Some Properties of the Extensions of the k – semiring Homomorphisms

Yang Sung-Ho

k - 반환의 준동형사상에 관한 몇가지 성질

梁 成 豪

Summary

This paper considers the extensions of the k-semiring homomorphisms. We prove that each k-semiring homomorphism $f: R \to S$ induces the unique ring homomorphism $\bar{f}: \bar{R} \to \bar{S}$ such that $\bar{f}(a) = f(a)$ for all $a \in R$ and there exists a homomorphism φ: Hom(R, S) $\to$ Hom($\bar{R}$, $\bar{S}$) and that f is an isomorphism if and only if $\bar{f}$ is an isomorphism and study their some properties.

1. Introduction and preliminaries

One of the more interesting aspects of any algebraic structure is the study of homomorphisms of that structure. It is usually interesting to see what properties of a structure are preserved under homomorphisms.

Louis Dale [2] was concerned with extending certain halfring homomorphisms to the homomorphisms of the ring of difference of the halfring. Moreover Y.B. Chun, H.S. Kim, and H.B. Kim [1] constructed the extension ring of a k-semiring by adding a set to the k-semiring and giving adequate operations.

In this paper, we will be concerned with extending the k-semiring homomorphisms to the homomorphisms of the extension ring of the k-semiring and determining what properties of the k-semiring homomorphism are preserved under the extension.

We must first introduce the extension ring of the k-semiring.

Let R be a k-semiring. Let R' be a set of the same cardinality with R-{0} such that $R \cap R' = \phi$ and let denote the image of a $\in$R-{0} under a given bijection by a'. Let $\oplus$ and Θ denote addition and multiplication respectively on a set $\bar{R} = R \cup R'$ as follows;

$$a \oplus b = \begin{cases} a+b & \text{if } a, b \in R \\ (x+y)' & \text{if } a = x', b = y' \in R' \\ c & \text{if } a \in R, b = y' \in R', a = y+c \\ c' & \text{if } a \in R, b = y' \in R', a+c = y \end{cases}$$

where c is the unique element in R such that either a=y+c or a+c=y but not both, and

$$a \Theta b = \begin{cases} ab & \text{if } a, b \in R \\ xy & \text{if } a = x', b = y' \in R' \\ (ay)' & \text{if } a \in R, b = y' \in R'. \end{cases}$$

It can be shown that these operations are well defined.

Theorem(1-1). If R is a k-semiring, then $(\bar{R},\oplus,\odot)$ is a ring, called the extension ring of R.

Proof. Refer to [1].

Remark (1-2). Let $\ominus a$ denote the additive inverse of any element $a\in\bar{R}$ and write $a\oplus(\ominus b)$ simply as $a\ominus b$. Then it is clear that $a'=-a$ and $a=\ominus a'$ for all $a\in R$.

2. The extensions of the k-semiring homomorphisms

In this section, we assume that k-semirings always have the extension rings.

Theorem (2-1). If $f: R\to S$ is a k-semiring homomorphism, then there exists the unique ring homomorphism $\bar{f}: \bar{R}\to\bar{S}$ such that $\bar{f}(a)=f(a)$ for all $a\in R$.

Proof. Define $\bar{f}: \bar{R}\to\bar{S}$ by $\bar{f}(a)=f(a)$ for all $a\in R$ and $\bar{f}(x')=(f(x))'$ for all $x'\in R'$.

If $a=b\in R$, then $\bar{f}(a)=f(a)=f(b)=\bar{f}(b)$. If $a=b=x'\in R'$, then $\bar{f}(a)=\bar{f}(x')=(f(x))'=\bar{f}(b)$. Thus $\bar{f}$ is well defined.

We claim that $\bar{f}$ is a homomorphism.

If a, $b\in R$, then $\bar{f}(a\oplus b)=\bar{f}(a+b)=f(a+b)=f(a)+f(b)=f(a)\oplus f(b)=\bar{f}(a)+\bar{f}(b)$. If $a=x'$, $b=y'\in R'$, then $\bar{f}(a\oplus b)=\bar{f}((x+y)')=(f(x+y))'=(f(x)+f(y))'=(f(x)\oplus f(y))'=\ominus(f(x)\oplus f(y))=\ominus f(x)\ominus f(y)=(f(x))'\oplus(f(y))'=\bar{f}(x')\oplus\bar{f}(y')=\bar{f}(a)\oplus\bar{f}(b)$
If $a\in R$, $b=y'\in R'$, $a=y+c$, then $\bar{f}(a\oplus b)=\bar{f}(c)=f(c)=f(a)\oplus(f(y))'=f(a)\oplus\bar{f}(y')=\bar{f}(a)\oplus\bar{f}(b)$ since $f(a)=f(y)+f(c)$. If $a\in R$ $b=y'\in R'$, $a+c=y$, then $\bar{f}(a\oplus b)=\bar{f}(c')=(f(c))'=f(a)\oplus(f(y))'=\bar{f}(a)\oplus\bar{f}(y')=\bar{f}(a)\oplus\bar{f}(b)$ since $f(a)+f(c)=f(y)$. If a, $b\in R$, then $\bar{f}(a\odot b)=\bar{f}(ab)=f(ab)=f(a)f(b)=f(a)\odot f(b)=\bar{f}(a)\odot\bar{f}(b)$. If $a=x'$, $b=y'\in R'$, then $\bar{f}(a\odot b)=\bar{f}(xy)=f(xy)=f(x)f(y)=(f(x))'\odot(f(y))'=\bar{f}(x')\odot\bar{f}(y')=\bar{f}(a)\odot\bar{f}(b)$. If $a\in R$, $b=y'\in R'$, then $\bar{f}(a\odot b)=\bar{f}((ay)')=(f(ay))'=(f(a)f(y))'=f(a)\odot(f(y))'=\bar{f}(a)\odot\bar{f}(y')=\bar{f}(a)\odot\bar{f}(b)$. Thus $\bar{f}$ is a homomorphism.

If $g:\bar{R}\to\bar{S}$ is another homomorphsm such that $g(a)=f(a)$ for all $a\in R$, then $g(x')=g(\ominus x)=\ominus g(x)=\ominus f(x)=(f(x))'=\bar{f}(x')$ for all $x'\in R'$. Thus $g=\bar{f}$. Hence $\bar{f}$ is the unique homomorphism.

Definition (2-2). If $f:R\to S$ is a k-semiring homomorphism, then the map $\bar{f}:\bar{R}\to\bar{S}$ given in theorem (2-1) is called the extension of f to $\bar{R}$.

By theorem (2-1), each k-semiring homomorphism $f:R\to S$ induces the unique ring homomorphism $\bar{f}:\bar{R}\to\bar{S}$. It is clear that Hom(R,S) is a commutative monoid under addition defined by $(f+g)(a)=f(a)+g(a)$ for each $a\in R$. Likewise Hom $(\bar{R},\bar{S})$ is an abelian group.

Theorem (2-3). If R and S are k-semirings, then the map φ: Hom(R,S) $\to$ Hom$(\bar{R},\bar{S})$ given by $\varphi(f)=\bar{f}$ is a homomorphism.

Proof. By the uniqueness of $\bar{f}$ in theorem (2-1), it is clear that φ is well defined. If f and g are in Hom(R,S), then $\varphi(f+g)=\overline{f+g}$ and $\varphi(f)+\varphi(g)=\bar{f}+\bar{g}$. Since $(\overline{f+g})(a)=(f+g)(a)=f(a)+g(a)=\bar{f}(a)+\bar{g}(a)=(\bar{f}+\bar{g})(a)$ for all $a\in R$ and $(\overline{f+g})(x')=((f+g)(x))'=(f(x)+g(x))'=(f(x))'+(g(x))'=\bar{f}(x')+\bar{g}(x')=(\bar{f}+\bar{g})(x')$ for all $x'\in R'$, $\overline{f+g}=\bar{f}+\bar{g}$. Thus φ is a homomorphism.

Theorem (2-4). If $f:R\to S$ is a k-semiring homomorphism, then f is an isomorphism if and only if $\bar{f}$ is an isomorphism.

Proof. If $\bar{f}$ is injective, it is clear that f is injective since $f(a)=\bar{f}(a)$ for all $a\in R$. Suppose that f is injective and $\bar{f}(a)=\bar{f}(b)$.
If a, $b\in R$, then it is clear that a=b. If $a=x'$, $b=y'\in R'$, then $\bar{f}(x')=\bar{f}(y')$ implies $(f(x))'=(f(y))'$. So, $f(x)=f(y)$. i.e. x=y. Thus $a=x'=y'=b$. If $a\in R$, $b=y'\in R$, then $f(a)=\bar{f}(y')=(f(y))'$ implies $f(a+y)=f(a)+f(y)=0$. Since f is injective, $a+y=0$.
Thus $a=y'=b$.
Now if $\bar{f}$ is surjective and f is not surjective, then $S-f(R)\neq\phi$. If $s\in S-f(R)\subset\bar{S}$, then there exists $x'\in R'$ such that $\bar{f}(x')=s$ since $\bar{f}$ is surjective. Thus $(f(x))'=\bar{f}(x')=s$. So, $s\in S'\cap S$.
This is contradict to $S'\cap S=\phi$. Hence f is surjective. Suppose that f is surjective and y' is an element in S'. Then y is an element in S. Since f is surjective, there exists $x\in R$ such that $f(x)=y$. So,

$x' \epsilon R' \subset \bar{R}$ and $\bar{f}(x')=(f(x))'=y'$

Theorem (2-5). Let $f:R \to S$ and $g:S \to L$ be the k-semiring homomorphisms. Then $\overline{gf}=\bar{g}\,\bar{f}$.

Proof. If $x \epsilon R$, then $\overline{(gf)}(x)=(gf)(x)=g(f(x))=\bar{g}(f(x))=\bar{g}(\bar{f}(x))=(\bar{g}\bar{f})(x)$. If $x' \epsilon R'$ then $(\overline{gf})(x')=((gf)(x))'=(g(f(x)))'=\bar{g}((f(x))')=\bar{g}(\bar{f}(x'))=(\bar{g}\bar{f})(x')$.

Corollary (2-6). If $f:R \to R$ is a k-semiring homomorphism, then

(1) $\bar{1}_R=1_{\bar{R}}$, and

(2) $(\bar{f})^{-1}=(\overline{f^{-1}})$ if f^{-1} exists.

Proof. (1) if $x \epsilon R$, then $\bar{1}_R(x)=1_R(x)=x=1_{\bar{R}}(x)$. If $x' \epsilon R'$ then $\bar{1}_R(x')=(1_R(x))'=x'=1_R(x')$.

(2) By (1) and theorem (2-5), $\bar{f}(\overline{f^{-1}})=\overline{ff^{-1}}=\bar{1}_R=1_{\bar{R}}$ and $(\overline{f^{-1}})\,\bar{f}=(\overline{f^{-1}f})=\bar{1}_R=1_{\bar{R}}$. It follows that $(\bar{f})^{-1}=(\overline{f^{-1}})$ if f^{-1} exists.

References

[1] Y.B. Chun, H.S. Kim, and H.B. Kim, *A study on the structure of a semiring*, J. of N. S.R.I. Vol. 2 (1983). Yonsei University.

[2] Louis Dale, *Extending certain semiring homomorphisms to ring homomorphisms*, Kyungpook Math. J. Vol. 23, No. 1, June, 1983, 13-18.

국 문 초 록

k-반환의 준동형사상에 관한 몇가지 성질

이 논문에서는 f가 k-반환의 준동형사상이면 f의 확대 환 준동형사상은 유일하게 존재하며 Hom(R, S)에서 Hom($\bar{R}$, $\bar{S}$)로의 준동형사상도 존재함을 보였다. 그리고 f가 동형사상이 될 필요 충분조건은 f의 확대 환 준동형사상이 동형사상임을 밝혔다.