every x, y where $x\vee y=\max(x,y)$

(2.3b) $x\oplus y\leq x\wedge y$ for every x, y where $x\wedge y=\min(x,y)$

Definition 2.3 A pair, consisting of a t-norm and s-norm are said to be dual or associated if

$$x\oplus y=1-((1-x)\otimes(1-y))\ \forall x,y\in[0,1]$$

or

$$x\otimes y=1-((1-x)\oplus(1-y))\ \forall x,y\in[0,1]$$

We then also say s norm $\oplus$ is associated to t norm $\otimes$ or that $\otimes$ is associated to $\oplus$.

Associated t norms and s norms are also called conjugate in the literature.

Remark 1. The smallest t norm and its corresponding s norm are as follows :

$$x\otimes y=\begin{cases}x & \text{if } y=1\\ y & \text{if } x=1\\ 0 & \text{otherwise}\end{cases}$$

$$x\oplus y=\begin{cases}x & \text{if } y=0\\ y & \text{if } x=0\\ 1 & \text{otherwise}\end{cases}$$

Remark 2. The largest t norm is

$$x \otimes y = \min(x, y).$$

And also the smallest s norm is

$$x \oplus y = \max(x, y).$$

Definition 2.4 A t norm $\otimes$ is called Archimedian if it is continuous and sastisfies $x \otimes x < x$ whenever $0 < x < 1$.

Definition 2.5 A t norm $\otimes$ is called Archimedian if it is continuous and sastisfies $x \otimes y < x \otimes z$ whenever $x > 0$ and $y < z$.

From the definintion of a t norm $\otimes$ the following property is derived immediately :

$$x \otimes 0 = 0$$

More over each pair $(\otimes, \oplus)$ of mutually corresponding t-norms and s-norms satisfies the following equation

$$(x \otimes y) + (x \oplus y) = x + y.$$

Since a t norm $\otimes$ and its corresponding s-norm $\oplus$ are binary operation on $[0,1]$, their associativity allows then to be extended to n-any operations

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n : [0,1]^n \to [0,1]$$

and

$$x_1 \oplus x_2 \oplus \cdots \oplus x_n : [0,1]^n \to [0,1].$$

In that follows we shall write $T(x_1, \cdots, x_n)$ and $S(x_1, \cdots, x_n)$ instead of $x_1 \otimes \cdots \otimes x_n$ and $x_1 \oplus \cdots \oplus x_n$, respectively.

The extension to an infinitely operation is also possible. For each sequence $(x_n)_{n \in N}$ in $[0,1]$, the sequence $(T(x_1, \cdots, x_n))_{n \in N}$ is nonincreasing. Therefore its limit

$$T(x_1, x_2, \cdots) = \lim_{n \to \infty} T(x_1, x_2, \cdots, x_n)$$

always exists.

Proposition 2.2. Let $\otimes$ be an Archimedean t-norm and let $(x_n)_{n \in N}$ be a constant sequence in $[0,1]$, i.e. $x_n = a$ for all $n \in N$. Then we have $\lim_{n \to \infty} T(x_1, x_2, \cdots x_n) = 0$.

Proof. Assume that $a \neq 0$

Consider the function $h : X \to [0,1]$ defined by $h(x) = x \otimes x$ putting $h^{\cdot} = h$, $h^{n+1} = h \otimes h^n$.

we have for every $x \in (0,1)$ $h(x) < x$. $h^{n+1}(x) \le h^n(x)$ by Archimedian property.

Then for $b = \lim_{n \to 0} h^n(a)$ we get

$$h(b) = h(\lim_{n \to \infty} h^n(a)) = \lim_{n \to \infty} h^{n+1}(a) = b$$

which implies b=0

Since the sequence $(h^n(a))_{n \in N}$ is a subsequence of the convergent sequence. $(T(x_1, \cdots, x_n))_{n \in N}$ the result follows.

3. Determinant of a fuzzy matrix

Definition 3.1 For fuzzy matrices $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times p}$ and $C = [c_{ij}]_{n \times p}$ the following operations are defined

(3.1a) $B + C = [b_{ij} \oplus c_{ij}]$ where $\oplus$ is a s norm.

(3.1b) $AB=[\sum_{k=1\oplus}^{n} a_{ik}\otimes b_{kj}]$ where $\otimes$ is a t norm,

(3.1c) $A'=[a_{ji}]$ (the transpose of A).

(3.1d) $A_k=[a_{ij}^k]$, $A^{k+1}=A^k\otimes A$,

(3.1e) $A^0=I_n$ where I_n is the usual identity matrix.

Definition 3.2 Let $A=[a_{ij}]$ be an $n\times n$ matrix over the interval [0,1], $\otimes$ be a t norm and $\oplus$ be a s norm. The determinant of an $n\times n$ fuzzy matrix A with respect to $\otimes$ and $\oplus$ is defined as follows :

$$|A|=\sum_{\sigma\in S_n\oplus} a_{1\sigma(1)}\otimes a_{2\sigma(2)}\otimes\cdots\otimes a_{n\sigma(n)}$$

where S_n denotes the symmetric group of all permutations of indices $(1,2,\cdots,n)$.

Proposition 3.1 If a fuzzy matrix B is obtained from an $n\times n$ fuzzy A by multiplying the ith row of A by $k\in(0,1]$, then $k|A|=B$.

Proof. By definition

$$\begin{aligned}|B| &= \sum_{\sigma\in S_n\oplus} b_{1\sigma(1)}\otimes b_{2\sigma(2)}\otimes\cdots\otimes b_{n\sigma(n)}\\ &= \sum_{\sigma\in S_n\oplus} a_{1\sigma(1)}\otimes\cdots ka_{i\sigma(i)}\cdots\otimes a_{n\sigma(n)}\\ &= k\sum_{\sigma\in S_n} a_{1\sigma(1)}\otimes\cdots a_{i\sigma(i)}\cdots\otimes a_{n\sigma(n)}\\ &= k|A|\end{aligned}$$

Proposition 3.2 Let A be an $n\times n$ fuzzy matrix.

If A contains a zero row(column) , then |A|=0

Proposition 3.3 For any square fuzzy matrix A, Let $A_{i,j}$ be the determinant of the submatrix obtained by deleting row i and column j of A.

Then for any i and j

$$|A|=\sum_{k=1\oplus}^{n} a_{ik}\otimes A_{ik}=\sum_{k=1\oplus}^{n} a_{kj}\otimes A_{kj}$$

Example 1. For a fuzzy matrix

$$A=\begin{bmatrix}0.5 & 0.3 & 0.8\\ 0.6 & 0.2 & 0.9\\ 0.0 & 0.7 & 0.4\end{bmatrix}$$

we calculate the determinant $|A|$ using t norm $\otimes$ and s norm $\oplus$ as follows:

$$\begin{aligned}|A|=0.5\otimes&\begin{vmatrix}0.2 & 0.9\\ 0.7 & 0.4\end{vmatrix}\\ \oplus 0.3\otimes&\begin{vmatrix}0.6 & 0.9\\ 0.0 & 0.4\end{vmatrix}\\ \oplus 0.8\otimes&\begin{vmatrix}0.6 & 0.2\\ 0.0 & 0.7\end{vmatrix}\end{aligned}$$

1) For $x\otimes y=\min(x,y)$ and $x\oplus y=\max(x,y)$

$$\begin{aligned}|A| &= 0.5\otimes(0.2\oplus 0.7)\oplus 0.3\otimes(0.4\oplus 0.0)\\ &\quad \oplus 0.8\otimes(0.6\oplus 0.0)\\ &= 0.5\otimes 0.7\oplus 0.3\otimes 0.4\oplus 0.8\otimes 0.6\\ &= 0.5\oplus 0.3\oplus 0.6=0.6\end{aligned}$$

2) For $x\otimes y=xy$ and $x\oplus y=x+y-xy$

$$\begin{aligned}|A| &= 0.5\otimes(0.2+0.7-0.2\times 0.7)\oplus 0.3\otimes\\ &\quad (0.4+0.0-0.0)\oplus 0.8\otimes(0.6+0.0\\ &\quad -0.0)\\ &= 0.5\otimes(0.9-0.14)\oplus 0.3\otimes 0.4\oplus 0.8\\ &\quad \otimes 0.6\\ &= 0.38\oplus 0.12\oplus 0.48=0.716288\end{aligned}$$

3) For $x \otimes y = \max(0, x+y-1)$

$x \oplus y = \min(1, x+y)$

$|A| = 0.5 \otimes 0.6 \oplus 0.3 \otimes 0.0 \oplus 0.8 \otimes 0.3$

$= 0.1 \oplus 0 \oplus 0.1 = 0.1 + 0.1 = 0.2$

4) For $x \otimes y = \frac{xy}{x+y-xy}$

$x \oplus y = \frac{x+y-2xy}{1-xy}$

$|A| \fallingdotseq (0.5 \otimes 0.67) \oplus (0.3 \otimes 0.316) \oplus (0.8 \otimes 0.477)$

$= 0.401 \oplus 0.182 \oplus 0.426$

$= 0.62$

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