Filteraton 位相空間에서의 거리화와 완비성에 관하여

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On the Metrization and Completion for the Filteration Topology

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Summary

In this paper, we treat to the problem of metrization for filteration topology and find to the condition for the completion of the filteration topology.

Preliminary

One of the classical problems of topology involves finding condition on a topological space (E, $\underline{E}$) such that one can define a metric d on E x E such that the metric $\underline{E}_d$ induced on E by d identically $\underline{E}$.

In this paper, we treat to the problem of metrization for the filterization for the completion of the fiterization topology.

In this section, we establish basic teminology and recall certain known results relevant to our discussion. Let N be a set of all natural numbers and R is a ring. Let E be a left R-module.

DEFINITION (1.1) A family $\{E_n | n\in N\}$ is called the filteration on E, if E_n's are submodule of E such that $E_n \supset E_{n+1}$ for $n\in N$, that is, a filteration on E is a decreasing sequence of submodules of E.

Using the above definition, we have

PROPOSITION (1.2) Let $\{E_n\}$ be a given filteration on E, E is a set of subset V of E where for $v\in V$ there exists a natural number n_0 such that $v+E_{n_0}$ is a subset of V. Here $\underline{E}$ is called the filteration topology.

The following consequence of the above proposition reflects a basic properties of the filteration topology.

PROPOSITION (1.3) In the above proposition, $\{v+E_n \mid n\in N, v\in E\}$ form a base for the filteration topology (E, $\underline{E}$).

COROLLARY (1.4) The filteration topology is a first countable space.

PROPOSITION (1.5) Let K be a submodule of E. Then E is open in E if and only if $E_n \subset K$ for some $n\in N$. Furthermore, if a submodule is open then it is also closed.

COROLLARY (1.6) In the above proposition (1.5), each of the submodules E_n are both open and closed in $\underline{E}$.

Here, we introduce the important property of the filteration topology.

PROPOSITION (1.7) Let E be a left R-module with filteration $\{E_n\}$. Then $\underline{E}$ is a Hausdorff space if and only if $\bigcap_{n\in N} E_n=0$.

DEFINITION (1.8) Let E be a left R-module with the filteration E_n. A sequence $\langle P_n \rangle$ in E is said to converge if there is a $p\in E$ with the following property: For each natural number k, there is a natural number n_0 such that $n>n_0$ implies that $p-p_n\in E_k$.

And we write $\langle p_n\rangle \to p$. A sequence $\langle p_n\rangle$ in E is called a Cauchy sequence if given any natural number k, there is a natural number n_0 such that $m,n>n_0$ implies that $p_m-p_n\in E_k$.

PROPOSITION (1.9) If a sequence $\langle p_n \rangle$ of E be a convergent sequence, then it is also a Cauchy sequence.

Let Z be the set of all integers. Then Z is a left Z-module. Let $\langle p_n \rangle$ be the sequence of all prime numbers such that $p_i>p_j$ for $i>j$, $Z_n = (\prod_{i=1}^{n} p_i)Z$ and $x_n = \prod_{i=1}^{n} p_i$. Then Z_n is a submodule of E and $E_n \supset E_{n+1}$ for $n\in N$. Moreover, a sequence $\langle x_n \rangle$ is a Cauchy sequence but not convergent sequence. Hence, the converse to the above proposition is not true.

DEFINITION (1.10) A left R-module E with the filteration $\{E_n\}$ is said to complete if E is a Hausdorff space and every Cauchy sequence converges to some element of E.

Metrization for the Filteration Topology

In this section, we find that the filteration topology is a pseudometric space and that the filteration topology is a metric space if it is a Hausdorff space.

DEFINITION (2.1) Let E be a left R-module with the filteration $\{E_n\}$. For each non-empty finite subset H of N, we defined the subspace E_H of E by $E_H = \sum_{n \in N} E_n$ and the real number $p_H = \sum_{n \in N} 2^{-n}$. Here, for $n \in N$, $n \lhd H$ means that $n \lhd k$ for all $k \in N$.

By the above definition, we have directly following lemmas.

LEMMA (2.2) Let H be a non-empty finite subset of N and $n \in N$. If $p_H < 2^{-n}$ then $n < H$, then $E_H \subset E_n$.

LEMMA (2.3) Let K_1 and K_2 be non-empty finite subset of N with $p_{k_1} + p_{k_2} < 1$. Then there exists a non-empty finite subset K of N such that $p_{k_1} + p_{k_2} = p_k$.

Using the above lemma, we have

THEOREM (2.4) Let E be a left R-module with a filteration E_n. Then a filteration topology $\underline{E}$ is a pseudo-metric space.

PROOF. We define the real-valued function d: $E \times E \to R$ by $d(x,y) = 1$ if x-y is not containned in any E_H and by $d(x,y) = \inf_H \{ p_H \mid x\text{-}y \in E_H \}$ otherwise. Let $x,y,z \in E$. Then since $x\text{-}y \in E_n$ implies $y\text{-}x \in E_n$, $d(x,y) = d(y,x)$ and by the definition, $d(x,y) \geqslant 0$ is obious. Now, we must be show that $d(x,z) \leqslant d(x,y) + d(y,z)$. If $d(x,y) + d(y,z) \geqslant 1$ then the given inequlity is trivial. Hence suppose that $d(x,y) + d(y,z) < 1$. Then there exists a positive real number ϵ such that $d(x,y) + d(y,z) + 2\epsilon < 1$. By the definition, there exists non-empty finite subsets K_1, K_2 of N such that $x\text{-}y \in E_{k_1}$, $y\text{-}z \in E_{k_2}$ and $p_{k_1} < d(x,y) + \epsilon$, $p_{k_2} < d(y,z) + \epsilon$. Since $p_{k_1} + p_{k_2} < 1$, by the lemma (2.3), the exists a unique finite subset K of N for which $p_k = p_{k_1} + p_{k_2}$. And since $E_{n+1} + E_{n+j} \subset E_n$ for $n,i,j \in N$, by the lemma (2.2), $E_{k_1} + E_{k_2} \subset E_k$. It follows that $x\text{-}z = (x\text{-}y) + (y\text{-}z) \in E_k$ and hence $d(x,z) \leqslant p_k = p_{k_1} + p_{k_2} < d(x,y) + d(y,z) + 2\epsilon < 1$. Since ϵ was arbitrary, we have $d(x,z) \leqslant d(x,y) + d(y,z)$. This completes the proof.

Let us write $x * y$ if and only if $d(x,y) = 0$. It is clear that this is an equivalence reration in E which partitions E into equivalence classes. moreover, $d(x,y) = 0$ if and only if $0 = \inf_H \{ p_H \mid x\text{-}y \in E_H \}$ if and only if $x\text{-}y \in E_n$ for all $n \in N$ if and only if $x\text{-}y \in \bigcap_{n \in N} E_n$. Since $\bigcap_{n \in N} E_n$ is a submodule of E, we obtain a quotiant module $E / \bigcap_{n \in N} E_n$. Moreover, we have $E / \bigcap_{n \in N} E_n = E / *$. Here, if $a = x + \bigcap_{n \in N} E_n$, $b = y + \bigcap_{n \in N} E_n$, define $d^*(a,b) = d(x,y)$, then $(E / \bigcap_{n \in N} E_n, d^*)$ is a metric space.

THEOREM (2.5) Let E be a left R-module with a filteration E_n. If $\underline{E}$ is a Hausdorff space.

PROOF. Since $\underline{E}$ is a pseudo-metric space, we must show that for $x,y \in E$, $d(x,y) = 0$ implies $x = y$. Hence suppose that $d(x,y) = 0$. Then $0 = \inf_H \{ p_H \mid x\text{-}y \in E_H \}$ which implies that for 2^{-n}, there exists a non-empty finite subset H of N such that $p_H < 2^{-n}$. By the lemma (2.2), $n < H$ and $x\text{-}y \in E_H \subset E_n$. Hence for $n \in N$, $x\text{-}y \in E_n$, that is $x\text{-}y \in \bigcap_{n \in N} E_n$. Since $\bigcap_{n \in N} E_n = 0$, $x = y$ which implies that d is a metric on E.

Compactness, Completion and Quotientness

In this section, we find to the some properties of the compactness, completion and quotientness for the filteration topology.

DEFINITION (3.1). Let $\underline{E}$ be a filteration topology. Associate to each a E and to each non-zero element r of R the translation T_a and the multiplication operator M_r, by the formulas $T_a(x) = a+x$, $M_r(x) = rx$ $(x \in E)$.

A useful property of the filteration topology is as follows:

THEOREM (3.2). Let $\underline{E}$ be a filteration topology. Then T_a is a homeomorphism and M_r a continuous function. In particular, if r is invertible, then M_r is a homeomorphism.

PROOF. The R-module axioms implies that T_a is bijective and $(T_a)^{-1} = T_{-a}$. Let V be open with $a+x \in V$. If $y \in (T_a)^{-1}(V)$ then $y \in V_{-a}$, i.e, $y+a \in V$. Hence there exists $n_0 \in N$ such that $(y+a) + E_{n_0} \subset V$. Then $y+E_{n_0} \subset (T_a)^{-1}(V)$ which implies that $(T_a)^{-1}(V)$ is open. Therefore T_a is a continuous function. Similarly, we have $(T_a^{*})^{-1}(V)$ is continuous. This prove that T_a is a homeomorphism. Now, let W be open with $rx \in V$. If $y \in (M_r)^{-1}(V)$ then $ry \in V$. Hence there exists $n_1 \in N$ such that $ry+E_{n_1} \subset V$. Here, $y+E_n$ is open and $y \in y+E_n$. Since $M_r(y+E_{n_1}) = ry+E_n \subset V$, $y+E_{n_1} \subset (M_r)^{-1}(V)$ which implies that $(M_r)^{-1}(V)$ is open. Therefore, M_r is continuous. Suppose that r is invertible. Then M_r is bijective and $(M_r)^{-1} = M_r{-1}$. Hence, M_r is a homeomorphism.

In the above results, we have easily following property.

COROLLARY (3.3) In the filteration topology, the module operations are continuous.

LEMMA (3.4) Let $\underline{E}$ be a filteration topology and A open