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SPECTRAL MAPPING THEOREM FOR THE WEYL SPECTRUM

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ABSTRACT. In this paper we show that the Weyl spectrum of a M-hyponormal operator satisfies the spectral mapping theorem for analytic functions and then answer an old question of Oberai. Also we show that the set of operators T satisfying $w(T) = \sigma_e(T)$ is closed in $B(H)$, and invariant under compact perturbation. In particular we show that the Weyl spectrum of a operator T satisfying $w(T) = \sigma_e(T)$ satisfies the spectral mapping theorem for analytic functions.

0. Introduction

Let H be an infinite dimensional Hilbert space and write $B(H)$ for the set of all bounded linear operators on H and $\mathcal{K}$ for the set of all compact operators on H. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T, $\pi_0(T)$ for the set of eigenvalues of T, and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. If K is a subset of $\mathbb{C}$, we write iso K for the set of isolated points of K. An operator $T \in B(H)$ is said to *Fredholm* if its range ran T is closed and both the null space ker T and ker T^* are finite dimensional. The *index* of a Fredholm operator T, denoted by $i(T)$, is defined by

$$i(T) = \dim\ker T - \dim\ker T^*.$$

It was well known ([3]) that $i : \mathcal{F} \to \mathbb{Z} \cup \{\pm\infty\}$ is a continuous function where the set $\mathcal{F}$ of Fredholm operators has the norm topology and $\mathbb{Z} \cup \{\pm\infty\}$ has

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the discrete topology. The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called a *Weyl* operator. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}.$$

It was shown ([1]) that for any operator T, $\sigma_e(T) \subset w(T) \subset \sigma(T)$,

$$w(T) = \cap_{K \in \mathcal{K}} \sigma(T + K)$$

and $w(T)$ is a nonempty compact subset of $\mathbb{C}$.

For example, define an operator T on l_2 by

$$T(x_1, x_2, \cdots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots).$$

Then $\sigma(T) = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, and $w(T) = \sigma_e(T) = \{0\}$ since T is compact. Hence $w(T) = \sigma_e(T)$. However, consider the weighted shift U on l_2 given by

$$U(x_1, x_2, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Then U is hyponormal, $w(U) = \sigma(U) = D(=$ the closed unit disc) and $\sigma_e(U) = C(=$ the unit circle). Hence $w(U) \neq \sigma_e(U)$ and so we note that $w(U) \neq \sigma_e(U)$, even if T is hyponormal.

Recall ([12]) that an operator $T \in B(H)$ is said to be $M-$*hyponormal* if there exists $M > 0$ such that

$$\|(T - z)^* x\| \leq M \|(T - z)x\| \tag{1}$$

for all x in H and for all $z \in \mathbb{C}$.

Every hyponormal operator is $M-$hyponormal, but the converse is not true in general: for example, consider the weighted shift S on l_2 given by

$$S(x_1, x_2, \cdots) = (0, 2x_1, x_2, x_3, \cdots).$$

If T is Fredholm, then by (1)

$$T \ M\text{−hyponormal} \Longrightarrow \ i(T) \leq 0. \tag{2}$$

It was known that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at T([10]). However if $T_n \to T$ with $T_nT = TT_n$ for all $n \in \mathbb{N}$ then

$$\lim \omega(T_n) = \omega(T). \tag{3}$$

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic funcions: if f is analytic on a neighborhood of $\sigma(T)$ then

$$\omega(f(T)) \subset f(\omega(T)). \tag{4}$$

The inclusion (4) may be proper(see [2, Example 3.3]). If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal since $f(T)$ is also normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions.

In this paper we show that the Weyl spectrum of a $M-$hyponormal operator satisfies the spectral mapping theorem for analytic functions and then answer an old question of Oberai. Also we show that the set of operators T satisfying $w(T) = \sigma_e(T)$ is closed in $B(H)$, and invariant under compact perturbation. In particular we show that the Weyl spectrum of a operator T satisfying $w(T) = \sigma_e(T)$ satisfies the spectral mapping theorem for analytic functions.

1. Weyl spectrum and spectral mapping theorems

Theorem 1. *If S and T are commuting $M-$hyponormal operators, then*

$$S, T \ \text{Weyl} \iff ST \ \text{Weyl}. \tag{5}$$

Proof. If S, T are Weyl, then S, T are Fredholm and $i(S) = i(T) = 0$. By [3], ST is Fredholm and by the index product theorem, $i(ST) = i(S) + i(T) = 0$. Hence ST is Weyl.

For the backward implication of (5) we note that if $ST = TS$, then $\ker S \cup \ker T \subseteq \ker ST$ and $\ker S^* \cup \ker T^* \subseteq \ker(ST)^*$. If ST is Weyl, then $\dim\ker S, \dim\ker T < \infty$ and $\dim\ker S^*, \dim\ker T^* < \infty$. Also ran S and ran T are closed by [5, Theorem 3.2.2]. Hence S, T are Fredholm. Since S and T are $M-$hyponormal, by (1) $i(S) = i(T) = 0$ since $0 = i(ST) = i(S) + i(T)$.

If the "$M-$hyponormal" condition is dropped in the above theorem, then the backward implication may fail even though T_1 and T_2 commute: For example, if U is the unilateral shift on l_2, consider the following operators on $l_2 \oplus l_2 : T_1 = U \oplus I$ and $T_2 = I \oplus U^*$.

Theorem 2. *If T is $M-$hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

Proof. Suppose that p is any polynomial. Let

$$P(T) - \lambda I = a_0(T - \mu_1 I)\cdots(T - \mu_n I).$$

Since T is $M-$hyponormal, $T-\mu_i I$ are commuting $M-$hyponormal operators for each $i = 1, 2, \cdots, n$. It thus follows from Theorem 1 that

$$\begin{aligned}\lambda \notin \omega(p(T)) &\Longleftrightarrow p(T) - \lambda I = \text{Weyl} \\ &\Longleftrightarrow a_0(T - \mu_1 I)\cdots(T - \mu_n I) = \text{Weyl} \\ &\Longleftrightarrow T - \mu_i I = \text{Weyl for each } i = 1, 2, \cdots, n \\ &\Longleftrightarrow \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \cdots, n \\ &\Longleftrightarrow \lambda \notin p(\omega(T))\end{aligned}$$

which says that $\omega(p(T)) = p(\omega(T))$. If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([3]), there is a sequence (p_n) of polynomials such that $f_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by [8]

$$f(\omega(T)) = \lim p_n(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T)).$$

Corollary 3. *If T is hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

We say that *Weyl's theorem holds for T* if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

There are several classes of operators including hyponormal operators for which Weyl's theorem holds. Oberai has raised the following question: Does there exist a hyponormal operator T such taht Weyl's theorem does not hold for T^2 ? Note that T^2 may not be hyponormal even if T is hyponormal([4, Problem 209]). We will show that Weyl's theorem holds for $p(T)$ when T is hyponormal.

Recall ([9]) that $T \in B(H)$ is said to be isoloid if iso $\sigma(T) \subset \pi_0(T)$.

Theorem 4. *([9]) Let $T \in B(H)$ be isoloid. Then for any polynomial $p(t)$, $p(\sigma(T) - \pi_{oo}(T)) = \sigma(p(T)) - \pi_{oo}(p(T))$.*

Corollary 5. *If $T \in B(H)$ is hyponormal, then for any polynomial p on a neighborhood of $\sigma(T)$ Weyl's theorem holds for $p(T)$.*

Proof. By [10], T is isoloid and Weyl's theorem holds for any hyponormal operator. Hence by Theorem 2 and Theorem 4,

$$w(p(T)) = p(w(T)) = p(\sigma(T) - \pi_{00}(T)) = \sigma(p(T)) - \pi_{00}(p(T))$$

Therefore Weyl's theorem holds for $p(T)$.

Lemma 6. *([1],[3]) For any operator T in $B(H)$,*

$$\omega(T) = \sigma_e(T) \cup \theta(T) \qquad \textit{(disjoint union)},$$

where $\theta(T) = \{\lambda : T - \lambda$ is Fredholm and $i(T - \lambda) \neq 0\}$.

For example, if U is the simple unilateral shift, then $\sigma_e(U) = \{\lambda : |\lambda| = 1\}$, and $\theta(U) = \{\lambda : |\lambda| < 1\}$.

The above Lemma clearly show that $\sigma_e(T) = \omega(T)$ if and only if the open set $\theta(T)$ is empty

Theorem 6. *The set of operators T satisfying $w(T) = \sigma_e(T)$ is closed in $B(H)$ and invariant under compact perturbations.*

Proof. Suppose $w(T_n) = \sigma_e(T_n)$ for each n and $T_n \to T$ in norm topology. It suffices to show that $\sigma_e(T) = w(T)$. If $\sigma_e(T) \neq w(T)$, then by Lemma 5 there exists $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm of nonzero index. By [6, Theorem 4.5.17], there exists an $\epsilon > 0$ such that if $\|T - \lambda - S\| < \epsilon$, then S is a Fredholm operator. Also there exists an integer N_1 such that for $n \geq N_1$ we have

$$\|(T - \lambda) - (T_n - \lambda)\| < \frac{\epsilon}{2}.$$

Thus $T_n - \lambda$ is Fredholm for $n \geq N_1$. Since the index i is continuous, there exists an integer N_2 such that for $n \geq N_2$, $i(T_n - \lambda) \neq 0$. Hence for $n \geq N = \max(N_1, N_2)$, $T_n - \lambda$ is Fredholm of nonzero index and so $\sigma_e(T_n) \neq w(T_n)$ by Lemma 5. This is a contradiction. Thus $\sigma_e(T) = w(T)$.

If $T \in W$ and K is compact, $w(T + K) = w(T)$ by [1, Corollary 2.7] and $\sigma_e(T) = \sigma_e(T + K)$. Thus the set of operators T satisfying $w(T) = \sigma_e(T)$ is invariant under compact perturbations.

Lemma 7. *([3]) If T is Fredholm and K is compact in $B(H)$, then $T+K$ is Fredholm and $i(T+K)=i(T)$.*

Theorem 8. *If T in $B(H)$ is of the form normal + compact, then $w(T) = \sigma_e(T)$.*

Proof. Let $T = N + K$, where N is normal and K is compact. If $w(T) \neq \sigma_e(T)$, then by Lemma 6, there exists $\lambda \in \mathbb{C}$ such that $T-\lambda$ is Fredholm of nonzero index. But by Lemma 7, $T-\lambda-K$ is Fredholm and $i(T-\lambda) = i(T-\lambda-K) = i(N-\lambda) = 0$. This is a contradiction.

From this theorem we know that the unilateral shift U is not of the form normal + compact.

Theorem 9. *$w(T) = \sigma_e(T)$ if and only if there exists a compact operator K such that $\sigma(T+K) = \sigma_e(T)$.*

Proof. If $\sigma(T+K) = \sigma_e(T)$ for some compact operator K, then

$$w(T) = \cap_{K\in\mathcal{K}}\sigma(T+K) \subseteq \sigma_e(T).$$

Since $\sigma_e(T) \subset w(T)$, $w(T) = \sigma_e(T)$.

Conversely if $\sigma_e(T) = w(T)$, then by [11, Theorem 4] there exists a compact operator K such that $\sigma(T+K) = w(T)$. Hence $\sigma(T+K) = w(T) = \sigma_e(T)$ for some compact operator K.

Theorem 10. *If T satisfies $w(T) = \sigma_e(T)$ and f is analytic on a neighborhood of $\sigma(T)$, then $w(f(T)) = f(w(T))$.*

Proof. Suppose that p is any polynomial. Then $\pi(p(T) = p(\pi(T))$ where π denotes the natural map of $B(H)$ onto $B(H)/\mathcal{K}$. By the spectral mapping theorem,

$$p(w(T)) = p(\sigma_e(T)) = \sigma_e(p(T)) \subseteq w(p(T)).$$

But for any operator $T \in B(H)$, $w(p(T)) \subseteq p(w(T))$([1, Theorem 3.2]). Therefore $w(p(T)) = p(w(T))$ for any polynomial p.

If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([3]), there is a sequence (p_n) of polynomials such that $f_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by [8]

$$w(f(T)) = \lim w(p_n(T)) = \lim p_n(w(T)) = f(w(T)).$$

Theorem 11. *If T satisfies $w(T) = \sigma_e(T)$ and f is analytic on a neighborhood of $\sigma(T)$, then $w(f(T)) = \sigma_e(f(T))$.*

Proof. Suppose that p is any polynomial. Then by Theorem 10 and the spectral mapping theorem, $w(p(T)) = p(w(T))$ and $w(p(T)) = p(w(T)) = p(\sigma_e(T)) = \sigma_e(p(T))$.

If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([3]), there is a sequence (p_n) of polynomials such that $f_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by [7] and Theorem 10,

$$\begin{aligned} w(f(T)) = f(w(T)) &= \lim_{n\to\infty} p_n(w(T)) = \lim_{n\to\infty} p_n(\sigma_e(T)) \\ &= \lim_{n\to\infty} \sigma_e(p_n(T)) = \sigma_e(f(T)). \end{aligned}$$

Thus $f(T)$ satisfies $w(f(T)) = \sigma_e(f(T))$.

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