J. of Basic Sciences, Cheju Nat. Univ.
9(1), 17~28, 1996

기초과학연구 제주대학교
9(1), 17~28, 1996

THE CURVATURE OF A REGULAR CURVE UNDER INVERSION

HYUN, JIN-OH. KIM, SOON-CHAN

Department of Mathematics Education
Cheju National University

1. Introduction

In this paper, our study of the curvature will be restrictied to the regular curve in Euclidean space E^3 and we derive the formula which is a relation of the curvature of a regular curve under inversion and the one of the given regular curve. We show that if κ and $\bar{\kappa}$ are the curvatures of a unit speed curve α and the inversion curve of α, repectively, then the necessary and sufficient condition for the formula $\bar{\kappa} = \frac{\|\alpha(t)\|^2}{R^2}\kappa$ is that $\|\alpha(t)\| = At + B$ for some constants A and B with $At + B > 0$ for all t.

2. Definition and Some Properties of an Inversion

Let the sysmbol $(O)_R$ denote the sphere with center O and radius R.

Definition 2.1. Two points P and P' of E^3 are said to be *inverse* with respect to a given sphere $(O)_R$ if

$$OP \cdot OP' = R^2 \tag{2.1}$$

where P, P' are on the same side of O and O, P, P' are collinear.

A sphere $(O)_R$ is called the *sphere of inversion*, and the transformation which sends point P into P' is called an *inversion*. As point P moves on a curve C, its inverse point P' moves on a curve C' which is the inverse curve of C. But the center O of the sphere of inversion has no inverse point because if P is at the center O then $OP = 0$, which means that the relation $OP' = \dfrac{R^2}{OP}$ is meaningless.

From now on, we take the center O as an origin of the coordinate system in E^3, and denote the distance from O to a point $X \in E^3$ by $\|X\|$. Then we have the following properties.

Proposition 2.2.
(1) A line through O inverts into a line through O.
(2) A line not through O inverts into a circle through O.
(3) A circle through O inverts into a line not through O.
(4) A circle not through O inverts into a circle not through O.

Proposition 2.3. Let $\alpha : (a, b) \longrightarrow E^3$ be a regular curve. Define a mapping $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ by for all $X \in E^3 - \{(0,0,0)\}$

$$f(X) = \frac{R^2 X}{\langle X, X\rangle} = \frac{R^2 X}{\|X\|^2}, \tag{2.2}$$

Then
(1) f is an inversion,
(2) new curve $\bar{\alpha} = f \circ \alpha$ is regular, and
(3) the arc-length $\bar{s}(t)$ of a regular curve segment $\bar{\alpha}$ of α under inversion is given by the formula

$$\bar{s}(t) = R^2 \int_0^t \frac{1}{\|\alpha\|^2} \left\| \frac{d\alpha}{dt} \right\| dt. \tag{2.3}$$

Proof. (3) Since $\alpha(t) \neq 0$ for all $t \in (a, b)$, we have

$$\begin{aligned} \frac{d\bar{\alpha}}{dt} &= \frac{df(\alpha)}{dt} \\ &= \frac{d}{dt} \frac{R^2 \alpha}{\|\alpha\|^2} \\ &= \frac{R^2}{\|\alpha\|^2} \frac{d\alpha}{dt} - \frac{2R^2}{\|\alpha\|^4} \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \alpha; \end{aligned} \tag{2.4}$$

and so

$$\begin{aligned}\left\|\frac{d\bar{\alpha}}{dt}\right\|^2 &= \left\langle \frac{d\bar{\alpha}}{dt}, \frac{d\bar{\alpha}}{dt} \right\rangle \\ &= \left\langle \frac{R^2}{\|\alpha\|^2}\frac{d\alpha}{dt}, \frac{R^2}{\|\alpha\|^2}\frac{d\alpha}{dt} \right\rangle \\ &= \frac{R^4}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \right\rangle \\ &= \frac{R^4}{\|\alpha\|^4}\left\|\frac{d\alpha}{dt}\right\|^2.\end{aligned} \tag{2.5}$$

By using of (1.3), we get

$$\begin{aligned}\bar{s}(t) &= \int_0^t \left\|\frac{d\bar{\alpha}}{dt}\right\| dt \\ &= R^2 \int_0^t \frac{1}{\|\alpha\|^2} \left\|\frac{d\alpha}{dt}\right\| dt.\end{aligned}$$

3. The Curvature of a Regular Curve under Inversion

We derive the formula which is a relation of the curvature of a regular curve under inversion and the one of the given regular curve. We show that if κ and $\bar{\kappa}$ are the curvatures of a unit speed curve α and the inversion curve of α, repectively, then the necessary and sufficient condition for the formula $\bar{\kappa} = \dfrac{\|\alpha(t)\|^2}{R^2}\kappa$ is that $\|\alpha(t)\| = At + B$ for some constants A and B with $At + B > 0$ for all t.

Lemma 3.1. Let $\alpha : I \longrightarrow E^3$ be a regular curve, and let $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion of α. Then, for the new curve $\bar{\alpha} = f(\alpha)$,

$$\begin{aligned}(1)\ \frac{d^2\bar{\alpha}}{dt^2} =& \frac{R^2}{\|\alpha\|^2}\frac{d^2\alpha}{dt^2} - \frac{4R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt} \\ &- \frac{2R^2}{\|\alpha\|^4}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\|\frac{d\alpha}{dt}\right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right)\alpha.\end{aligned} \tag{3.1}$$

(2) $$\left\|\frac{d^2\bar{\alpha}}{dt^2}\right\|^2 = \frac{R^4}{\|\alpha\|^4}\left\|\frac{d^2\alpha}{dt^2}\right\|^2 + \frac{4R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^4 + \frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\|\frac{d\alpha}{dt}\right\|^2 - \frac{8R^4}{\|\alpha\|^6}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle. \tag{3.2}$$

(3) $$\left\langle\frac{d\bar{\alpha}}{dt},\frac{d^2\bar{\alpha}}{dt^2}\right\rangle = \frac{R^4}{\|\alpha\|^4}\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle - \frac{2R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle. \tag{3.3}$$

(4) $$\begin{aligned}\left\|\frac{d\bar{\alpha}}{dt}\times\frac{d^2\bar{\alpha}}{dt^2}\right\|^2 &= \frac{R^8}{\|\alpha\|^8}\left\|\frac{d\alpha}{dt}\times\frac{d^2\alpha}{dt^2}\right\|^2 + \frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^6 \\ &+ \frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle - \frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle \\ &- \frac{4R^8}{\|\alpha\|^{12}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2.\end{aligned} \tag{3.4}$$

Proof. (1) Differentation of (2.4) gives the following ;

$$\begin{aligned}\frac{d^2\bar{\alpha}}{dt^2} &= \frac{R^2\frac{d^2\alpha}{dt^2}\|\alpha\|^2 - 2R^2\langle\frac{d\alpha}{dt},\alpha\rangle\frac{d\alpha}{dt}}{\|\alpha\|^4} \\ &- \frac{2R^2\|\alpha\|^4\left[\left(\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle + \langle\frac{d\alpha}{dt},\frac{d\alpha}{dt}\rangle\right)\alpha + \langle\frac{d\alpha}{dt},\alpha\rangle\frac{d\alpha}{dt}\right]}{\|\alpha\|^8} \\ &+ \frac{8R^2\|\alpha\|^2\langle\frac{d\alpha}{dt},\alpha\rangle^2\alpha}{\|\alpha\|^8}\end{aligned}$$

$$= \frac{R^2}{\|\alpha\|^2}\frac{d^2\alpha}{dt^2} - \frac{2R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt} - \frac{2R^2}{\|\alpha\|^4}\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle \alpha$$

$$- \frac{2R^2}{\|\alpha\|^4}\left\| \frac{d\alpha}{dt} \right\|^2 \alpha - \frac{2R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt} + \frac{8R^2}{\|\alpha\|^6}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \alpha$$

$$= \frac{R^2}{\|\alpha\|^2}\frac{d^2\alpha}{dt^2} - \frac{4R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt}$$

$$- \frac{2R^2}{\|\alpha\|^4}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right) \alpha.$$

(2) From the formula (1), we get

$$\left\| \frac{d\bar{\alpha}}{dt} \right\|^2$$

$$= \left\langle \frac{d\bar{\alpha}}{dt}, \frac{d\bar{\alpha}}{dt} \right\rangle$$

$$= \frac{R^4}{\|\alpha\|^4}\left\| \frac{d^2\alpha}{dt^2} \right\|^2 + \frac{16R^4}{\|\alpha\|^8}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \left\| \frac{d\alpha}{dt} \right\|^2$$

$$+ \frac{4R^4}{\|\alpha\|^8}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right)^2 \|\alpha\|^2$$

$$- \frac{8R^4}{\|\alpha\|^6}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle$$

$$+ \frac{16R^4}{\|\alpha\|^8}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right)$$

$$- \frac{4R^4}{\|\alpha\|^6}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right) \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle$$

$$= \frac{R^4}{\|\alpha\|^4}\left\| \frac{d^2\alpha}{dt^2} \right\|^2 + \frac{16R^4}{\|\alpha\|^8}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \left\| \frac{d\alpha}{dt} \right\|^2 + \frac{4R^4}{\|\alpha\|^6}\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle^2$$

$$+ \frac{4R^4}{\|\alpha\|^6}\left\| \frac{d\alpha}{dt} \right\|^4 + \frac{64R^4}{\|\alpha\|^{10}}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^4 + \frac{8R^4}{\|\alpha\|^6}\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle \left\| \frac{d\alpha}{dt} \right\|^2$$

$$
\begin{aligned}
&-\frac{32R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\|\frac{d\alpha}{dt}\right\|^2-\frac{32R^4}{\|\alpha\|^8}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\\
&-\frac{8R^4}{\|\alpha\|^6}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle+\frac{16R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\\
&+\frac{16R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\|\frac{d\alpha}{dt}\right\|^2-\frac{64R^4}{\|\alpha\|^{10}}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^4-\frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle^2\\
&-\frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\|\frac{d\alpha}{dt}\right\|^2+\frac{16R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\\
=&\frac{R^4}{\|\alpha\|^4}\left\|\frac{d^2\alpha}{dt^2}\right\|^2+\frac{4R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^4+\frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\|\frac{d\alpha}{dt}\right\|^2\\
&-\frac{8R^4}{\|\alpha\|^6}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle.
\end{aligned}
$$

(3) Differentiating both sides of (2.5), we have

$$
\begin{aligned}
2\left\langle\frac{d\bar{\alpha}}{dt},\frac{d^2\bar{\alpha}}{dt^2}\right\rangle&=\frac{2R^4\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle\|\alpha\|^4-4R^4\left\|\frac{d\alpha}{dt}\right\|^2\|\alpha\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle}{\|\alpha\|^8}\\
&=\frac{2R^4}{\|\alpha\|^4}\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle-\frac{4R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle.
\end{aligned}
$$

Hence we have

$$
\left\langle\frac{d\bar{\alpha}}{dt},\frac{d^2\bar{\alpha}}{dt^2}\right\rangle=\frac{R^4}{\|\alpha\|^4}\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle-\frac{2R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle.
$$

(4) From the formulas (2.5), (3.2), and (3.3), we obtain

$$
\begin{aligned}
&\left\| \frac{d\bar{\alpha}}{dt} \times \frac{d^2\bar{\alpha}}{dt^2} \right\|^2 \\
=&\left\| \frac{d\bar{\alpha}}{dt} \right\|^2 \left\| \frac{d^2\bar{\alpha}}{dt^2} \right\|^2 - \left\langle \frac{d\bar{\alpha}}{dt}, \frac{d^2\bar{\alpha}}{dt^2} \right\rangle^2 \\
=&\frac{R^8}{\|\alpha\|^8} \left\| \frac{d\alpha}{dt} \right\|^2 \left\| \frac{d^2\alpha}{dt^2} \right\|^2 + \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^6 + \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle \\
&- \frac{8R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^2 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle - \frac{R^8}{\|\alpha\|^8} \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle^2 \\
&+ \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^2 \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle - \frac{4R^8}{\|\alpha\|^{12}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \\
=&\frac{R^8}{\|\alpha\|^8} \left\| \frac{d\alpha}{dt} \right\|^2 \left\| \frac{d^2\alpha}{dt^2} \right\|^2 - \frac{R^8}{\|\alpha\|^8} \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle^2 + \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^6 \\
&+ \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle - \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^2 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle \\
&- \frac{4R^8}{\|\alpha\|^{12}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2. \\
=&\frac{R^8}{\|\alpha\|^8} \left\| \frac{d\alpha}{dt} \times \frac{d^2\alpha}{dt^2} \right\|^2 + \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^6 \\
&+ \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle - \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^2 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle \\
&- \frac{4R^8}{\|\alpha\|^{12}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2.
\end{aligned}
$$

Theorem 3.2. Let $\alpha : I \longrightarrow E^3$ be a regular curve with curvature κ, and let $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion of α. Then the curvature

$\bar{\kappa}$ of $\bar{\alpha} = f(\alpha)$ under inversion is computed by the following formula

$$\begin{aligned}\bar{\kappa}^2 =&\frac{\|\alpha\|^4}{R^4}\kappa^2 + \frac{4\|\alpha\|^2}{R^4} + \frac{4}{R^4\|\frac{d\alpha}{dt}\|^2}\left(\|\alpha\|^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle - \left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\right)\\ &-\frac{4\|\alpha\|^2}{R^4\|\frac{d\alpha}{dt}\|^4}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle.\end{aligned} \tag{3.5}$$

Proof. By using of the formulas (1.6), (2.5), and Lemma 3.1, we have

$$\begin{aligned}\bar{\kappa}^2 =&\frac{\left\|\frac{d\bar{\alpha}}{dt}\times\frac{d^2\bar{\alpha}}{dt^2}\right\|^2}{\|\frac{d\bar{\alpha}}{dt}\|^6}\\ =&\frac{\frac{R^8}{\|\alpha\|^8}\left\|\frac{d\alpha}{dt}\times\frac{d^2\alpha}{dt^2}\right\|^2}{\frac{R^{12}}{\|\alpha\|^{12}}\|\frac{d\alpha}{dt}\|^6} + \frac{\frac{4R^8}{\|\alpha\|^{10}}\|\frac{d\alpha}{dt}\|^6 + \frac{4R^8}{\|\alpha\|^{10}}\|\frac{d\alpha}{dt}\|^4\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle}{\frac{R^{12}}{\|\alpha\|^{12}}\|\frac{d\alpha}{dt}\|^6}\\ &-\frac{\frac{4R^8}{\|\alpha\|^{12}}\|\frac{d\alpha}{dt}\|^4\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2 + \frac{4R^8}{\|\alpha\|^{10}}\|\frac{d\alpha}{dt}\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle}{\frac{R^{12}}{\|\alpha\|^{12}}\|\frac{d\alpha}{dt}\|^6}\\ =&\frac{\|\alpha\|^4}{R^4}\frac{\left\|\frac{d\alpha}{dt}\times\frac{d^2\alpha}{dt^2}\right\|^2}{\|\frac{d\alpha}{dt}\|^6} + \frac{4\|\alpha\|^2}{R^4} + \frac{4\|\alpha\|^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle}{R^4\|\frac{d\alpha}{dt}\|^2} - \frac{4\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2}{R^4\|\frac{d\alpha}{dt}\|^2}\\ &-\frac{4\|\alpha\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle}{R^4\|\frac{d\alpha}{dt}\|^4}\\ =&\frac{\|\alpha\|^4}{R^4}\kappa^2 + \frac{4\|\alpha\|^2}{R^4} + \frac{4}{R^4\|\frac{d\alpha}{dt}\|^2}\left(\|\alpha\|^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle - \left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\right)\\ &-\frac{4\|\alpha\|^2}{R^4\|\frac{d\alpha}{dt}\|^4}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle.\end{aligned}$$

Corollary 3.3. Let α be a unit speed curve with curvature κ. Then the curvature $\bar{\kappa}$ of $\bar{\alpha}$ under inversion is computed by the following;

$$\bar{\kappa}^2 = \frac{\|\alpha\|^4}{R^4}\kappa^2 + \frac{4\|\alpha\|^2}{R^4} + \frac{4}{R^4}\|\alpha\|^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle - \frac{4}{R^4}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2. \tag{3.6}$$

Proof. Let α be a unit speed curve. Then $\left\|\frac{d\alpha}{dt}\right\| = 1$; so $\left\|\frac{d\alpha}{dt}\right\|^2 = 1$. Hence $\left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle = 0$ by differentiation of $\left\|\frac{d\alpha}{dt}\right\|^2 = 1$. From the formula (3.5), we get the formula (3.6).

Theorem 3.4. Let $\alpha : (a, b) \longrightarrow E^3$ be a unit speed curve with curvature κ and let $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion. Also, let $\bar{\kappa}$ be the curvature of $\bar{\alpha} = f \circ \alpha$. Then, for any $t \in (a, b)$,

$$\bar{\kappa} = \frac{\|\alpha(t)\|^2}{R^2}\kappa \qquad \text{if and only if} \qquad \|\alpha(t)\| = At + B$$

for some constants A, B with $At + B > 0$ for all t.

Proof. Let $\bar{\kappa} = \frac{\|\alpha\|^2}{R^2}\kappa$. Then, by Corollary 3.3,

$$\|\alpha\|^2 + \|\alpha\|^2 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle - \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 = 0. \tag{3.7}$$

Put $g(t) = \langle \alpha(t), \alpha(t) \rangle$. Then g is a differentiable real-valued function and $g(t) > 0$ for all t. Differentiating both sides of the formula

$$\langle \alpha(t), \alpha(t) \rangle = g(t),$$

we have

$$\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle = \frac{1}{2}g', \tag{3.8}$$

where g' denotes the derivative of g with respect to t. Since α is a unit speed curve, differentiating both sides of (3.8), we have

$$\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle = \frac{1}{2}g'' - 1. \tag{3.9}$$

Substituting the formulas (3.8) and (3.9) to the formula (3.7), we get the differential equation

$$2gg'' - (g')^2 = 0. \tag{3.10}$$

Case 1 : If $g' = 0$, then there exists a positive constant B such that $g(t) = B$ since $g(t) > 0$ for all t.

Case 2 : If $g' \neq 0$, then, from the formula (3.10),

$$2\frac{g''}{g'} = \frac{g'}{g}.$$

Hence

$$(2\ln|g'|)' = (\ln|g|)';$$

and so

$$\ln(g')^2 = \ln C_1 g,$$

where C_1 is a positive constant. Therefore we obtain

$$(g')^2 = C_1 g.$$

Simplifying this equation, we get

$$\frac{g'}{\sqrt{g}} = \pm\sqrt{C_1}.$$

By integrating both sides of this equation, we obtain

$$\sqrt{g} = \pm\frac{\sqrt{C_1}}{2}t + \frac{C_2}{2},$$

where C_2 is a constant. To get $\|\alpha(t)\| = \sqrt{g(t)} = At + B$, we choose $\pm\frac{\sqrt{C_1}}{2} = A$ and $\frac{C_2}{2} = B$ which are satisfied the inequality $At + B > 0$ for all t. Then we are done.

Conversely, let $\|\alpha(t)\| = At + B$; so $\|\alpha(t)\|^2 = (At + B)^2$. Then, by differentiating both sides of the above equation, we get

$$\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle = A(At + B).$$

By differentiating both sides of the above equation, we have

$$\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \right\rangle = A^2.$$

Since $\left\|\frac{d\alpha}{dt}\right\| = 1$, we have

$$\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle = A^2 - 1.$$

By Corollary 3.3, we obtain

$$\begin{aligned}\bar{\kappa}^2 &= \frac{\|\alpha\|^4}{R^4}\kappa^2 + \frac{4\|\alpha\|^2}{R^4} + \frac{4}{R^4}\|\alpha\|^2(A^2-1) - \frac{4}{R^4}A^2\|\alpha\|^2 \\ &= \frac{\|\alpha\|^4}{R^4}\kappa^2.\end{aligned}$$

Hence our proof is completed.

REFERENCES

[1] Yang,Chang-Hong(1993), *On the arc length under inversion*, Mathematics Education Major Graduate School of Education,Cheju National University Cheju,Korea.

[2] Moon,Young-Bong(1994), *The area of regular surfaces under inversion*, Mathematics Education Major Graduate School of Education,Cheju National University Cheju,Korea.

[3] Mandredo P. Do Carmo(1976), *Differential Geometry of Curves and Surfaces*, Prentice-Hall.

[4] Chuan-Chih Hsiung(1981), *A First Course in Differential Geometry*, Jhon Wiley & Sons.

[5] Claire Fisher Adler(1967), *Modern Geometry*, McGraw-Hill.

[6] W.R.Utz(1967), *A Short Course in Differential Equations*, McGraw-Hill.