Recurrence Relations for the Moments of Discrete Order Statistics

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이산순서 통계의 적률에 관한 점화식

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I. Introduction

Suppose $X_1, X_2, \cdots, X_n$ are n independent variables, each discrete cumulative distribution function P(x) over $x=0,1,2\cdots$. Let $X_{r:n}$ $(r=1, 2, \cdots, n)$ be the rth orerder statistic for these variates and let $F_{r:n}(x)$ be the c.d.f of $X_{r:n}$. Then the c.d.f $F_{r:n}(x)$ is given by

$$(1.1)\ F_{r:n}(x)=\sum_{i=r}^{n}\binom{n}{i}P^i(x)[1-P(x)]^{n-i}$$

From the relation between binomial sums and the incomplite beta function, we write (1.1) as

$$(1.2)\ F_{r:n}(x)=I_{P(x)}(n,n-r+1)$$

Where $I_p(a,b)=\frac{1}{B(a,b)}\int_0^p t^{a-1}(1-t)^{b-1}dt$

The bivariate joint c.d.f of $X_{r:n}$ and $X_{s:n}$ $(1\le r<s\le n)$ is conveniently denoted by $F_{rs:n}(x, y)$. Then $F_{rs:n}(x, y)$ is obtained by a direct argument. we have for $x<y$

$$(1.3)\ F_{rs:n}(x,y)=\sum_{j=s}^{n}\sum_{i=r}^{j}\frac{n!}{i!(j-i)!(n-j)!}$$

$$\cdot P^i(x)[P(y)-P(x)]^{j-i}[1-P(y)]^{n-j}$$

Also for $x\ge y$ the inequality $X_{s:n}\le y$ implies $X_{r:n}\le x$, so that

$$(1.4)\ F_{rs:n}(x,y)=F_{s:n}(y).$$

Generally we may remark that a similar argument leads to the multivariate joint c.d.f of the

$X_{n_1:n}, X_{n_2:n}, \cdots, X_{n_k:n}(1\le n_1<n_2<\cdots<n_k\le n)$. We have for $x_1<x_2<\cdots<x_k$

$$(1.5)\ F_{n_1n_2\cdots n_k:n}(x_1,x_2,\cdots,x_k)$$

$$=n!\sum_{s_k=n_k}^{n}\sum_{s_{k-1}=n_{k-1}}^{s_k}\cdots\sum_{s_1=n_1}^{s_2}\frac{P^{s_1}(x_1)}{s_1!}$$

$$\cdot\left\{\prod_{i=1}^{k-1}\frac{[P(x_{i+1})-P(x_i)]^{s_{i+1}-s_i}}{(s_{i+1}-s_i)!}\right\}$$

$$\cdot\frac{[1-P(x_k)]^{n-s_k}}{(n-s_k)!}$$

if $X_i\ge x_j(1\le i<j\le k)$, then we obtain

$$(1.6)\ F_{n_1n_2\cdots n_k:n}(x_1,x_2,\cdots,x_k)$$

$$=F_{n_1\cdots n_{i-1}n_{i+1}\cdots n_k:n}(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)$$

II. Probability Functions

Suppose that p(x) is the probability function corresponding to the c.d.f P(x) over x= 0,1,2 ⋯,. Let $f_{r:n}(x)$ be the p.f of $X_{r:n}$. Then from (1.2) we have the expressions

(2.1) $$f_{r:n}(x)=F_{r:n}(x)-F_{r:n}(x-1)$$
$$=I_{P(x)}(r,n-r+1)-I_{P(x-1)}(r,n-r+1)$$

the bivariate p.f $f_{rs:n}(x,y)$ and the multivariate p.f $f_{n_1n_2\cdots n_k:n}(x_1,x_2,\cdots,x_k)$ follow from (1.4)~(1.6). For $x_1 < x_2 < \cdots < x_k$, since

$$f_{n_1n_2\cdots n_k:n}(x_1x_2,\cdots,x_k)$$
$$=\sum_{\substack{s_i=0,1\\1\le i\le k}}(-1)^{\sum_{i=1}^{k}s_i}$$
$$\cdot F_{n_1n_2\cdots n_k:n}(x_1-s_1,x_2-s_2,\cdots,x_k-s_k),$$

it has been definded that $x_o=-1$, $X_{k+1}=\infty$ $n_0=o$ and $n_{k+1}=n+1$ we have

$$f_{n_1n_2\cdots n_k:n}(x_1,x_2,\cdots,x_k)$$
$$=n!\sum\{\pi_{i=1}^{k}\frac{[p(x_i)]^{s_i+t_i+1}}{(s_i+t_i+1)!}\}$$
$$\cdot\{\pi_{j=1}^{k}\frac{[p(x_{j+1}-1)-P(x_j)]^{n_{j+1}-n_j-s_{j+1}-t_j-1}}{(n_{j+1}-n_j-s_{j+1}-t_j-1)!}\}$$

where Σ denotes the sumation over non-negative integral values of s_1, t_1, s_2, t_2, ⋯, s_k, t_k subject to $s_i+t_i\le n_{i+1}-n_i-1(i=1,2,\cdots,k)$, and it is that $t_o=0$ and $s_{k+1}=0$. Setting

$$C^n_{n_1n_2\cdots n_k}=\frac{n!}{\pi_{i=1}^{k}(n_{i+1}-n_i-1)!}$$

we may write

$$f_{n_1n_2\cdots n_k:n}(x_1,x_2,\cdots,x_k)$$
$$=C^n_{n_1n_2\cdots n_k}\sum_{s_1=o}^{n_1-1}\sum_{t_k=o}^{n-n_k}\sum_{s,t}\binom{n_1-1}{s_1}$$
$$\cdot\binom{n-n_k}{t_k}\frac{\pi_{i=1}^{k-1}(n_{i+1}-n_i-1)!}{\pi_{i=1}^{k-1}[(n_{i+1}-n_i-s_{i+1}-t_i-1)!s_{i+1}!t_i!]}$$
$$\cdot\{\pi_{i=0}^{k}[P(x_i-1)-P(x_i)]^{n_{i+1}-n_i-s_{i+1}-t_i-1}\}$$
$$\cdot\{\pi_{i=0}^{k}[p(x_i)]^{s_i+t_i+1}\}$$
$$\cdot\int_0^1\int_0^1\cdots\int_0^1 z_1^{s_1}(1-z_1)^{t_1}[\pi_{i=2}^{k}z_i^{t_i}(1-z_i)^{s_i}]$$
$$\cdot dz_1dz_2\cdots dz_k$$

where the sumation Σ subject to $t_i=m_{i+1}-n_i-1(i=1,2\cdots,k-1)$ and $s_i=n_i-n_{i-1}-t_{i-1}-1(i=2,3,\cdots,k)$. Interchanging the sumation and integral signs, we simplify this equation repeatedly. Putting $v_i=P(x_i)-z_i p(x_i)(i=k,k-1,\cdots 2)$, and $v_1=P(x_1-1)+z_1p(x_i)$, we have

$$\int_0^1\sum_{t_i=0}^{n_{i+1}-n_i-1}\sum_{s_i=0}^{n_i-n_{i-1}-t_{i-1}-1}\{\binom{n_{i+1}-n_i-1}{t_{i+1}}$$
$$/(n_i-n_{i-1}-s_i-t_{i-1}-1)!s_i!\}[v_{i+1}-P(x_i)]^{n_{i+1}-n_i-t_i-1}$$
$$\cdot[p(x_i-1)-P(x_{i-1})]^{n_{i+1}-n_i-t_i-1}[p(x_i)]^{s_i+t_i+1}$$
$$\cdot z_i^{t_i}(1-z_i)^{s_i}dz_i$$
$$=\frac{1}{(n_i-n_{i-1}-t_{i-1}-1)!}\int_{P(x_i-1)}^{P(x_i)}(v_{i+1}-v_i)^{n_{i+1}-n_{i-1}}$$
$$\cdot[v_i-P(x_{i-1})]^{n_i-n_{i-1}-t_{i-1}-1}dv_i$$

for i=k, k−1, ⋯, 2 and

$$\int_0^1\sum_{s_1=0}^{n_1-1}\sum_{t_1=0}^{n_2-n_1-1}\binom{n_1-1}{s_1}\binom{n_2-n_1-1}{t_1}$$
$$\cdot[v_2-P(x_1)]^{n_2-n_1-t_1-1}[P(x_1-1)]^{n_1-s_1-1}$$

$$\cdot [p(x_1)]^{s_1+t_1+1} z_1^{s_1}(1-z_1)^{t_1} dz_1$$

$$= \int_{P(x_1-1)}^{P(x_1)} v_1^{n_1-1}(v_2-v_1)^{n_2-n_1-1} dv_1.$$

therefore, we obtain

$$(2.2) \quad f_{n_1 n_2 \cdots n_k : n}(x_1, x_2, \cdots, x_k)$$

$$= C^{n}_{n_1 n_2 \cdots n_k} \int_{P(x_k-1)}^{P(x_k)} \int_{P(x_{k-1}-1)}^{P(x_{k-1})} \cdots$$

$$\int_{P(x_1-1)}^{P(x_1)} v_1^{n_1-1} \{ \prod_{i=1}^{k-1} (v_{i+1}-v_i)^{n_{i+1}-n_i-1} \}$$

$$\cdot (1-v_k)^{n-n_k} dv_1 dv_2 \cdots dv_k$$

the right hand side is the Dirichlet integral. this probability function may be extended as follows. For $x_1 \le x_2 \le \cdots \le x_k$,

$$(2.3) \quad f_{n_1 n_2 \cdots n_k : n}(x_1, x_2, \cdots, x_k) = c^{n}_{n_1 n_2 \cdots n_k}$$

$$\int_{P(x_{k-1}-1)}^{P(x_k)} \int_{P(x_{k-1})}^{Q_{k-1}} \cdots \int_{P(x_1-1)}^{Q_1} v_1^{n_1-1}$$

$$\cdot \{ \prod_{i=1}^{k-1} (v_{i+1}-v_i)^{n_{i+1}-n_i-1} \} (1-v_k)^{n-n_k}$$

$$\cdot dv_1 dv_2 \cdots dv_k$$

where $Q_i = \min \{v_{i+1}, P(x_i)\}$ $(i=1,2 \cdots, K-1)$. we derive the relationship of the bivariate p.f $f_{rs}(x,y)$ for $x=y$ in particular $f_{rs}(x,x)$

$$= \sum_{i=0}^{r-1} \sum_{j=0}^{n-s} \frac{n!}{(r-i-1)!(s-r+i+j+1)!(n-s-j)}$$

$$\cdot [P(x-1)]^{r-i-1} [p(x)]^{s-r+i+j+1} [1-P(x)]^{n-s-j}$$

$$= C^{n}_{rs} \sum_{i=0}^{r-1} \sum_{j=0}^{n-s} \binom{r-1}{i} \binom{n-s}{j} [P(x-1)]^{r-i-1}$$

$$\cdot [p(x)]^{s-r+i+j+1} [1-P(x)]^{n-s-j}$$

$$\cdot \int_0^1 \int_0^1 z^i (1-z_1)^{s-r-1} z_2^j (1-z_2)^{s-r+i} dz_2 dz_1$$

$$= C^{n}_{rs} \sum_{i=0}^{r-1} \binom{r-1}{i} [P(x-1)]^{r-i-1} \int_0^1 [1-P(x)$$

$$+ z_2 p(x)]^{n-s} [P(x) - z_2 p(x)]^{s-r+i} p(x) dz_2$$

$$\cdot \int_0^1 z_1^i (1-z_1)\, dz_1.$$

putting $v = P(x) - z_2 p(x)$, we have

$$f_{rs}(x,x) = C^{n}_{rs} \sum_{i=0}^{r-1} \binom{r-1}{i} [P(x-1)]^{r-i-1}$$

$$\cdot \int_{P(x-1)}^{P(x)} (1-v)^{n-s} [v - P(x-1)]^{s-r+i} dv \int_0^1 z_1^i$$

$$\cdot (1-z_1)^{s-r-1} dz_1$$

$$= C^{n}_{rs} \int_{P(x-1)}^{P(x)} \{ \int_0^1 [P(x-1) + z_1 v - z_1 P(x-1)]^{r-1}$$

$$\cdot [v - P(x-1) - z_1 v + z_1 P(x-1)]^{s-r-1}$$

$$\cdot [v - P(x+1)] dz_1 \} (1-v)^{n-s} dv.$$

Putting $u = P(x-1) + zv - zP(x-1)$, we obtain the equation

$$(2.4) \quad f_{rs}(x,x) = \int_{P(x-1)}^{P(x)} \int_{P(x-1)}^{v}$$

$$u^{r-1} (v-u)^{s-r-1} (1-v)^{n-s} du dv.$$

Accordingly for $x \le y$ the bivariate p·f $f_{rs}(x,y)$ may be written as

$$(2.5) \quad f_{rs}(x,y) = \int_{P(y-1)}^{P(y)} \int_{P(x-1)}^{Q} u^{r-1}$$

$$\cdot (v-u)^{s-r-1} (1-v)^{n-s} du dv$$

where $Q = \min \{v, p(x)\}$.

using the equation (2,5), we may easily find the p.f $fw_n(w)$ and c.d.f $Fw_n(w)$ of the range $W_n = X_n - X_1$. when $w > 0$,

$$(2.6) \quad fw_n(w) = n(n-1) \sum_{x=0}^{\infty} \int_{P(x-1)}^{P(x)} \int_{P(x+w+1)}^{P(x+w)}$$

$(v-u)^{n-2}dvdu$

$$= \sum_{x=0}^{\infty}\{P(x+w)-P(x-1)]^n-[P(x+w)-P(x)]^n$$

$$+[P(x+w-1)-P(x)]^n-[P(x+w-1)$$

$$-P(x-1)]^n\}$$

and when $w=0$.

$$(2.7) \quad fw_n(o)=n(n-1)\sum_{x=0}^{\infty}\int_{P(x-1)}^{P(x)}$$

$$\int_u^{P(x)}(v-u)^{n-2}dvdu= \sum_{x=0}^{\infty}[p(x)]^n$$

so that we have

$$(2.8) \quad Fw_n(w)$$

$$=[P(w)]^n+\sum_{x=0}^{\infty}\{[P(x+w+1)-P(x)]^n$$

$$-[P(x+w)-P(x)]^n\}$$

for $w \geq 0$.

Also we may find the p.f $fw_{r,r+1}(w)$ and c.d.f $Fw_{r,r+1}(w)$ of $W_{r,r+1}=X_{r+1}-X_r$ from the equation (2,5). Since for $x<y$

$$f_{r,r+1:n}(x,y)$$

$$=\binom{n}{r}[P^r(x)-P^r(x-1)]\{[1-P(y-1)]^{n-r}$$

$$-[1-P(y)]^{n-r}\},$$

$$(2.9) \quad fw_{r,r+1}(w)$$

$$=\binom{n}{r}\sum_{x=0}^{\infty}[P^r(x)-P^r(x-1)]$$

$$\{[1-P(x+w-1)]^{n-r}-[1-P(x+w)]^{n-r}\}$$

for $w>0$. From

$$f_{r,r+1}(x,x)$$

$$=\frac{n!}{(r-1)!(n-r)!}\int_{P(x-1)}^{P(x)}u^{r-1}(1-u)^{n-r}du$$

$$-\binom{n}{r}[P^r(x)-P^r(x-1)][1-P(x)]^{n-r},$$

$$(2.10) \quad fw_{r,r+1}(0)=1-\binom{n}{r}\sum_{x=0}^{\infty}[P^r(x-1)$$

$$-p^r(x-1)][1-P(x)]^{n-r}$$

Therefroe we have

$$(2.11) \quad Fw_{r,r+1}(w)=1-\binom{n}{r}\sum_{x=0}^{\infty}[P^r(x)$$

$$-p^r(x-1)][1-P(x+w)]^{n-r}$$

for $w \geq 0$.

Ⅲ. Moments and recurrence Relations

We write the population mean, variance, kth raw moments and kth factorial moments as

$$(3.1) \quad \mu=\varepsilon X,\ \delta^2=\text{var}X,\ \mu^{(k)}=\varepsilon(X^k),$$

$$\mu_{[k]}=\varepsilon(X^{[k]})$$

The moments of ordered stastics is defined

$$(3.2) \quad \mu_{r:n}=\varepsilon X_{r:n},\ \mu^{(a)}{}_{r:n}=\varepsilon(X^a_{r:n}),$$

$$\delta^2_{r:n}=\text{var}X_{r:n},\ \mu_{rs:n}=\varepsilon(X_{r:n}X_{s:n}),$$

$$\mu^{(a)}_{rsin}=\varepsilon(X^a_{r:n}X^a_{s:n}),\ \mu^{(a\ b)}_{rs:n}$$

$$=\varepsilon(X^a_{r:n}X^b_{r:n}),\ \delta^2_{rs:n}=\text{cov}(X_{r:n},X_{s:n}),$$

$$\mu_{n_1n_2\cdots n_k:n}=\varepsilon(X_{n_1:n}X_{n_2:n}\cdots X_{n_k:n})$$

$$\mu^{(a)}_{n_1n_2\cdots n_k:n}=\varepsilon(X^a_{n_1:n}X^a_{n_2:n}\cdots X^a_{n_k:n}),$$

$$\mu^{(a_1,a_2,\cdots a_k)}_{n_1n_2\cdots n_k:n}$$

$$=\varepsilon(X^{a_1}_{n_1:n}X^{a_2}_{n_2:n}\cdots X^{a_k}_{n_k:n}).$$

To obtain the mean and variance of an ordered statistic, we consider the following Lemma.

Lemma 1. Suppose that $p(x)(i=0,1,2,\cdots)$ is the discrete parent of which c.d.f is $P(x)$. Let $q(x)=1-P(x)$ and define the generatng functions

$$\theta(s)=\sum_{x=0}^{\infty} p(x)s^{x}, \quad \phi(s)=\sum_{x=0}^{\infty} q(x)s^{x}.$$

If the kth factorial moment $\mu_{[k]}$ exists, then

$$(3.3) \quad \mu_{[k]}=k\phi^{(k-1)}(1)$$

therefore the mean and variance are given by

$$(3.3)' \quad \mu=\mu_{[1]}=\sum_{x=0}^{\infty}[1-P(x)]$$

$$\delta^2=\mu_{[2]}+\mu(1-\mu)$$

$$=2\sum_{x=0}^{\infty}x[1-P(x)].$$

Applying these results to the moments of $X_{r:n}$, W_n and W_{rr+1}, from (1.2), (2.8) and (2.11) we obtain as

Theorem 2.

$$(3.4) \quad \mu_{r:n}=\sum_{x=0}^{\infty}[1-I_{P(x)}(r,n-r+1)]$$

$$\delta^2_{r:n}=2\sum_{x=0}^{\infty}x[1-I_{P(x)}(r,n-r+1)]+\mu_{r:n}(1-\mu_{r:n})$$

$$(3.5) \quad \varepsilon W_n=\sum_{x=0}^{\infty}\{1-P^n(x)-[1-P(x)]^n$$

$$\mathrm{var}\, W_n=2\sum_{y=0}^{\infty}\sum_{x=0}^{y}\{1-P^n(y)-[1-P(x)]^n+[P(y)-P(x)]^n\}-\varepsilon W_n(1-\varepsilon W_n)$$

$$(3.6) \quad \varepsilon W_{r\,r+1}=\binom{n}{r}\sum_{y=0}^{\infty}\sum_{x=0}^{y}[P^r(x)-P^r(x-1)][1-P(y)]^{n-r}$$

$$\mathrm{var}\, W_{r\,r+1}=\binom{n}{r}\sum_{y=0}^{\infty}\sum_{x=0}^{y}(y-x)[P^r(x)-P^r(x-1)][1-P(y)]^{n-r}+\varepsilon W_{r\,r+1}(1-\varepsilon W_{r\,r+1}),$$

Proof. For any c.d.f $P(x)$ the existence εx implies

$$\lim_{x\to-\infty} xP(x)=\lim_{x\to\infty} x(1-P(x))=0$$

we use this result. From (2, 14)

$$\varepsilon W_n(W_n-1)=2\sum_{w=0}^{\infty}w[1-F_{W_n}(w)]$$

$$=2\sum_{x=0}^{\infty}x[1-P^n(x)]-2\sum_{x=0}^{\infty}\sum_{w=0}^{\infty}w\cdot\{[P(x+w+1)-P(x)]^n-[P(x+w)-P(x)]^n\}.$$

But

$$\sum_{w=0}^{\infty}W\{[P(x+w+1)-P(x)]^n-[P(x+w)-P(x)^n\}$$

$$=\sum_{w=0}^{\infty}\{w([1-P(x)]^n-[P(x+w)-p(x)]^n)-(w+1)([1-P(x)]^n-[P(x+w+1)-P(x)]^n)\}+\sum_{w=0}^{\infty}([1-P(x)]^n-[P(x+w+1)-P(x)^n)$$

$$=\sum_{w=0}^{\infty}([1-P(x)]^n-[P(x+w+1)-P(x)]^n)$$

$$=\sum_{w=0}^{\infty}([1-P(x)]^n-[P(x+w)-P(x)]^n)-[1-P(x)]^n=\sum_{y=0}^{\infty}\{[1-P(x)]^n-[P(y)-P(x)]^n\}-[1-P(x)]^n$$

Hence

$$\varepsilon W_n(W_n-1)$$

$$=2\sum_{y=0}^{\infty}\{y[1-P^n(y)]+[1-P(y)]^n\}-2\sum_{y=0}^{\infty}\sum_{x=0}^{y}\{[1-P(x)]^n-[P(y)-P(x)]^n)$$

therefore

$$\mathrm{var} W_n=2\sum_{y=0}^{\infty}\sum_{x=0}^{y}\{1-P^n(y)-[1-P(x)]^n+[P(y)-P(x)]^n\}-\varepsilon W_n(1+\varepsilon W_n).$$

The basic relationship between ordered stastics and unordered statistics is

$$(3.7)\quad \sum_{n_i \neq n_j} X^{a_1}_{n_1:n} X^{a_2}_{n_2:n} \cdots X^{a_k}_{n_k:n} = \sum_{n_i \neq n_j} X^{a_1}_{n_1} X^{a_2}_{n_2} \cdots X^{a_k}_{n_k}$$

Where the sign $\sum_{n_i \neq n_j}$ is the sumation of all terms corresponding to the permutations $n_1, n_2, \cdots, n_k$ which consists of different numbers of 1,2 ⋯,, n. The lefthand side is only a rearrangement of the right hand side. Using this relation we have

Teeorem 3.

$$(3.8)\quad \sum_{n_i \neq n_j} \mu^{(a_1, a_2 \cdots a_k)}_{n_1 n_2 \cdots n_k : n} = n^{[k]} \mu^{(a_1)} \mu^{(a_2)} \cdots \mu^{(a_k)}$$

$$(3.9)\quad \sum_{n_1=1}^{n-k+1} \sum_{n_2=n_1+1}^{n-k+2} \cdots \sum_{n_k=n_{k-1}+1}^{n} \mu^{(a)}_{n_1 n_2 \cdots n_k : n} = \binom{n}{k} \{\mu^{(a)}\}^k$$

Corollary.

$$(3.10)\quad \sum_{r=1}^{n} \mu^{(a)}_{r:n} = n\mu^{(a)}$$

$$(3.11)\quad \sum_{r=1}^{n} \sum_{s=1}^{n} \delta_{rs:n} = n\delta^2$$

We consider contraction for sample size.

Theorem 4.

$$(3.12)\quad \sum_{n_1=1}^{n-k+1} \sum_{n_2=n_1+1}^{n-k+2} \cdots \sum_{n_k=n_{k-1}+1}^{n} \mu^{(a_1, a_2, \cdots a_k)}_{n_1 n_2 \cdots n_k : n} = \binom{n}{k} \mu^{(a_1, a_2, \cdots a_k)}_{1,2,\cdots k:k}$$

Proof. Since

$$C^{n}_{n_1 n_2 \cdots n_k} = n^{[k]} \prod_{i=1}^{k} \binom{n_{i+1}-i-1}{n-i},$$

$$C^{k}_{1,2,\cdots,k} = k!$$

where $n_{k+1} = n+1$, and in (2,3)

$$\sum_{n_i=i}^{n_{i+1}-1} \binom{n_{i+1}-i-1}{n_i - i} v_i^{\,n_i - i} (v_{i+1} - v_i)^{n_{i+1}-n_i-1}$$

$$= v_{i+1}^{\,n_{i+1}-i-1} \quad (i=1,2,\cdots,k)$$

where $v_{k+1}=1$, Theorem 4 follows.

We have the following recurrecne relations between the moments of order statistics.

Theorem 5.

$$(3.13)\quad \sum_{i=0}^{k} (n_{i+1} - n_i) \mu^{(a_1, \cdots a_i, a_{i+1} \cdots a_k)}_{n_1' \cdots n_i' n_{i+1} \cdots n_k : n} = n \mu^{(a_1, a_2 \cdots a_k)}_{n_1' n_2' \cdots n_k' : n-1}$$

where $n_0=1$, $n_{k+1}=n+1$ and $n_i'=n_i-1$

$(i=0,1,2,\cdots k)$

Proof. Since

$$n C^{n-1}_{n_1' n_2' \cdots n_k'} = (n_{i+1} - n_i) C^{n}_{n_1' \cdots n_i' . n_{i+1} \cdots n_k}$$

and in(2,9)

$$\sum_{i=0}^{k} (v_{i+1} - v_i) = 1$$

where $V_o=0$ and $V_{k+1}=1$, Theorem 5 follows.

Applying Theorem 5 we have

Corolary.

$$(3.14)\quad (n-r)\mu^{(a)}_{r:n} + r\mu^{(a)}_{r+1:n} = n\mu^{(a)}_{r:n-1}$$

$$(3.15)\quad \mu^{(a)}_{r:n} = \sum_{i=n-r+1}^{n} \binom{i-1}{n-r} \binom{n}{i} (-1)^{r+i-n-1} u^{(a)}_{1:i}$$

(3.16) $(n-r)^{[n]}\mu^{(a)}_{r:n} = \sum_{i=0}^{m}(-r)^{[i]}n^{[m-i]}$

$\cdot\binom{m}{i}\mu^{(a)}_{r+1:n-m+i}$

(3.17) $\mu^{(a)}_{r:n} = \sum_{i=r}^{n}\binom{i-1}{r-1}\binom{n}{i}(-1)^{i-r}\mu^{(a)}_{i:i}$

(3.18) $\binom{n}{m}\mu_{r:m} = \sum_{i=0}^{n-m}\binom{n-r-i}{m-r}\binom{r+i-1}{i}\mu_{r+i:n}$

(3.19) $\sum_{i=1}^{n}\frac{1}{i}\mu^{(a)}_{i:n} = \sum_{i=1}^{n}\frac{1}{i}\mu^{(a)}_{1:i}$

(3.20) $\sum_{i=1}^{n}\frac{1}{n-i+1}\mu^{(a)}_{i:n} = \sum_{i=1}^{n}\frac{1}{i}\mu^{(a)}_{i:i}$

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國 文 抄 錄

이산 순서 확률 변수들의 결합 확률 함수는 디리클레 적분으로 표시할 수 있다.
본 논문에서는 이를 이용하여 이산 순서통계량들의 적률에 관한 몇가지 관계식을 규명하였다.