Restricted Semi-Local Rings

Baik Seung-il

制限된 半局所環

白 承 一

요 약

半單純環의 성질을 조사하고 그 환에서의 멱등원이 原始元이 될 조건은 正則환에서 原始性과 동치임을 보임. 그리고 Artin환의 성질을 이용하여 右-Artin환이 半單純環이 되기 위해서는 환 R의 각 加群이 Proper large部分 加群을 갖지 않는 것이라는 것을 보이고 그 환이 제한된 半局所環(RSL-ring)임을 보임. 제한된 半局所環이 멱零元반단순일 때 Artin환이 됨을 이용하여 환 R이 제한된 半準素環(RSP-ring)이 됨을 조사하고, 새로운 환 즉 제한된 半單純環(RSS-ring)이 제한된 반국소환이고, 제한된 반준소환이며 환 R이 멱영원 반단순일때 그 역이 성립함을 보이므로 제한된 반국소환과 제한된 반준소환 그리고 제한된 반단순환 사이의 관계를 조사함.

I. Introduction and Preliminaries

A ring R which has no nonzero nilpotent ideals is called Nil-semisimple(introduced by D. M. Burton). If, in addition, R is right Artin, R is Semi-simple. Thus the radical of a semi-simple ring is zero and in fact this is sometimes taken as the definition of semisim plicity. A ring R is Local if all the noninvertible elements form a proper ideal. A local ring thus has precisely one maximal ideal, which also is the unique maximal right ideal. Note: The ring R is local if and only if R/J(R) is a skew-field [2].

More generally, R is said to be a Semi-local ring if R/J(R) is a semi-simple ring. Note that a semi-local ring has only finitely many maximal right ideals. A commutative ring is semi-local if and only if the number of maximal ideals is finite [2]. A ring R is Semi-primary if the Jacobson radical J(R) is nilpotent and R/J(R) is Artin. If R has the property that R/I is semi-primary for each ideal $I \neq O$ of R, we call R a Restricted semi-primary ring (introduced by Kenneth E. Hummel) or *RSP* ring for brevity.

The concept of a commutative ring all of whose factors are Artin (*RM*-rings) was introduced by *I. S. Cohn* in [5], and later noncommutative *RM*-rings were considered in [1] by A. J. Ornstein.

If R has the property that it is a semi-local and idempotents can be lifted modulo J(R), we call R a Restricted semi-local(or semi-perfect) or RSL-ring for brevity.

II. Theorems and Lemmas

An element e of a ring R is Idempotent if $e^2=e$.

LEMMA-1. If R is a right Artin ring and I is a non-nilpotent minimal right ideal in R, then I has a nonzero idempotent.

PROOF. Let a be a non-nilpotent element of I. Then $aR \subset I$ and is non-nilpotent since $a^2 \in aR$: thus $aR=I$ by minimality. Similarly $a^2R=I$. Thus there is an $a_1 \in aR$ such that $a=aa_1$. Then $aa_1^2=aa_1=a$ so $a(a_1-a_1^2) \in \{a\}_r \cap aR$, where $\{a\}_r$ is the set of right annihilators of a.

Now we let $a_2=a+a_1-a_1a$ so that $aa_2=a^2+aa_1-aa_1a=a^2+a-a^2=a$. Also

$$a_2(a_1-a_1^2)=aa_1+a_1^2-a_1aa_1-aa_1^2-a_1^3+a_1aa_1^2$$
$$=a+a_1^2-a_1a-a-a_1^3+a_1a=a_1^2-a_1^3.$$

Since $aa_2=a$, a_2 is not nilpotent. Hence $a_2R=aR=I$ and $\{a_2\}_r \cap aR \subset \{a\}_r \cap aR$. Either $a_1^2=a_1^3$ or $a_1^2\neq a_1^3$. If $a_1^2=a_1^3$, then $(a_1^2)^2=a_1^3a_1=a_1^2a_1=a_1^3=a_1^2$ so a_1^2 is idempotent and we are finished. On the other hand, if $a_1^2\neq a_1^3$, then $a_2(a_1-a_2^2)\neq 0$ and $a_1-a_1^2 \notin \{a_2\}_r \cap aR$. Therefore, $\{a_2\}_r \cap aR \subset \{a\}_r \subset aR$.

We can now repeat the process with a_2 playing the role of a. We obtain elements $a_3, a_4 \in I$ such that either $a_3^2=a_3^3$ or $a_3^2\neq a_3^3$ and $\{a_4\}_r \cap aR \subset \{a_2\}_r \cap aR$. If $a_3^2=a_3^3$, a_3^2 is our desired idempotent. If $a_3^2\neq a_3^3$, then the containment is strict. Hence if an idempotent is not obtained after a finite number of steps, we have an infinite descending chain of right ideals, contradicting the fact that R is right Artin.

If there are not nonzero nilpotent ideals in an Artin ring, we can obtain the following result;

THEOREM-1. Any nonzero right ideal in a semi-simple ring has a unique idempotent generator.

PROOF. Let I be nonzero right ideal of R. Then I is non-nilpotent and I has a nonzero idempotent element. Using the minimal condition, we choose a nonzero idempotent $e \in I$ such that $\{e\}_r \cap I$ is as small as possible. Suppose $\{e\}_r \cap I \neq (0)$. Then $\{e\}_r \cap I$ is non-nilpotent and hence cotains a nonzero idempotent e_1. Let $e_2=e+e_1-e_1e$. We note that $e_2\neq 0$. Then $e_2 \in I$ and since $ee_1=0$, we have

$$e_2^2=e^2+ee_1-ee_1e+e_1e-e_1^2-e_1^2e-e_1e^2-e_1ee_1+e_1ee_1e=e+e_1e_1e$$
$$=e_2$$

Moreover, $\{e_2\}_r \cap I \subset \{e\}_r \cap I$, since $ee_2=e+ee_1-ee_1e=e$, and so if $e_2x=0$, we have $ex=ee_2x=0$. But $ee_1=0$, so that $e_1 \in \{e\}_r \cap I$, and $e_2e_1=ee_1+e_1-e_1ee_1=e_1\neq 0$ and hence $e_1 \notin \{e_2\}_r \cap I$. Thus $\{e_2\}_r \cap I \subset \{e\}_r \cap I$, which is a contradiction. Hence we have $\{e\}_r \cap I=(0)$. Now we let $x \in I$. Then $e(x-ex)=ex-e^2x=ex-ex=0$, so $x-ex \in \{e\}_r \cap I=(0)$ and therefore $ex=x$. Thus $I=eR$ and $e^2=e$. Here, clearly $I_1=\{e\}_l$ and $(I_1\cap I)^2 \subset II_1=(0)$. Hence $I_1 \cap I=(0)$ since R is semi-simple and $I_1 \cap I$ is a left ideal in R. For each $x \in I$, $(x-xe)e=0$ so $x-xe \in \{e\}_l \cap I=I_1 \cap I=(0)$. Thus $x=xe$ for all $x \in I$. Also, for any $x \in I$, $x \in eR$, that is, $x=er$ for some $r \in R$, so that $ex=e^2r=er=x$. Hence e is a two-sided identity in the ring I and as such as is unique.

By the above Lemma and Theorem we obtain the followings.

COROLLARY-1. Any semi-simple ring R is a right Noetherian.

COROLLARY-2. A semi-simple ring R has an identity.

COROLLARY-3. A commutative semi-simple ring R is a principal ideal ring.

LEMMA-2. If A is an ideal in a ring R, then $J(R)=J(R) \cap A$.

PROOF. Since every element of $A \cap J(R)$ is left quasi-regular, we have $A \subset J(R) \subset J(R)$. Suppose that $J(R)=(0)$. Let $P=\{x \in R \mid Ax=(0)\}$. P is clearly a right ideal of R. $AJ(A)$ is a left ideal of R and $AJ(A) \subset J(A)$ and so $AJ(A)$ is left quasi-regular. Thus $AJ(A) \subset J(R)=(0)$. Then $J(A) \subset P \cap A$. But if $x \in P \cap A$ and $x^2=0$, then $x \in J(A)$ since every nil left ideal of R is contained in J(R). Hence $J(A)=P \cap A$. Therefore, J(A) is a right ideal of R. But every element of J(A) is right quasi-regular as an element of A and hence as an element of R; therefore J(A) is a right quasi-regular right ideal of R. Thus $J(A) \subset J(R)=(0)$.

Now we consider the general case. $(A+J(R))$

$/J(R)$ is an ideal in the semi-simple ring $R/J(R)$. Therefore, $J((A+J(R))/J(R))=(0)$ and so $J(A/(A \cap J(R)))=(0)$. Hence $J(A) \subset A \cap J(R)$.

THEOREM-2. If A is an ideal in a semi-simple ring R, then A is also semi-simple.

Let $e_1, e_2, \cdots, e_n$ be nonzero idempotents in a ring R. They are mutually orthogonal if $e_ie_j=0$ whenever $i \neq j$. In this case $e=e_1+e_2+\cdots+e_n$ is also an idempotent. An idempotent is Primitive if it cannot be written as the sum of two orthogonal idempotents. It is well known that:

Remark-1. Let R be a semi-simple. Then an idempotent $0 \neq e \in R$ is primitive if and only if eR is a minimal right ideal of R.

Remark-2. In a semi-simple ring R, an idempotent $e \neq 0$ is primitive if and only if eRe forms a division ring.

A ring R is called Regular if for every $a \in R$ there is some $x \in R$ such that $axa=a$. Now, we have the following theorem.

THEOREM-3. Let R be a regular ring. Then an idempotent $0 \neq e$ in R is primitive if and only if eRe is a division ring.

PROOF. Suppose e is primitive in R and a is nonzero element in eRe. Then Re is minimal and $a \in Re$ and so $Ra \subset Re$. Hence $Ra=Re$ or $Ra=(0)$. But $a=ea \in Ra$, so that $Ra \neq (0)$. Therefore $Ra=Re$. Thus $e \in Ra$, ie, there is an $x \in R$ such that $e=xa$. Then exe is a left inverse in eRe for a, since $exea=ex(ea)=exa=ee=e$. Hence eRe is a division ring.

Coversely, if eRe is a division ring and I is a left ideal of R with $I \subset Re$. Then eI is a left ideal in eRe. Hence either $eI=(0)$ or $eI=eRe$. If $eI=(0)$, then $I^2 \subset ReI=(0)$ and $I=(0)$ since R is regular, R has no nonzero nilpotent ideal [6]. Now suppose that $eI=eRe$. Then there is an $x \in I$ such that $ex \in eRe$ and $ex \neq 0$. Also, $exe=ex$ since e is the identity for eRe. Moreover, ex has an inverse in eRe, say eye. Then $(eye)(exe)=e$ and $e \in Rexe=Rex \subset I$. Then $Re \subset I$ and $I=Re$, so that Re is a minimal left ideal of R. Hence e is a primitive [Remark-1].

By the preceding theorem and Remark-2, we obtain the following.

THEOREM-4. In a semi-simple ring R, an idempotent $e \neq 0$ is primitive if and only if an idempotent $e \neq 0$ is primitive in a regular ring R.

LEMMA-3. In a ring R having exactly one maximal ideal M, the only idempotent are 0 and 1.

PROOF. Suppose that there exists an idempotent $a \in R$ with $a \neq 0, 1$. Then $a^2=a$ implies $a(1-a)=0$ so that a and $1-a$ are both zero divisors. Hence, neither the element a nor $1-a$ is invertible in R since no divisor of zero can possess a multiplicative inverse in R. But this means that the principal ideals (a) and $(1-a)$ are both proper ideals of R. As such, they must be contained in M, the sole maximal of R. Hence a and $1-a$ lie in M, whence $1=a+(1-a) \in M$. This leads at once to the cotradiction that $M=R$.

Let R be a local ring, then R has precisely unique one maximal ideal [preliminary].

THEOREM-5. If R is a local ring, then the only idempotent in R are 0 and 1.

Let us first show that the chain codition are not destroyed by homemorphism.

LEMMA-4. If R is an Artin ring, then any homomorphic image of R is also Artin.

PROOF. Let Φ be a homomorphism of the Artin ring R onto the ring R^* and consider any decending chain $I_1^* \supset I_2^* \supset \cdots \supset I_n^* \supset \cdots$ of ideals of R^*. Put $I_k=\Phi^{-1}(I_k^*)$, for $k=1, 2, \cdots$. Then $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ forms an decending chain of ideals of R and there is some n such that $I_m=I_n$ for all $m \geq n$. Since f is an onto mapping, we have $\Phi(I_k)=I_k^*$. Hence, $I_m^*=I_n^*$ whenever $m \geq n$, so that the original chain also stabilizes at some point.

Letting Φ be the natural mapping, we have as a theorem;

THEOREM-6. If I is an ideal of the Artin ring R, then the quotient ring R/I is Artin.

COROLLARY-4. If I is an ideal of the Noetherian ring R, then the quotient ring R/I is also Noetherian.

LEMMA-5. For any ring R, radical of the quotient ring R/J(R) is zero.

PROOF. Let J=J(R). Suppose that the coset $a+J \in J(R/J)$. Hence, $(1+J)-(r+J)(a+J)=1-ra+J$ is invertible in R/J for each choice of $r \in R$. Accordingly, there exists a coset $b+J$ such that $(1-ra+J)(b+J)=1+J$. This is plainly equivalent to requiring $1-(b-rab) \in J$. And, we conclude that the element $b-rab=1-1(1-b+rab)$ has an inverse c in R. But then $(1-ra)(bc)=(b-rab)c=1$, so that $1-ra$ possesses a multiplicative inverse in R. As this argument holds for every $r \in R$, it follows that $a \in J(R)=J$. Hence $a+J=J$.

LEMMA-6. If R is an Artin ring, then J(R) forms a nilpotent ideal.

POOF. Let J=J(R). Then $J \supset J^2 \supset \cdots$, and so there is a positive integer n such that $J^n=J^{n+1}=\cdots=J^{2n}=J^{2n+1}=\cdots$. Assume $J^n \neq (0)$. Then J^n is contained in the family

$F=\{L|L$ is a left ideal in R, $L \subset J^n, J^nL \neq (0)\}$.

Hence $F \neq \phi$. Let Lm be minimal in F. There exists $1 \in Lm$ such that $J^n1 \neq (0)$. Since $J^n=J^{2n}$, J^n1 is a member of F. Furthermore $J^n1 \subset Lm$ and therefore $J^n1=Lm$. Hence there exists $x \in J^n \subset J$ such that $x1=1$. Since x is quasiregular, there exists $y \in R$ such that $0=x+y-yx$. This implies that $0=11x-y(1-1x)=1-(x+y-yx)1=1$ a contradiction since $J^n1 \neq (0)$. Therefore $J^n=(0)$.

COROLLARY-5. If R is a right Artin ring, then J(R) is the unique largest nilpotent ideal in R.

PROOF. Any nilpotent ideal is contained in the prime radical and this is contained in radical of R.

Here, by Theorem-6, Lemma-5 and Lemma-6

THEOREM-7. If R is a right Artin ring, then R is a semi-local ring.

THEOREM-8. If R is a right Artin ring, then R is a semi-primary.

A module M is Completely reducible if every submodule of M is a direct summand of M.

LEMMA-7. If an R-module M is the sum of irreducible submodules, then M is completely reducible.

PROOF. Let N be a submodule of M and N^* a submodule of M maximal with respect to the property that $N \cap N^*=(0)$. We must show that $M=N+N^*$. Suppose not. Then there exists m in M such that $m \notin N+N^*$. We have $m=m_1+\cdots+m_s$, $m_i \in M_i$, an irreducible submodule, $i=1, \cdots, s$. Some $m_j \notin N^*+N$ and there exists an irreducible submodule M_j such that $M_j \not\subset N+N^*$. Because M_j is irreducible, $M_j \cap (N+N^*)=(0)$. But then $N^* \subset N^*+M_j$ and $(N^*+M_j) \cap N=(0)$, contradicting the maximality of N^*. Thus $N+N^*=M$,

THEOREM-9. A right Artin ring R with identity is semi-simle if and only if every right R-module has no proper large submodule.

PROOF. If R is semisimple, we have $R=e_1R\oplus\cdots\oplus e_nR$, where the e_iR are minimal right ideals of R. If M is an R module, we can write $M=\sum_{m \in M}\sum_{i=1}^{n} me_iR$. Each me_iR is clearly a submodule, but the sum is not necessarily direct. Each e_iR is an irreducible R-module, so that each me_iR is either irreducible or else $me_iR=(0)$. Thus by the Lemma-7., M being the sum if irreducible submodules, is completely reducible. Then a large submodule of M has nonzero intersection with every nonzero submodule of M, hence contains every irreducible submodule of M, hence contains SocM=M. so that M has no proper large submodule. Conversely, if every R-module M has no proper large submodule, and let B be any submodule of M. Then we have $C \subset M$ such that $B \cap C = \{0\}$ and B+C is large. Thus, by condition, B+C=M, Hence, M is completely reducible. Then R is completely reducible. Let J(R) be its radical. Then $R = J(R) \oplus N$, N some right ideal of R. Then $1 = x + x^*$, $x \in J(R)$, $x^* \in N$. Then $x - x^2 = x^*x \in J(R) \cap N$. Hence $x - x^2 = 0$ and $x = x^2 = \cdots = 0$ since $x \in J(R)$ and hence is nilpotent. Thus $x^* = 1$ and N=R. Therefore J(R)=0, ie. R is semi-simple.

According to the fact that if every right R-module M is completely reducible, then R is also completely reducible. We have the following equivalent statement.

THEOREM-10. 1) R is semi-simple
2) R is completely reducible
3) R is right Artin and regular
4) R is right Noetherian and regular

Let N be a two-sided ideal of an arbitrary ring R. We say that idempetents can be lifted module N if for every idempotent $f \in R/N$ there exists an idempotent $e \in R$ such that $\bar{e} = f$. This means that the idempotents of R/I can be lifted if for each element $u \in R$ such that $u^2 - u \in I$ there exists some element $e^2 = e \in R$ with $e - u \in I$.

LEMMA-8. If N is a nil ideal of an arbitrary ring R, then idempotents can be lifted module N.

PROOF. Suppose f is an idempotent of R/N. Choose $u \in R$ such that $\bar{u} = f$. Then $u^2 - u \in N$, and hence $(u^2 - u)^r = 0$ for some r. Hence, we obtain $0 = u^r(1-u)^r = u^r - u^{r+1}\, g(u)$ …(1), where $g = g(u)$ is a polynomial in u. Now put $e = u^r g^r$. By the use of (1) we get $e^2 = u^{2r} g^{2r} = u^{r-1}\, u^{r+1} g$. $g^{2r-1} = u^{r-1} u^r 2^{2r-1} = u^{2r-1} g^{2r-1} = \cdots = u^r g^r = e$. We also have $\bar{e} = f$, because (1) gives $f = f\bar{g} = f\bar{g}^r$.

THEOREM-11. If R is an Artin ring, then idempotents can be lifted module J(R).

PROOF. Since R is an Artin ring, J(R) is nilpotent. So that J(R) is a nil ideal in R. Hence, by Lemma-8, idempotents can be lifted modulo J(R).

THEOREM-12. Every right Artin ring is Restricted semi-local ring.

PROOF. Let R be a right Artin, then so is R/J(R). And, by Lemma-5, R/J(R) is semi-simple. Moreover idempotents can be lifted modulo J(R), by Theorem-11, since the radical is nil. Hence R is RSL-ring.

COROLLARY-6. Any semi-simple ring is a *RSL*-ring.

COROLLARY-7. Any semi-primary ring is a *RSL*-ring.

A projective cover of M is a minimal epimorphism of a projective module onto M. We call a ring R right RSL if every cyclic right R-module has a projective cover. This definition of RSL is equivalent two the definition in the introduction.

LEMMA-9. Let I be a two-sided ideal in R. Then if $P \longrightarrow A \longrightarrow (0)$ is a R-projective cover of a R/I-module A, the induced map $P/IP \longrightarrow A \longrightarrow (0)$ is a R/I-projective cover of A.

PROOF. Let $K = \ker(P \longrightarrow A)$. Since $IA = (0)$, $IP \subset K$ and the second map is well defined. Moreover, P/IP is R/I-projective. If $S/IP + K/IP = P/IP$ then $S + K = P$, so $S = P$ and therefore $S/IP = P/IP$; ie. $P/IP \longrightarrow A$ is minimal.

From this Lemma, we have the following theorem.

THEOREM-13. If R is a *RSL*-ring and I is an ideal in R, then R/I is also *RSL*-ring.

LEMMA-10. Suppose $(0) \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow (0)$ is exact with P projective and $P(A) \longrightarrow A$

⟶(0) is a projective cover. Then we can write P=P(A) ⊕ P* with P* ⊂ K and K ∩ P(A) superfluous in P(A).

PROOF. Since P is projective, there exists a map P⟶P(A) making P⟶A⟶(0) commutative.
P(A)

Since im(P ⟶ P(A))+(P(A) ⟶A)= P(A), im(P⟶P(A))=P(A), so P⟶P(A) is an epimorphism and therefore splits. Thus, identifying P(A) with a direct summand of P, we may write P=P(A)⊕ P*, where P*=ker(P⟶P(A)) ⊂ ker (P⟶A)=K. Moreover, P⟶A induces the given minimal epimorphism P(A) ⟶A on P(A), and the induced kernel is K∩ P(A).

From this the last statement follows.

LEMMA-11. If I is a right ideal of R, then R⟶R/I⟶(0) is minimal if and only if I ⊂ J(R). Moreover, if R is right *RSL*-ring, either I ⊂ J(R) or I contains a nonzero direct summand of R.

PROOF. I is superfluous in R if and only if I is comaximal with no proper right ideal, ie. if and only if I is contained in every maximal right ideal. Suppose now that R is right RSL-ring, so that R/I has projective cover. Then, by Lemma-10, we can write R=P(R/I) ⊕ P* with P* ⊂ I and I ∩ P(R/I) superfluous in P(R/I). If P* ≠ (0), we are finished. Otherwise P(R/I)=R, so I ⊂ J(R) by the first part of this Lemma.

THEOREM-14. If R is nil-semisimple and right *RSL*-ring, then R is an Artin.

PROOF. We shall prove this by showing that R equalsits right sccle S. If not, S ⊂ M for some maximal right ideal M. Applying Lemma-10 to the exact sequence (0)⟶M⟶R⟶R/M⟶(0) we have R=P⊕Q with Q ⊂ M and M ∩ P superfluous in P. The latter condition guarantees that M ∩ P can contain no direct summand of P, so also of R. Hence, by the nil-semisimplicity and by Lemma-11, M ∩ P=(0). But then P ≅ R/M so P ⊂ S:Contradiction.

THEOREM-15. If R is a nil-semisimple, and *RSL*-ring, then R is *RSP*-ring.

PROOF. By Theorem-14, R is an Artin. Hence R/I is an Artin for every ideal I ≠0 of R. Then R/I is a semi-primary ring(by Theorem-8). Thus R is a *RSP*-ring.

Now, we call a ring R is a Restricted semi-simple(or*RSS*-ring for brevity) if R/I is a semi-simple for ideal I ≠0 of R. Then we have the following Theorems.

THEOREM-16. If R is a *RSS*-ring, then R is *R*SL-ring.

THEOREM-17. If R is a *RSS*-ring, then R is a *RS*P-ring.

PBOOF. Since R is a *RSS*-ring, R/I is an Artin. Then J(R/I) is nilpotent and R/I/J(R/I) is an Artin. Hence R is a *RSP*-ring.

THEOREM-18. If R is a nil-semisimple and *RSL*(or RSP)-ring, then R is a *RSS*-ring.

PROOF. By Theorem-14, R is an Artin. Thus R/I also Artin for ideal I≠0 of R. Hence R/I is a semi-simple.

References

Avraham J. Ornstein, 1968. Rings with Restricted Minimum condition, Proc. Amer. Math. Soc. V-19p 1145~1150. [1]

B. Stenstrom, 1975, Rings of Quotients. Springer. [2]

Carl Faith, 1966, On Köthe Rings, Math Annalen V-164 p. 207~212. [3]

Frank W. Anderson. 1974. Rings and Categories of Modules, Springer. [4]

I. S. Cohen, 1950, Commutative Rings with Restricted Minimum condition, Duke Math. J. V-17. p. 27~42. [5]

Mary Gray, 1970. A Radical Approach To Algebra, Addison Wesley. [6]

Robert Gilmer, 1972. Multiplicative Ideal Theory, Dekker [7]