The Isomomorphism of Relative Ideals

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상대적 Ideals 의 동형사상

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Introduction

In [1] J.M. Howie has explained the basic properties of semigroup and studied the congruence on a semigroup and proved the isomorphism of the quotient set of a semigroup by the congruence relation.

In [2] T.K. Dutta has defined the relative ideal and studied the properties of relative ideal.

Now we will review the properties of a semigroup and relative ideal. And we will apply the isomorphism of the quotient set of a semigroup by the congruence relation to the isomorphism of the quotient set of relative ideal by the Rees congruence relation.

I. Definitions and Preliminarlies

Definition (1-1). We will say that $(S,\cdot)$ is a semigroup if $(xy)z = x(yz)$ for any $x,y,z, \in S$.

Definition (1-2). If a semigroup $(S,\cdot)$ has the additional property that $xy = yx$ for any $x,y \in S$, it is called a commutative semigroup.

Definition (1-3). If a semigroup $(S,\cdot)$ has an element 1 such that $x1 = 1x$ for any $x \in S$, 1 is called an identity (element) of S and S is called a semigroup with identity, or monoid.

Definition (1-4). If A and B are subsets of a semigroup, we write $AB = \{ab : a \in A, b \in B\}$ and $\{a\}B = aB = \{ab : b \in B\}$ for $a \in S$.

Definition (1-5). If $(S,\cdot)$ is a semigroup, then a nonempty subset T of S is called a subsemigroup of S if $xy \in T$ for any $x,y \in T$.

Definition (1-6). A nomempty subset A of a semigroup S is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$, and an idela if it is both a left and right ideal.

Definition (1-7). If x is a nonempty set, then a subset ρ of $X \times X$ is called a relation on X. $X \times X$ is called a universal relation and $1x = \{(x,x) : x \in X\}$ is called the equality relation.

Definition (1-8). Let $\beta(S)$ be the set of all relations on X and let $\rho, \sigma \in \beta(X)$. Then we define a binary operation on $\beta(X)$ as follows; if ρ, $\sigma \in \beta(X)$, then $\rho \circ \sigma = \{(x,y) \in X \times X : \exists z \in X \ni (x,y) \in \rho$ and $(z,y) \in \sigma\}$.

Definition (1-9). $\rho^{-1} = \{(x,y) \in X \times X : (y,x) \in \rho\}$ is called the inverse of ρ

Definition (1-10). A relatin ρ is called an equivalence relation if (i) $(x,x) \in \rho$ for every $x \in X$: reflexive (ii) $\rho = \rho^{-1}$: symmetric (iii) $\rho \circ \rho \subseteq \rho$: transitive.

Definition (1-11). $X/\rho = \{x\rho : x \in X\}$ is called the quotient set with an equivalence ρ. $\rho^{\#}$ is called the natural mapping from X onto X/ρ defined by $x\rho^{\#} = x\rho$ for any $x \in X$.

Definition (1-12). Let $(S,\cdot)$ be a semigroup. A relation R on S is called left compatible if $(s,t)\in R \Rightarrow (as, at)\in R$ and right compatible if $(s,t)\in R \Rightarrow (sa, ta)\in R$ for any $s,t,a\in S$. R is called compatibel if $(s,s')\in R$ and $(t,t')\in R \Rightarrow (st,s't')\in R$ for any $s,t,s',t'\in S$. A compatible equivalence relation is called a congruence.

Proposition (1-13). Let S be a semigroup and let ρ be a congruence on as emigroup S. Then $S/\rho = \{x\rho : x\in S\}$ is a semigroup.

Definition (1-14). If ϑ is a mapping from a semigroup $(S,\cdot)$ into a semigroup $(T,\cdot)$ we say that ϑ is a homomorphism if $(xy)\vartheta = (x\vartheta)(y\vartheta)$ for any $x,y\in S$. We refer to S as the domain of ϑ, to T as the codomain of ϑ, and to the subset $S\vartheta = \{s\vartheta : s\in S\}$ of T as the range of ϑ. If ϑ is one-one we shall call it a monomorphism, and if it is both one-one and onto we shall call it an isomorphism. $Ker\vartheta = \vartheta\cdot\vartheta^{-1} = \{(a,b)\in S\times S : a\vartheta = b\vartheta\}$.

Proposition (1-15). If ρ is a congruence on a semigroup S, then S/ρ is a semigroup w.r.t the operation $(a\rho)(b\rho) = (ab)\rho$ and the mapping ϕ:S defined by $x\rho^{\#} = x\rho$ for any $x\in S$ is a homomorphism. If $\phi: S\rightarrow T$ is a homomorphism, where S and T are semigroups, then the relation $Ker\ \phi = \phi\cdot\phi^{-1} = \{(a,b)\in S\times S : a\phi = b\phi\}$ is a congruence on S and there is a monomorphimsm $\alpha: S/ker\phi \rightarrow T$ such that ran (α) = ran (ϕ) and the diagram commutes.

Proposition (1-16). Let ρ be a congruence on a semigroup S. If $\phi : S \rightarrow T$ is a homomorphism such that $\rho \subseteq Ker\ \phi$ then there is a unique homomorphism $\beta : S/\rho \rightarrow T$ such that ran (β) = ran (ϕ) and the diagram commutes.

Proposition (1-17). Let ρ, σ be congruences on a semirgroup S such that $\rho\subseteq\sigma$. Then $\sigma/\rho = \{(x\rho, y\rho)\in S/\rho\times S/\rho : (x,y)\in\sigma\}$ is a congruence on S/ρ, and $(S/\rho)/(\sigma/\rho) \simeq S/\sigma$.

II. Relative Ideal for Semigroup

Definition (2-1). Let S be a semigroup and T be a subsemigroup of S. A nonempty subset A of S is called a left T-ideal if $TA\subseteq A$. The right T-ideal is defined anlogously. A nomempty subset A of S is called a T-ideal if it is both left and right T-idea.

Example (2-2). Let M_2 be the set of all 2×2 nonsingulr metrices over the field of rational numbers. Then M_2 is a group w.r.t matrix multiplication. Let $T = \{\begin{pmatrix} a & o \\ o & b\end{pmatrix} : a, b \text{ are integers}\}$ and $A = \{\begin{pmatrix} e & f \\ g & h\end{pmatrix} : e,f,g,h \text{ are even integers}\}$. Then A is a left T-ideal as well as a right T-ideal of M_2

Remark (2-3). Let S be a semigroup. Then every ideal in S is a S-idal.

Propositon (2-4). Let S be a semigroup and A be a left (right) T_1-ideal and a left (right) T_2-ideal with $T_1\cap T_2 \neq \phi$. Then A is also a left (right) $T_1\cap T_2$-ideal.

Proof: Let $x,y\in T_1\cap T_2$. Then $x,y\in T_1$ and $x,y\in T_2$. Thus $xy\in T_1$ and $xy\in T_2$ and $T_1\cap T_2$ is a subsemigroup of S. Since $(T_1\cap T_2)A\subseteq T_1A\subseteq A$, so A is a left $T_1\cap T_2$-ideal. In right case we can easily prove.

Corollary (2-5). Let S be a semigroup and let A be a T_1-ideal and T_2-ideal with $T_1\cap T_2\neq\phi$ Then A is a $T_1\cap T_2$-ideal.

Proposition (2-6). Let S be a semigroup and let A be a left T_1-ideal and right T_2-ideal with $T_1\cap T_2\neq\phi$ Then A is a $T_1\cap T_2$-ideal.

Proof: Since $T_1\cap T_2$ is a subsemigroup and $A(T_1\cap T_2)\subseteq AT_2\subseteq A$ and $(T_1\cap T_2)A\subseteq T_1A\subseteq A$. By the definition A is a $T_1\cap T_2$-ideal.

Proposition (2-7). Let S be a semigroup and let A and B be a left (right) T-ideal. Then $A\cap B$ and $A\cup B$ are also left (right) T-ideals.

Proof: Since $TA\subseteq A$ and $TB\subseteq B$, so $T(A\cap B)\subseteq TA\subseteq A$ and $T(A\cap B)\subseteq TB\subseteq B$. Thus $T(A\cap B)\subseteq A\cap B$. If $x\in T(A\cup B)$, $\exists\ t\in T$, $a\in A\cup B$, $\cdot\ni\cdot$ $x = ta$. Here if $a\in A$, then $x = ta\in TA$ and if $a\in B$, then $x = ta\in TB$. Thus $T(A\cup B)\subseteq (TA)\cup(TB)$ and $TA\subseteq A$ and $TB\subseteq B$. Hence $T(A\cup B)\subseteq (TA)\cup(TB)$. In right case we can complete the proof (by same method).

Corollary (2-8). Let S be a semigroup and let A and B be a T-ideals. Then $A\cap B$ and $A\cup B$ are also T-ideals

Remark (2-9). Let S and T be semigroup. Then the direct product $S \times T = \{(s,t) : s \in S, t \in T\}$ is a semigroup for $(s,t)(s',t') = (ss',tt')$. Now we can define $(S \times T)(A \times B) = SA \times TB$ and $(A \times B)(S \times T) = AS \times BT$, where A and B are subsets of S and T, respectively. If A and B are subsemigroups of S and T, respectively, then $A \times B$ is a subsemigroup of $S \times T$. And let A and B be left (right) ideal of S and T, respectively, then $A \times B$ is a left (right) ideal of $S \times T$. Furthermore let S and U be semigroup and let T,V be subsemigroup of S and U, respectively and let A be a T-ideal and B be a V-ideal. Then $A \times B$ is a $T \times V$-ideal in $S \times U$.

Definition (2-10). A semigroup S is said to have the properties α,ß or ϱ if the relation $L \cap L_2 = L_1L_2$, $R_1 \cap R_2 = R_1R_2$ or $L_1 \cap R_1 = L_1R_1$ hold for left T-ideals L_1, L_2 and right T-ideals R_1, R_2 of S.

Lemma. Let S be a semigroup having property ϱ (α or ß) and T be a subsemigroup of S. Then T is a normal subsemigroup of S.

Proposition (2-11). Let M is a monoid having property ϱ (α orß) and T be a subsemigroup with identity of M. Then$\{ mT : m \in M\}$ is a monoid.

Proof: Consider an operation as folow $(mT)(nT) = mnT$ for any $m,n \in M$. Then the operation is well defined since T is a normal subsemigroup of M and T has an identity. And associative property is evident since M is associative. Now $eT = T$ is an identity in $\{mT : m \in M\}$, where e is an identity in M. Hence $\{mT : m \quad M\}$ is a monoid.

Proposition (2-12). Let I be a T-ideal and a subsemigroup of a semigroup S and let $I \cup T$ be a subsemigroup of S. Then$\rho_I^{TUI} = (IXI) \cup 1_{TUI}$ is a congruence on $T \cup I$.

Proof: For any $x \in T \cup I$ $(x,x) \in 1_{TUI} \subseteq \rho_I^{TUI}$. If $(a,b) \in \rho_I^{TUI}$, then $(a,b) \in I \times I$ or $(a,b) \in 1_{TUI}$. Thus $(b,a) \in I \times I$ or $(b,a) \in 1_{TUI}$, that is, $(b,a) \in \rho_I^{TUI}$ If $(a,b) \in \rho_I^{TUI}$ and $(b,c) \in \rho_I^{TUI}$, then $(a,b) \in I \times I$ or $(a,b) \in 1_{TUI}$ and $(b,c) \in I \times I$ or $(b,c) \in 1_{TUI}$. Thus $(a,c) \in \rho_I^{TUI}$ for every case. If $(a,b) \in \rho_I^{TUI}$ and $(a',b') \in \rho_I^{TUI}$, then $(aa',bb') \in \rho_I^{TUI}$ since I is a subsemigroup of S and I is a T-ideal. Hence ρ_I^{TUI} is a congruence on $T \cup I$.

Remark (2-13). Let S be a semigroup and I be a T-ideal and let $I \subseteq T$. Then I is a subsemigroup of S and $T \cup I = T$ is a subsemigroup of S. Thus $\rho_I^{TUI} = \rho_I^T$ is a congruence on T. Furthermore let I be an ideal of S. Then I is a S-ideal since we can take T to be S. Thus ρ_I is a congruence on S.

Proposition (2-14). Let I be a T-ideal and a subsemigroup of a semigroup S and let $T \cup I$ be a subsemigroup of S. Then ${}^{TUI}/\rho_I^{TUI}$ is a semigroup with zero element I and ${}^{TUI}/\rho_I^{TUI} = \{I\} \cup \{\{x\} : x \in (T \cup I)\text{-}I\}$.

Proof: By Proposition 1.13. ${}^{TUI}/\rho_I^{TUI}$ is a semigroup of the quotient sets with operation $(x\rho_I^{TUI})(y\rho_I^{TUI}) = xy\rho_I^{TUI}$. Now we must show that I is a zero element in ${}^{TUI}/\rho_I^{TUI}$ and ${}^{TUI}/\rho_I^{TUI} = \{I\}u\ \{\{x\} : x \in (T \cup I)-I\}$. For any $x,y \in I$ $x\rho_I^{TUI} = I$ and $y\rho_I^{TUI} = I$. Here $(x\rho_I^{TUI})(y\rho_I^{TUI}) = (xy)\rho_I^{TUI} = I$ since x and y belong to I. And if $x \in (T \cup I)-I$ and $y \in I$, then $x\rho_I^{TUI} = \{x\}$ and $\{x\} I = (x\rho_I^{TUI})(y\rho_I^{TUI}) = xy\rho_I^{TUI} = I$ and $I\{x\} = I$ for any $y \in I$. That is, $\alpha I = \alpha I = I$ for any $\alpha \in {}^{TUI}/\rho_I^{TUI}$. Second ${}^{TUI}/\rho_I^{TUI} = \{I\} \cup \{\{x\} : x \in (T \cup I)-I\}$. By the definition ${}^{TUI}/\rho_I^{TUI} = \{x\rho_I^{TUI} : x \in TUI\}$. Here if $x \in I$, then $x\rho_I^{TUI} = I$ since $\rho_I^{TUI} = (IxI) \cup 1_{TUI}$ and if $x \notin I$, then $x\rho_I^{TUI} = x$. Thus Thus ${}^{TUI}/\rho_I^{TUI} = \{I\} \cup \{x\} : x \in (TUI)-I\}$.

Proposition(2-15). Let I,J be T-ideal of a semigroup S such that $I \subseteq J \subseteq T$. Then $T/\rho_J^T \simeq (T/\rho_I^T)/(\rho_J^T/\rho_I^T)$.

Proof : Define β as follows ;$(a\rho_I^T)\beta = a\rho_J^T$ for any $a \in T$. Then $[(a\rho_I^T)(b\rho_I^T)]\beta = (ab\rho_I^T)\beta = ab\rho_J^T = (a\rho_J^T)(b\rho_J^T) = (a\rho_I^T)\beta(b\rho_I^T)\beta$. And $\text{Ker}\ \beta = \beta o \beta^{-1} = \{(a\rho_I^T, b\rho_I^T) \in T/\rho_I^T \times T/\rho_I^T : (a\rho_I^T)\beta = (b\rho_I^T)\beta\} = \{(a\rho_I^T, b\rho_I^T) \in T/\rho_I^T \times T/\rho_I^T : a\rho_J^T = b\rho_J^T\} = \rho_J^T/\rho_J^T$. Now we define α as follow $[(a\rho_I^T)\rho_J^T/\rho_I^T]\alpha = a\rho_J^T$.

Hence $\alpha : (T/\rho_I^T)/(\rho_J^T/\rho_I^T) \rightarrow T/\rho_J^T$ is an isomorphism.

Literature cited

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國 文 抄 錄

본 논문에서는 Congruence 관계에 의한 반군들의 Quotient 집합에 대한 동형을 Rees Congruence 관계에 의한 상대적 Ideals의 Quotient 집합에 적용시켜 보았다.