Stochastic Model for the Population Growth*

Kim, Hae Kyung • Han, Young Hyeon

人口變動에 關한 確率 모델

金 海 璟 • 韓 溶 鉉

ABSTRACT

The problem of finding a stochastic model for process of human population which fluctuates according to the births, deaths, immigrations and emigrations, is considered. In particular; the paper deals with the numerical prediction of human population and its some probabilistic properties.

1. Introduction

Births and deaths along with migrations are the main factors which can influence directly the size of population in a given region, or its geographic distribution. Anything which affects the numbers of people in a given region manifests itself in changes of one or more of these three processes. Stochastic models for population growth date back to Galton and Watson (1874), who considered the problem of extinction of family names and thus initiated the theory of branching processes. Subsequently, McKendrich (1914) investigated continuous generating models for population growth under restricted conditions. The main purpose of the present paper is to derive stochastic model for the process of human population which fluctuates according to the natality, mortality, immigration and emigration.

Let N_t be a random variable which represents the size of the population at time t in given region. Obviously the stochastic process $\{N_t : t\epsilon[0, \infty)\}$ is a time-homogeneous Markov process having a discrete state space. Throughout this paper, we shall obtain the probability generating function, mean and variance of the random variable N_t.

2. Generating Function

Suppose that a population in a given region, consists of members which can give birth to new members with mean rate α, can die with mean rate β, can immigrate from other populations with mean rate γ, and can emigrate to the others with mean rare δ. And we assume that there is no interaction among the members in the population. The quantities α, β, γ and δ may be regarded respectively as the mean birth, death, immigration and emigration rates per individual, in any short time interval, and are consequently referred to as the mean birth, death, im migration and emigration rates respectively, these would arise if, in the short time

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interval, each individual in the population, independently of the others, has approximately probability α of giving birth to a new individual, probility β of dying, probability γ of immigrating, and probability δ of emigrating. Similarly, for the time interval $(t, t+h)$ having length h, each individual has approximately probabilies $\alpha h+0(h)$, $\beta h+0(h)$, $\gamma h+0(h)$, and $\delta h+0(h)$, respectively.

In order to describe the population, we define the random variables, $U_i(i=1,2,3\cdots)$, V_1 and V_2 as follows. $U_i=1$ if the i-th individual gives birth in $(t, t+h)$, $U_i=-1$ if the i-th individual dies in $(t, t+h)$ and $U_i=0$ otherwise. $V_1=1$ if there is an immigrant in $(t, t+h)$ and $V_1=0$ otherwise. And $V_2=-1$ if there is an emigrant in $(t, t+h)$ and $V_2=0$ otherwise. Then we have

$$(2.1)\quad N_{t+h}=N_t+\sum_{i=1}^{N_t}U_i+\sum_{i=1}^{2}V_i$$

where N_t denotes the population size at time t.

On the other hand, the probability density functions of U_i, V_1, and V_2 are

$$f_{U_i}(x)=\begin{cases}\alpha h+0(h) & if\ x=1\\ \beta h+0(h) & if\ x=-1\\ 1-(\alpha+\beta)h+0(h) & if\ x=0\\ 0 & \text{otherwise}\end{cases}$$

$$(2.2)\quad f_{V_1}(x)=\begin{cases}\gamma h+0(h) & if\ x=1\\ 1-\gamma h+0(h) & if\ x=0\\ 0 & \text{otherwise}\end{cases}$$

and

$$f_{V_2}(z)=\begin{cases}\delta h+0(h) & if\ x=-1\\ 1-\delta h+0(h) & if\ x=0\\ 0 & \text{otherwise}\end{cases}$$

Using the above observations we also have the probability generating functions of U_i, V_1 and V_2.

$$G_{U_i}(z)=E(z^{U_i})=1+\{\alpha z-(\alpha+\beta)+\beta z^{-1}\}\cdot h+0(h)$$

$$(2.3)\quad G_{V_1}(z)=E(z^{V_1})=1-(\gamma-\gamma z)\cdot h+0(h)$$

$$G_{V_2}(z)=E(z^{V_2})=1-(\delta-\delta z^{-1})\cdot h+0(h)$$

Let $H=\sum_{i=1}^{n}U_i+\sum_{i=1}^{2}V_i$. We see that

$$\begin{aligned}G_H(z)=E(z^H)&=[G_{U_i}(t)]^n\cdot G_{V_1}(t)\cdot G_{V_2}(t)\\ &=[1+n\{\alpha z-(\alpha+\beta)+\beta z^{-1}\}\cdot h+0(h)]\\ &-[1+n\{\alpha z-(\alpha+\beta)+\beta z^{-1}\}\cdot h+0(h)]\\ &\cdot[\{(\gamma+\delta)-\gamma z-\delta z^{-1}\}\cdot h+0(h)]\end{aligned}$$

from the binomial expansion, $h^k=0(h)$ where $k=2,3\cdots$ and independence of the ramdom variables. We have therefore

$$\begin{aligned}(2.4)\quad G_H(z)=&[\{(n\alpha+\gamma)\cdot h+0(h)\}]\cdot z\\ &+[1-\{n(\alpha+\beta)+(\gamma+\delta)\}\cdot h+0(h)]z^0\\ &+[(n\beta+\delta)\cdot h+0(h)]\cdot z^{-1}\end{aligned}$$

Let $p_{n,m}(h)=P(N_{t+h}=m\mid N_t=n)$. Then $p_{n,m}(h)=P(H=m-n)$.

We therefore see that (2.4) implies.

$$(2.5)\quad P_{n,m}(h)=\begin{cases}(n\alpha+\gamma)\cdot h+0(h) & if\ m=n+1\\ 1-\{n(\alpha+\beta)+(\gamma+\delta)\}\cdot h+0(h) & if\ m=n\\ (n\beta+\delta)\cdot h+.(h)\ ip & if\ m=n-1\\ 0(h) & \text{otherwise}\end{cases}$$

Let $p_n(t)=P(N_t=n)$ $n=0,1,2\cdots$. In view of the above fact (2.5), we are led to the following relation for $p_n(t+h)$:

$$\begin{aligned}p_n(t+h)=&p_{n-1}(t)\cdot[\{(n-1)\alpha+\gamma\}\cdot h+0(h)]\\ &+p_n(t)[1-\{n(\alpha+\beta)+(\gamma+\delta)\}\cdot h+0(h)]\\ &+p_{n+1}(t)[\{(n+1)\beta+\delta\}\cdot h+0(h)]+0(h)\end{aligned}$$

$$\begin{aligned}p_n(t+h)=&p_n(t)-\{n(\alpha+\beta)+(\gamma+\delta)\}\cdot h\cdot p_n(t)\\ &+\{(n-1)\alpha+\gamma\}\cdot h\cdot p_{n-1}(t)+\{(n+1)\beta+\\ &\delta\}\cdot h\cdot p_{n+1}(t)+0(h)\end{aligned}$$

By transposing the term $p_n(t)$, dividing by h, and passing to the limit, we obtain the system of differential equations

$$\begin{aligned}(2.6)\quad p_n'(t)=&\{(n-1)\alpha+\gamma\}\cdot p_{n-1}(t)-\{n(\alpha+\beta)+\\ &(\gamma+\delta)\}\cdot p_n(t)\\ &+(n+1)\beta+\delta\}\cdot p_{n+1}(t)\end{aligned}$$

The equation holds for $n=1,2,3\cdots$. For $n=0$ we have

$$p_0'(t)=-(\gamma+\delta)p_0(t)+(\beta+\delta)\cdot p_1(t)$$

since 0 is an absorbing state.

We now obtain the solution of the differential equations. Let us use the method of probability generating functions to find the distribution $p_n(t)$. As before, we put

$$G(z,t)=\sum_{n=0}^{\infty}\cdot z^n\cdot p_n(t)$$

We multiphy equation (2.6) by z^n;

$$z^n \cdot p'_n(t) = \alpha z^2(n-1)z^{n-1}p_{n-1}(t) + \gamma \cdot z \cdot z^{n-1}p_{n-1}(t)$$
$$-(\alpha+\beta)\ z \cdot nz^{n-1}p^n(t) - (\gamma+\delta)z^n \cdot p_n(t)$$
$$+\beta(n+1)\ z^n \cdot p_{n+1}(t) + \delta \cdot z^{-1} \cdot z^{n+1} \cdot p_{n+1}(t)$$

We then sum over $n=0, 1, 2\cdots$ and abbreviate to G the G(z, t). Then we have

(2.7) $$\frac{\partial G}{\partial t} + (\alpha z-\beta)\cdot(1-z)\cdot\frac{\partial G}{\partial z} = (\gamma z-\delta)\cdot (1-z^{-1})\cdot G$$

We shall find the solution of the partial differential equation for G. We solve the charcteristic equations

$$\frac{dt}{1} = \frac{dz}{(\alpha z-\beta)(1-z)} = \frac{dG}{(\gamma z-\delta)(1-z^{-1})\cdot G}$$

then we have

$$(\alpha-\beta)dt = \frac{\alpha dz}{\alpha z-\beta} + \frac{dz}{1-z}$$

(2.8) $$\left(\gamma-\left(\frac{\alpha\cdot\delta}{\beta}\right)\frac{dz}{dz-\beta} + \frac{\delta}{\beta}\ \frac{dz}{z} = -\frac{dG}{G}\right.$$

Therefore

$$\frac{\alpha z-\beta}{1-z}\ e^{-(\alpha-\beta).t} = c_1$$

$$z^{\frac{\delta}{\beta}}\ (dy-\beta)^{\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}\cdot G = c_2$$

Since the first order differential equation can have only one constant, two constants must be functions of one anothers. Therefore the general solution of (2.7) is

(2.8)

$$z^{\frac{\delta}{\beta}}\ (\alpha z-\beta)^{\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}\cdot G = g\left(\frac{\alpha z-\beta}{1-z}e^{-(\alpha-\beta).t}\right)$$

where g is an arbitary function. We now assume that $N=n$, that is, $G(z, 0)=z$. Then we can determine g from the initial condition.

$$z^{\frac{\delta}{\beta}}\ (\alpha z-\beta)^{\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}\cdot z^n = g\left(\frac{dz-\beta}{1-z}\right)$$

Putting $w=\frac{dz-\beta}{1-z}$ so $z=\frac{\beta+w}{\alpha+w}$

$$g(w) = \left(\frac{\beta+w}{\alpha+w}\right)^{n-\frac{\delta}{\beta}}\ \left(\frac{(\alpha-\beta)w}{\alpha+w}\right)^{\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}$$

Therefore

$$G(z, t) = \frac{(\beta e^{(\alpha-\beta)t}+w)^{n+\frac{\delta}{\beta}}}{(\alpha e^{(\alpha-\beta t)}+w)^{n+\frac{\gamma}{\alpha}}}\cdot\frac{(\alpha+w)^{\frac{\gamma}{\alpha}}}{(\beta+w)^{\frac{\delta}{\beta}}}$$

where $w=\frac{\alpha z-\beta}{1-z}$

And substitute w into (2.9), after much calculations, we have

(2.10) $$G(z, t) = \left[\frac{D(t)-B(t)z}{C(t)-A(t)z}\right]^n \left[\frac{\beta-\alpha}{C(t)-A(t)z}\right]^{\frac{\gamma}{\alpha}} \left[\frac{C(t)-A(t)z}{\beta-\alpha z}\right]^{\frac{\delta}{\beta}}$$

Where $A(t)=\alpha-\alpha\cdot e^{(\alpha-\beta)t}$

$B(t)=\alpha-\beta\cdot e^{(\alpha-\beta)t}$

$C(t)=\beta-\alpha\cdot e^{(\alpha-\beta)t}$

and $D(t)=\beta-\beta\cdot e^{(\alpha-\beta)t}$

3. Expected Value and Variance of Population size

To obtain expression for the expected value $E(N_t)$ and variance $V(N_t)$ of population size at time t, we differentiate the probability generating function (2.10) with respect to z. That is,

(3.1) $$E(N_t) = \left.\frac{\partial G(z, t)}{\partial z}\right|_{z=1}$$

(3.2) $$V(N_t) = \left.\frac{\delta^2 G(z, t)}{\partial z^2}\right|_{z=1} - \left(\left.\frac{\partial G(z, t)}{\partial z}\right|_{z=1}\right)^2 - \left.\frac{\partial G(z, t)}{\partial z}\right|_{z=1}$$

An alternative simple procedure for finding $E(N_t)$ and $V(N_t)$ is to differentiate partial differential equation (2.7) with respect to z. That is,

(3.3) $$\frac{\partial^2 G}{\partial z\cdot\partial t} + (\alpha z-\beta)(1-z)\frac{\partial^2 G}{\partial z^2} + \left\{ [(\beta-\alpha z) + \alpha(1-z) + (\delta-\gamma z)(1-\frac{1}{z})\right\}\frac{\partial G}{\partial z} = \left\{\gamma(1-\frac{1}{z}) + (\gamma z-\delta)\frac{1}{z^2}\right\}\cdot G$$

Since (3.1), We have

(3.4) $$\frac{dE_t}{dt} + (\beta-\alpha)E_t = (\gamma-\delta)$$

where $E(N_t)=E_t$

The solution E_t of the first order ordinary differential equation with inital condition; $G(z, 0)=n$ is

(3.5) $$E(N_t) = E_t = \frac{\gamma-\delta}{\beta-\alpha} + \left(n-\frac{\gamma-\delta}{\beta-\alpha}\right)e^{-(\beta-\alpha).t}$$

Similarly, we have from (3.3)

$$(3.6)\quad \frac{\partial^3 G}{\partial z^2 \partial t}\Big|_{z=1} + 2(\beta-\alpha)\frac{\partial^2 G}{\partial z^2}\Big|_{z=1} + 2(\delta-\gamma-\alpha)\frac{\partial G}{\partial z}\Big|_{z=1} = 2\cdot\delta$$

Since

$$\frac{\partial^3 G}{\partial z^2 \partial t}\Big|_{z=1} = \frac{dV_t}{dt} + (2E_t-1)\frac{dE_t}{dt}$$

$$\frac{\partial^2 G}{\partial z^2}\Big|_{z=1} = V_t + E^2{}_t - E_t$$

and $\frac{\partial G}{\partial z}\Big|_{z=1} = E_t$

where $V(N_t)=V_t$ we obtain ordinary differential equation;

$$(3.7)\quad \frac{dV_t}{dt} + 2(\beta-\alpha)V_t = (1-2E_t)E_t' - 2(\beta-\alpha)E_t^2 + 2(\beta-\delta-\gamma)E_t + 2\delta$$

Therefore we have V_t as general solution of (3.7).

$$V_t = \frac{\delta\beta-\alpha\delta}{(\beta-\alpha)^2} + \Big[\frac{1}{(\beta-\alpha)^2}\{2\alpha(\delta-\gamma)\} + \frac{1}{(\beta-\alpha)}\{2n\alpha-(\gamma-\delta)\} + n\Big] e^{-(\beta-\alpha)\cdot t} + G$$

And we have

$$V(N_t) = V_t = \frac{1}{(\beta-\alpha)^2}\{2\alpha(\delta-\gamma)\} + \frac{1}{(\beta-\alpha)}\{2n\alpha-(\gamma-\delta)\} + n\Big] \cdot (e^{-(\beta-\alpha)\cdot t} - 1)$$

since $V_0=0$.

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出生, 死亡, 移入 및 移出의 네 要因에 의하여 量과 質的 內容을 달리하는 人口運動의 量的 確率 모델을 결정하므로써 人口變動의 量的 豫測에 利用될 수 있도록 하였다.