J. of Basic Sciences, Cheju Nat'l Univ.
18(2), 63~76, 2005

기초과학연구 제주대학교
18(2), 63~76, 2005

Inequalities of some operators on fuzzy matrices

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Abstract

The fuzzy matrices have been popular topics for last decades because they are successfully used when fuzzy uncertainty occurs in various problems. In this paper, two new binary fuzzy operators $\uplus$ and $\circledast$ are introduced for fuzzy matrices. Several properties on $\uplus$ and $\circledast$ are presented in this paper. We also compare theses operators with existing operators.

Keywords: Fuzzy set, fuzzy matrix, fuzzy operators.

AMS Subject Classifications: 03E73, 04A72.

1 Introduction and Definitions

Matrices play major rule in various areas such as mathematics, physics, statistics, engineering, social sciences and many others. Fuzzy matrices arise in many applications, and several authors ([1]-[8] and therein) presented a number of results on fuzzy matrices.

We define some operators on fuzzy matrices whose elements are confined in the closed interval $\mathbb{F} = [0, 1]$. For all $x, y, \alpha, \lambda \in \mathbb{F}$, the following operators are defined:

(i) $x \vee y = \max\{x, y\}$,

(ii) $x \wedge y = \min\{x, y\}$,

(iii) $x \oplus y = x + y - x \cdot y$,

(iv) $x \odot y = x \cdot y$,

(v) $x^{(\alpha)} = \begin{cases} 1 & \text{if } x \geq \alpha \\ 0 & \text{if } x < \alpha \end{cases}$ and $x_{(\alpha)} = \begin{cases} x & \text{if } x \geq \alpha \\ 0 & \text{if } x < \alpha, \end{cases}$

(vi) $x^c = 1 - x$,

(vii) $x +_\lambda y = \lambda x + (1 - \lambda)y$,

where the operations "+, - , ·" are ordinary addition, subtraction and multiplication, respectively. In (v), $x^{(\alpha)}$ and $x_{(\alpha)}$ are called *upper α-cut* and *lower α-cut* of x, respectively.

Now, we define two new operators $\uplus$ and $\circledast$ as follows: for all $x, y \in \mathbb{F}$,

(viii) $x \uplus y = \begin{cases} 1 & \text{if } x > y \\ y & \text{if } x \leq y \end{cases}$,

(ix) $x \circledast y = x^2 + y - x \cdot y$.

We note that the values of the results of operators which are listed in the above (from (i) to (ix)) belong to $\mathbb{F}$.

Let $\mathcal{M}_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ matrices over $\mathbb{F}$. If $m = n$, we use the notation $\mathcal{M}_n(\mathbb{F})$ instead of $\mathcal{M}_{m,n}(\mathbb{F})$. The matrix I_n is the $n \times n$ identity matrix and the matrix J_n is the $n \times n$ matrix whose elements are all 1.

For all $A = [a_{i,j}], B = [b_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{F})$ and for all $\alpha \in \mathbb{F}$, the following operators are defined:

(i) $A \vee B = [a_{i,j} \vee b_{i,j}]$,

(ii) $A \wedge B = [a_{i,j} \wedge b_{i,j}]$,

(iii) $A \oplus B = [a_{i,j} \oplus b_{i,j}]$,

(iv) $A \odot B = [a_{i,j} \cdot b_{i,j}]$,

(v) $A^{(\alpha)} = [a_{i,j}{}^{(\alpha)}]$ and $A_{(\alpha)} = [a_{i,j_{(\alpha)}}]$

(vi) $A^c = [1 - a_{i,j}]$,

(vii) $A +_\lambda B = [a_{i,j} +_\lambda b_{i,j}]$,

(viii) $A \uplus B = [a_{i,j} \uplus b_{i,j}]$,

(ix) $A \circledast B = [a_{i,j} \circledast b_{i,j}]$,

(x) $A \leq B$ if and only if $a_{i,j} \leq b_{i,j}$ for all i and j.

Throughout this paper, we assume that $\alpha, \lambda \in \mathbb{F}$ so that $0 \leq \alpha, \lambda \leq 1$.

In [6], Shyamal and Pal characterized some properties of operators $\oplus$ and $\odot$ with pre-defined operators.

In this paper, we introduce new binary operators $\uplus$ and $\circledast$ on fuzzy matrices. Also, some properties of the fuzzy matrices over these new operators and some pre-defined operators are presented.

2 Some results

Now, we define some special types of matrices. An $n \times n$ fuzzy matrix $R = [r_{i,j}]$ is called

(1) *reflexive* if $r_{i,i} = 1$ for all $i = 1, \ldots, n$,

(2) *irreflexive* if $r_{i,i} = 0$ for all $i = 1, \ldots, n$,

(3) *nearly irreflexive* if $r_{i,i} \leq r_{i,j}$ for all $i, j = 1, \ldots, n$,

(4) *symmetric* if $r_{i,j} = r_{j,i}$ for all $i, j = 1, \ldots, n$.

Property 1. *For $A, B \in \mathcal{M}_n(\mathbb{F})$,*

(1) *if A and B are symmetric, then so are $A \uplus B$, $A \circledast B$ and $A +_\lambda B$,*

(2) *if A and B are nearly irreflexive, so are $A \circledast B$ and $A +_\lambda B$,*

(3) *$I_n \uplus A$ and $A \uplus I_n$ are reflexive,*

(4) *$I_n \circledast A$ is reflexive.*

Proof. Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$.

(1) Since A and B are symmetric, $a_{i,j} = a_{j,i}$ and $b_{i,j} = b_{j,i}$ for all $i, j = 1, \dots, n$. Thus, we have

$$a_{i,j} \uplus b_{i,j} = \begin{cases} 1 & \text{if } a_{i,j} > b_{i,j} \\ b_{i,j} & \text{if } a_{i,j} \le b_{i,j} \end{cases} = \begin{cases} 1 & \text{if } a_{j,i} > b_{j,i} \\ b_{j,i} & \text{if } a_{j,i} \le b_{j,i} \end{cases} = a_{j,i} \uplus b_{j,i},$$

$$a_{i,j} \circledast b_{i,j} = {a_{i,j}}^2 + b_{i,j} - a_{i,j} \cdot b_{i,j} = {a_{j,i}}^2 + b_{j,i} - a_{j,i} \cdot b_{j,i} = a_{j,i} \circledast b_{j,i}$$

and

$$a_{i,j} +_\lambda b_{i,j} = \lambda a_{i,j} + (1-\lambda) b_{i,j} = \lambda a_{j,i} + (1-\lambda) b_{j,i} = a_{j,i} +_\lambda b_{j,i}.$$

Hence $A \uplus B$, $A \circledast B$ and $A +_\lambda B$ are symmetric.

(2) If A and B are nearly irreflexive, then $a_{i,i} \le a_{i,j}$ and $b_{i,i} \le b_{i,j}$ for all $i, j = 1, \dots, n$. Let $c_{i,j}$ be the $(i,j)^{\text{th}}$ element of $A +_\lambda B$. Then we have

$$\begin{aligned} c_{i,j} - c_{i,i} &= (\lambda a_{i,j} + (1-\lambda) b_{i,j}) - (\lambda a_{i,i} + (1-\lambda) b_{i,i}) \\ &= \lambda (a_{i,j} - a_{i,i}) + (1-\lambda)(b_{i,j} - b_{i,i}) \ge 0 \end{aligned}$$

Hence $A +_\lambda B$ is nearly irreflexive.

(3) Let $I_n \uplus A = [r_{i,j}]$ and $A \uplus I_n = [g_{i,j}]$. Then we suffice to claim that $r_{i,i} = g_{i,i} = 1$ for all $i = 1, \dots, n$. By the definition of $\uplus$, we have that

$$r_{i,i} = \begin{cases} 1 & \text{if } 1 > a_{i,i} \\ a_{i,i} & \text{if } 1 \le a_{i,i} \end{cases} = 1 = \begin{cases} 1 & \text{if } a_{i,i} > 1 \\ 1 & \text{if } a_{i,i} \le 1 \end{cases} = g_{i,i}$$

for all $i = 1, \dots, n$.

(4) Let $I_n \circledast A = [h_{i,j}]$. Let $i \in \{1, \dots, n\}$ be arbitrary. Then $h_{i,i} = 1 \circledast a_{i,i} = 1^2 + a_{i,i} - 1 \cdot a_{i,i} = 1$. This shows that $I_n \circledast A$ is reflexive. ■

Remark 1. Note that $A \circledast I_n$ may be not reflexive; for if $a_{1,1} = 0.5$, then the $(1,1)^{\text{th}}$ entry of $A \circledast I_n$ is $0.5 \circledast 1 = 0.5^2 + 1 - 0.5 = 0.65 \neq 1$.

Property 2. *Let λ and μ be given in $\mathbb{F}$, and $A, B, C \in \mathcal{M}_{m,n}(\mathbb{F})$. Then*

(1) $A \uplus A = A$,

(2) $A \circledast A = A$,

(3) $A +_\lambda A = A$,

(4) $A +_\lambda B = B +_{(1-\lambda)} A$,

(5) $A +_\lambda (B +_\mu C) = (C +_{1-\mu} B) +_{1-\lambda} A$.

Proof. (1), (2), (3) and (4) are clear by the definitions of $\uplus$, $\circledast$ and $+_\lambda$. So, we will remain to prove (5).

(5) It follows from (4) that

$$A +_\lambda (B +_\mu C) = A +_\lambda (C +_{1-\mu} B) = (C +_{1-\mu} B) +_{1-\lambda} A. \qquad \blacksquare$$

Remark 2. If $\lambda = 0.5$, then $A +_\lambda B = B +_\lambda A$ is obvious. But, if $\lambda \in \mathbb{F} \setminus \{0.5\}$, then $A +_\lambda B = B +_\lambda A$ may be not true. For example, let $a_{1,1} = 1$ and $\lambda = b_{1,1} = 0.1$. Then $a_{1,1} +_\lambda b_{1,1} = 0.19$, while $b_{1,1} +_\lambda a_{1,1} = 0.91$.

Property 3. *Let A and B be matrices in $\mathcal{M}_{m,n}(\mathbb{F})$ with $A \leq B$. Then for any matrix $C \in \mathcal{M}_{m,n}(\mathbb{F})$, we have*

(1) $A \uplus C \leq B \uplus C$,

(2) $A +_\lambda C \leq B +_\lambda C$.

Proof. Let $A = [a_{i,j}]$, $B = [b_{i,j}]$ and $C = [c_{i,j}]$.

(1) Let $d_{i,j}$ and $e_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $A \uplus C$ and $B \uplus C$, respectively. Now, we claim that $d_{i,j} \leq e_{i,j}$. Note that

$$d_{i,j} = \begin{cases} 1 & \text{if } a_{i,j} > c_{i,j} \\ c_{i,j} & \text{if } a_{i,j} \leq c_{i,j} \end{cases} \quad \text{and} \quad e_{i,j} = \begin{cases} 1 & \text{if } b_{i,j} > c_{i,j} \\ c_{i,j} & \text{if } b_{i,j} \leq c_{i,j} \end{cases}.$$

If $b_{i,j} > c_{i,j}$, then $d_{i,j} \le 1 = e_{i,j}$ is clear. If $b_{i,j} \le c_{i,j}$, it follows from $a_{i,j} \le b_{i,j}$ that $a_{i,j} \le c_{i,j}$ so that $d_{i,j} = c_{i,j} = e_{i,j}$. Thus, we have $A \uplus C \le B \uplus C$.

(2) Let $x_{i,j}$ and $y_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $A +_\lambda C$ and $B +_\lambda C$, respectively. It follows from $a_{i,j} \le b_{i,j}$ that

$$y_{i,j} - x_{i,j} = (\lambda b_{i,j} + (1-\lambda)c_{i,j}) - (\lambda a_{i,j} + (1-\lambda)c_{i,j}) = \lambda(b_{i,j} - a_{i,j}) \ge 0.$$

Hence, $A +_\lambda C \le B +_\lambda C$. ∎

Remark 3. For the operator $\circledast$, Property 3 may be not true. For example, let $a_{1,1} = 0.1$, $b_{1,1} = 0.2$ and $c_{1,1} = 0.4$ be the $(1,1)^{\text{th}}$ elements of A, B and C with $A \le B$. Then we have that the value of $(1,1)^{\text{th}}$ element of $A \circledast C$ is 0.37, while that of $(1,1)^{\text{th}}$ element of $B \circledast C$ is 0.36.

Property 4. *Let A and B be matrices in $\mathcal{M}_{m,n}(\mathbb{F})$. Then, we have*

(1) $A \circledast B \le A \vee B \le A \uplus B$,

(2) $A \odot B \le A \wedge B \le A +_\lambda B \le A \vee B \le A \oplus B$.

Proof. Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$.

(1) Let $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $A \circledast B$, $A \vee B$ and $A \uplus B$, respectively. Then we have

$$y_{i,j} - x_{i,j} = \max\{a_{i,j},\ b_{i,j}\} - ({a_{i,j}}^2 + b_{i,j} - a_{i,j} \cdot b_{i,j}) \tag{2.1}$$

and

$$z_{i,j} - y_{i,j} = \begin{cases} 1 & \text{if } a_{i,j} > b_{i,j} \\ b_{i,j} & \text{if } a_{i,j} \le b_{i,j} \end{cases} - \max\{a_{i,j},\ b_{i,j}\} \tag{2.2}$$

Now, we will show that $x_{i,j} \le y_{i,j} \le z_{i,j}$. Two cases arise.

Case 1) $a_{i,j} > b_{i,j}$; then (2.1) and (2.2) become

$$y_{i,j} - x_{i,j} = (a_{i,j} - b_{i,j})(1 - a_{i,j}) \ge 0 \quad \text{and} \quad z_{i,j} - y_{i,j} = 1 - a_{i,j} \ge 0.$$

Thus, we have $x_{i,j} \le y_{i,j} \le z_{i,j}$ in this case.

Case 2) $a_{i,j} \le b_{i,j}$; then (2.1) and (2.2) become $y_{i,j} - x_{i,j} = a_{i,j}(b_{i,j} - a_{i,j}) \ge 0$ and $z_{i,j} - y_{i,j} = b_{i,j} - b_{i,j} = 0$. Thus, we also have $x_{i,j} \le y_{i,j} \le z_{i,j}$ in this case.

(2) Since $a_{i,j} \cdot b_{i,j} \le \min\{a_{i,j},\ b_{i,j}\}$, it follows that $A \odot B \le A \wedge B$. Notice that

$$a_{i,j} \oplus b_{i,j} = a_{i,j} + b_{i,j} - a_{i,j} \cdot b_{i,j} = \begin{cases} a_{i,j} + b_{i,j}(1 - a_{i,j}) \ge a_{i,j} \\ b_{i,j} + a_{i,j}(1 - b_{i,j}) \ge b_{i,j}, \end{cases}$$

and hence $a_{i,j} \oplus b_{i,j} \ge \max\{a_{i,j},\ b_{i,j}\}$. This implies that $A \vee B \le A \oplus B$. Thus, we suffice to show that $A \wedge B \le A +_\lambda B \le A \vee B$.

Let $c_{i,j}$, $d_{i,j}$ and $e_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $A \wedge B$, $A +_\lambda B$ and $A \vee B$, respectively. Then we have

$$d_{i,j} - c_{i,j} = \lambda a_{i,j} + (1-\lambda) b_{i,j} - \min\{a_{i,j}, b_{i,j}\} \tag{2.3}$$

and

$$e_{i,j} - d_{i,j} = \max\{a_{i,j}, b_{i,j}\} - \lambda a_{i,j} - (1-\lambda) b_{i,j}. \tag{2.4}$$

If $\min\{a_{i,j}, b_{i,j}\} = a_{i,j}$, then (2.3) and (2.4) become

$$d_{i,j} - c_{i,j} = \lambda a_{i,j} + (1-\lambda) b_{i,j} - a_{i,j} = (1-\lambda)(b_{i,j} - a_{i,j}) \ge 0$$

and

$$e_{i,j} - d_{i,j} = b_{i,j} - \lambda a_{i,j} - (1-\lambda) b_{i,j} = \lambda(b_{i,j} - a_{i,j}) \ge 0.$$

For the case of $\min\{a_{i,j}, b_{i,j}\} = b_{i,j}$, (2.3) and (2.4) become

$$d_{i,j} - c_{i,j} = \lambda a_{i,j} + (1-\lambda) b_{i,j} - b_{i,j} = \lambda(a_{i,j} - b_{i,j}) \ge 0$$

and

$$e_{i,j} - d_{i,j} = a_{i,j} - \lambda a_{i,j} - (1-\lambda) b_{i,j} = (1-\lambda)(a_{i,j} - b_{i,j}) \ge 0.$$

Thus, we have established that $c_{i,j} \le d_{i,j} \le e_{i,j}$. It follows that

$$A \wedge B \le A +_\lambda B \le A \vee B. \qquad \blacksquare$$

Property 5. *Let A, B and C be matrices in $\mathcal{M}_{m,n}(\mathbb{F})$. Then*

(1) $A +_\lambda (B \wedge C) = (A +_\lambda B) \wedge (A +_\lambda C)$,

(2) $A +_\lambda (B \vee C) = (A +_\lambda B) \vee (A +_\lambda C)$,

(3) $A \wedge (B +_\lambda C) \geq (A \wedge B) +_\lambda (A \wedge C)$,

(4) $A \vee (B +_\lambda C) \leq (A \vee B) +_\lambda (A \vee C)$.

Proof. Let $A = [a_{i,j}]$, $B = [b_{i,j}]$ and $C = [c_{i,j}]$.

(1) Let $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $A +_\lambda (B \wedge C)$, $A +_\lambda B$ and $A +_\lambda C$, respectively. Then we have

$$\begin{aligned} x_{i,j} &= \lambda a_{i,j} + (1-\lambda)\min\{b_{i,j},\ c_{i,j}\} \\ &= \lambda a_{i,j} + \min\{(1-\lambda)b_{i,j},\ (1-\lambda)c_{i,j}\} \\ &= \min\{\lambda a_{i,j} + (1-\lambda)b_{i,j},\ \lambda a_{i,j} + (1-\lambda)c_{i,j}\} \\ &= \min\{y_{i,j},\ z_{i,j}\}. \end{aligned}$$

Thus (1) is satisfied.

(2) Similar to (1).

(3) Let $x_{i,j}$ and $y_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $A \wedge (B +_\lambda C)$ and $(A \wedge B) +_\lambda (A \wedge C)$, respectively. Then we have

$$x_{i,j} = \min\{a_{i,j},\ \lambda b_{i,j} + (1-\lambda)c_{i,j}\}$$

and

$$y_{i,j} = \lambda \min\{a_{i,j},\ b_{i,j}\} + (1-\lambda)\min\{a_{i,j},\ c_{i,j}\}.$$

Now, we note that

$$y_{i,j} \leq \lambda a_{i,j} + (1-\lambda)a_{i,j} = a_{i,j}$$

and

$$y_{i,j} \leq \lambda b_{i,j} + (1-\lambda)c_{i,j}.$$

It follows that $y_{i,j} \leq x_{i,j}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Thus, (3) is satisfied.

(4) Similar to (3). ∎

3 Results on α-cut of fuzzy matrix

The upper α-cut fuzzy matrix is basically a Boolean fuzzy matrix. It represents only two values 0 and 1. But the lower α-cut fuzzy matrix is a multi-graded fuzzy matrix. If the elements of this matrix are less than α, then lower α-cut fuzzy matrix represents same value 0 and all other cases, it represents the actual values.

Property 6. [6]. *For any two fuzzy matrices A and B,*

(i) $(A \oplus B)^{(\alpha)} \geq A^{(\alpha)} \oplus B^{(\alpha)}$,

(ii) $(A \oplus B)_{(\alpha)} \geq A_{(\alpha)} \oplus B_{(\alpha)}$.

Now, we are interesting in α-cut for other fuzzy operators.

Property 7. *Let A and B be matrices in $\mathcal{M}_{m,n}(\mathbb{F})$. Then*

(1) $(A \odot B)^{(\alpha)} \leq A^{(\alpha)} \odot B^{(\alpha)}$,

(2) $(A \odot B)_{(\alpha)} \leq A_{(\alpha)} \odot B_{(\alpha)}$.

Proof. (1) Since $(a_{i,j} \cdot b_{i,j})^{(\alpha)} \in \{0, 1\}$, we suffice to show that if $(a_{i,j} \cdot b_{i,j})^{(\alpha)} = 1$, then ${a_{i,j}}^{(\alpha)} \cdot {b_{i,j}}^{(\alpha)} = 1$; this is, $a_{i,j} \cdot b_{i,j} \geq \alpha$ implies that $a_{i,j} \geq \alpha$ and $b_{i,j} \geq \alpha$. Suppose that $a_{i,j} \cdot b_{i,j} \geq \alpha$. It follows from

$$\min\{a_{i,j},\ b_{i,j}\} \geq a_{i,j} \cdot b_{i,j} \geq \alpha$$

that $a_{i,j} \geq \alpha$ and $b_{i,j} \geq \alpha$. Thus, we have $(A \odot B)^{(\alpha)} \leq A^{(\alpha)} \odot B^{(\alpha)}$.

(2) If $a_{i,j} \cdot b_{i,j} < \alpha$, there is nothing to prove. Assume that $a_{i,j} \cdot b_{i,j} \geq \alpha$. By the similar argument of (1), we have that $a_{i,j} \geq \alpha$ and $b_{i,j} \geq \alpha$, and hence $(A \odot B)_{(\alpha)} \leq A_{(\alpha)} \odot B_{(\alpha)}$. ∎

Property 8. *Let A and B be matrices in $\mathcal{M}_{m,n}(\mathbb{F})$ with $A \leq B$. Then*

(1) $(A \circledast B)^{(\alpha)} \leq A^{(\alpha)} \circledast B^{(\alpha)}$,

(2) $(A \circledast B)_{(\alpha)} \leq A_{(\alpha)} \circledast B_{(\alpha)}$,

(3) $(A \uplus B)^{(\alpha)} \leq A^{(\alpha)} \uplus B^{(\alpha)}$,

(4) $(A \uplus B)_{(\alpha)} \leq A_{(\alpha)} \uplus B_{(\alpha)}$.

Proof. (1) Notice that the value of ${a_{i,j}}^{(\alpha)} \circledast {b_{i,j}}^{(\alpha)}$ is either 0 or 1. Thus, we need show that if ${a_{i,j}}^{(\alpha)} \circledast {b_{i,j}}^{(\alpha)} = 0$, then $(a_{i,j} \circledast b_{i,j})^{(\alpha)} = 0$. Suppose that ${a_{i,j}}^{(\alpha)} \circledast {b_{i,j}}^{(\alpha)} = 0$. If ${b_{i,j}}^{(\alpha)} = 1$, then

$${a_{i,j}}^{(\alpha)} \circledast {b_{i,j}}^{(\alpha)} = {a_{i,j}}^{(\alpha)} \circledast 1 = {a_{i,j}}^{(\alpha)2} + 1 - {a_{i,j}}^{(\alpha)} = 1,$$

a contradiction. Thus, we have ${b_{i,j}}^{(\alpha)} = 0$, equivalently $b_{i,j} < \alpha$.

To prove $(a_{i,j} \circledast b_{i,j})^{(\alpha)} = 0$, we must show that $a_{i,j} \circledast b_{i,j} < \alpha$. It follows from $b_{i,j} < \alpha$ and $A \leq B$ that

$$\begin{aligned} \alpha - (a_{i,j} \circledast b_{i,j}) &= \alpha - ({a_{i,j}}^2 + b_{i,j} - a_{i,j} \cdot b_{i,j}) \\ &= (\alpha - b_{i,j}) + a_{i,j}\,(b_{i,j} - a_{i,j}) \geq 0. \end{aligned}$$

Thus, we have $(A \circledast B)^{(\alpha)} \leq A^{(\alpha)} \circledast B^{(\alpha)}$.

(2) Notice that

$$(a_{i,j} \circledast b_{i,j})_{(\alpha)} = \begin{cases} a_{i,j} \circledast b_{i,j} & \text{if } a_{i,j} \circledast b_{i,j} \geq \alpha \\ 0 & \text{if } a_{i,j} \circledast b_{i,j} < \alpha. \end{cases}$$

Thus, there is nothing to prove the case of $a_{i,j} \circledast b_{i,j} < \alpha$. Now, we will show that if $a_{i,j} \circledast b_{i,j} \geq \alpha$, then

$$a_{i,j(\alpha)} \circledast b_{i,j(\alpha)} \geq a_{i,j} \circledast b_{i,j}. \tag{3.1}$$

Suppose that $a_{i,j} \circledast b_{i,j} \geq \alpha$. If $a_{i,j} \geq \alpha$, then (3.1) is obvious because $b_{i,j} \geq a_{i,j}$. Thus, we may assume that $a_{i,j} < \alpha$. Two cases arise.

Case 1) $b_{i,j} \geq \alpha$; then we have

$$\begin{aligned} (a_{i,j(\alpha)} \circledast b_{i,j(\alpha)}) - (a_{i,j} \circledast b_{i,j}) &= (0 \circledast b_{i,j}) - (a_{i,j} \circledast b_{i,j}) \\ &= b_{i,j} - ({a_{i,j}}^2 + b_{i,j} - a_{i,j} \cdot b_{i,j}) \\ &= a_{i,j}\,(b_{i,j} - a_{i,j}) \geq 0, \end{aligned}$$

and hence (3.1) is satisfied.

Case 2) $b_{i,j} < \alpha$; then we have $a_{i,j} < \alpha$ and hence

$$\begin{aligned} \alpha - (a_{i,j} \circledast b_{i,j}) &= \alpha - ({a_{i,j}}^2 + b_{i,j} - a_{i,j} \cdot b_{i,j}) \\ &= (\alpha - b_{i,j}) + a_{i,j}(b_{i,j} - a_{i,j}) > 0, \end{aligned}$$

a contradiction to $a_{i,j} \circledast b_{i,j} \geq \alpha$. Hence, this case does not arise.

(3) Similar to the proof of (1), we suffice to show that if ${a_{i,j}}^{(\alpha)} \uplus {b_{i,j}}^{(\alpha)} = 0$, then $(a_{i,j} \uplus b_{i,j})^{(\alpha)} = 0$. Suppose that ${a_{i,j}}^{(\alpha)} \uplus {b_{i,j}}^{(\alpha)} = 0$. Then we can easily show that $a_{i,j} \leq b_{i,j} < \alpha$. By the definition of $a_{i,j} \uplus b_{i,j}$ with $a_{i,j} \leq b_{i,j}$, we obtain that $a_{i,j} \uplus b_{i,j} = b_{i,j}$. It follows from $b_{i,j} < \alpha$ that

$$(a_{i,j} \uplus b_{i,j})^{(\alpha)} = {b_{i,j}}^{(\alpha)} = 0.$$

Hence, we have $(A \uplus B)^{(\alpha)} \leq A^{(\alpha)} \uplus B^{(\alpha)}$.

(4) Similar to the proof of (2), we suffice to show that if $a_{i,j} \uplus b_{i,j} \geq \alpha$, then

$$a_{i,j_{(\alpha)}} \uplus b_{i,j_{(\alpha)}} \geq a_{i,j} \uplus b_{i,j}. \tag{3.2}$$

Suppose that $a_{i,j} \uplus b_{i,j} \geq \alpha$. Since $a_{i,j} \leq b_{i,j}$, by the definition of $a_{i,j} \uplus b_{i,j}$, we have $a_{i,j} \uplus b_{i,j} = b_{i,j} \geq \alpha$. But, if follows from $b_{i,j} \geq \alpha$ that $b_{i,j_{(\alpha)}} = b_{i,j}$ and so $a_{i,j_{(\alpha)}} \uplus b_{i,j_{(\alpha)}} = b_{i,j}$. Thus, (3.2) is satisfied. ■

Property 9. *Let A and B be matrices in $\mathcal{M}_{m,n}(\mathbb{F})$. If either* $\min\{a_{i,j},\ b_{i,j}\} \geq \alpha$ *or* $\max\{a_{i,j},\ b_{i,j}\} < \alpha$ *for all i and j, then*

(1) $(A +_\lambda B)^{(\alpha)} \geq A^{(\alpha)} +_\lambda B^{(\alpha)}$,

(2) $(A +_\lambda B)_{(\alpha)} \geq A_{(\alpha)} +_\lambda B_{(\alpha)}$.

Proof. We lose no generality in assuming that $\lambda, \alpha \in \mathbb{F} \setminus \{0, 1\}$.

(1) Let $x_{i,j}$ and $y_{i,j}$ be $(i,j)^{\text{th}}$ elements of $(A +_\lambda B)^{(\alpha)}$ and $A^{(\alpha)} +_\lambda B^{(\alpha)}$, respectively. Then we have that

$$x_{i,j} = (\lambda a_{i,j} + (1-\lambda) b_{i,j})^{(\alpha)} \ \text{ and } \ y_{i,j} = \lambda {a_{i,j}}^{(\alpha)} + (1-\lambda){b_{i,j}}^{(\alpha)}.$$

If $\min\{a_{i,j},\ b_{i,j}\} \geq \alpha$, then

$$\lambda a_{i,j} + (1-\lambda)b_{i,j} \geq \lambda\alpha + (1-\lambda)\alpha = \alpha,$$

equivalently $x_{i,j} = 1$. Thus, we have that $x_{i,j} \geq y_{i,j}$ in this case. For the case of $\max\{a_{i,j},\ b_{i,j}\} < \alpha$, we have

$$y_{i,j} = \lambda {a_{i,j}}^{(\alpha)} + (1-\lambda){b_{i,j}}^{(\alpha)} = 0,$$

and thus, we also obtain that $x_{i,j} \geq y_{i,j}$ in this case.

(2) The proof is quite similar to that of (1). ∎

Remark 4. In general, we can have that neither

$$(A +_\lambda B)^{(\alpha)} \geq A^{(\alpha)} +_\lambda B^{(\alpha)}$$

nor

$$(A +_\lambda B)^{(\alpha)} \leq A^{(\alpha)} +_\lambda B^{(\alpha)}.$$

For example, consider two 1×2 fuzzy matrices $A = [1 \quad 0.5]$ and $B = [1 \quad 0]$. If we take $\lambda = 0.1$ and $\alpha = 0.2$, then we have

$$(A +_\lambda B)^{(\alpha)} = [1 \quad 0] \quad \text{and} \quad A^{(\alpha)} +_\lambda B^{(\alpha)} = [\lambda \quad \lambda].$$

For the lower α-cut, the similar argument is established.

4 Results on complement of fuzzy matrix

The complement of a fuzzy matrix is used to analysis the complement nature of any system. Using the following results, we can study the complement nature of a system with the help of original fuzzy matrix.

The operator complement obey the De Morgan's laws for the operators $\oplus$ and $\odot$. This is established by Shyamal and Pal as following:

Property 10. [6]. *For the fuzzy matrices A and B,*

(i) $(A \oplus B)^c = A^c \odot B^c$,

(ii) $(A \odot B)^c = A^c \oplus B^c$,

(iii) $(A \oplus B)^c \leq A^c \oplus B^c$,

(iv) $(A \odot B)^c \geq A^c \odot B^c$.

Property 11. *Let λ be given in $\mathbb{F}$. Then for the matrices A and B in $\mathcal{M}_{m,n}(\mathbb{F})$,*

(1) $(A +_\lambda B)^c = A^c +_\lambda B^c$,

(2) $(A +_\lambda B)^c \geq A^c \odot B^c$,

(3) $(A \odot B)^c \geq A^c +_\lambda B^c$,

(4) $(A \uplus B)^c \leq A^c \uplus B^c$.

Proof. (1) Let $c_{i,j}$ and $d_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $(A +_\lambda B)^c$ and $A^c +_\lambda B^c$, respectively. Then we have

$$\begin{aligned} d_{i,j} &= \lambda(1 - a_{i,j}) + (1 - \lambda)(1 - b_{i,j}) \\ &= 1 - (\lambda a_{i,j} + (1 - \lambda) b_{i,j}) = c_{i,j}. \end{aligned}$$

Hence we have $(A +_\lambda B)^c = A^c +_\lambda B^c$.

(2) Note that if $A \leq B$, then $A^c \geq B^c$. Thus, we suffice to show that $A +_\lambda B \leq (A^c \odot B^c)^c$. It follows from Property 10-(ii) and Property 4-(2) that

$$(A^c \odot B^c)^c = A \oplus B \geq A +_\lambda B.$$

(3) By Property 10-(ii) and Property 3-(2), we have that $(A \odot B)^c = A^c \oplus B^c \geq A^c +_\lambda B^c$.

(4) Let $g_{i,j}$ and $h_{i,j}$ be the $(i,j)^{\text{th}}$ elements of $(A \uplus B)^c$ and $A^c \uplus B^c$, respectively. Then we have

$$g_{i,j} = 1 - (a_{i,j} \uplus b_{i,j}) \quad \text{and} \quad h_{i,j} = (1 - a_{i,j}) \uplus (1 - b_{i,j}).$$

Now, we claim that $g_{i,j} \leq h_{i,j}$. If $a_{i,j} > b_{i,j}$, then $g_{i,j} = 1 - (a_{i,j} \uplus b_{i,j}) = 1 - 1 = 0$, and hence $g_{i,j} \leq h_{i,j}$ in this case. For the case of $a_{i,j} \leq b_{i,j}$, we have $g_{i,j} = 1 - b_{i,j}$, while the value of $h_{i,j} = (1 - a_{i,j}) \uplus (1 - b_{i,j})$ is 1 or $1 - b_{i,j}$. Thus, we also have $g_{i,j} \leq h_{i,j}$ in this case. Therefore, we have $(A \uplus B)^c \leq A^c \uplus B^c$. ∎

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