On the Numerical Range of a Linear Operator

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線型 作用素의 數域에 관하여

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I. Introduction

For an operator on a Hilbert space the concept of numerical range was introduced, for finite dimensional spaces, by Toeplitz in 1918. No concept of numerical range appropriate to Banach spaces appeared until 1961 and 1962, when distinct, though related, concepts were introduced independently by Lumer and Bauer. In terms of Hahn–Banach theorem, Bauer defined the concept of numerical range on a finite dimensional Banach space, but the concept that he introduced is available without dimension.

Paschke (1973) defined a pre-Hilbert C^*-module, a right module over a C^*-algebra B which possess a B-valued inner product respecting module action. Under this inner product, Yang (1984) defined a spatial numerical range of an operator on a Hilbert C^*-module, and obtained its spectral and topological properties.

Berberian (1974) proved that for any element a of a unital C^*-algebra A, there exists a normalized state f on A such that $f(a^*a)=\|a^*a\|$. In terms of this result and C^*-valued inner product, we define the new concept of numerical range of an operator on a Hilbert C^*-module, and obtain its spectral and topological properties, and application of our concept to a unital C^*-algebra. In particular, we give that our numerical range is connected, but not closed.

Throughout this paper, we let B be a unital C^*-algebra, B' its dual space, X the Hilbert B-module with B-valued inner product $\langle , \rangle$, S(X) the unit sphere of X i.e., the set of all $x \in X$ such that $\|x\|_X = \|\langle x, x\rangle\| = 1$, and P the set of all positive linear functionals (normalized states) on B i.e., $P=\{f \in B' : f(e)=1=\|f\|\}$, where e denotes the unity of B. We also denote the action of B on a right B-module X by $(x,b) \to xb$ ($x \in X$, $b \in B$). A Hilbert B-module X

* This research is partially supported by the KOSEF research grant, 87～88.
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is assumed to have a vector space structure over the complex numbers **C** compatible with that of B in the sense that $\lambda(xb)=(\lambda x)b=x(\lambda b)$ ($x \in X$, $b \in B$, $\lambda \in \mathbf{C}$). We let B(X) the set of all bounded linear operators on X, and A(X) the set of all bounded linear operators on X which possess bounded adjoints with respect to the B-valued inner product. Without risk of confusion, we denote the operator norm on B(X) by $\|\ \|$.

II. Numerical ranges and spectra

From now on we let π be the natural projection of $X\times B'$ onto X, and π_p the subset of $X\times B'$ defined by

$\pi_p=\{(x,f) \in S(X)\times P : f(\langle x, x\rangle)=1\}$.

By Berberian (1974), $\pi_p\neq\emptyset$ if $X\neq\emptyset$. for each f in B′ we define f^* by $f^*(b)=\overline{f(b^*)}$ ($b \in B$). It is obvious that for each $(x,f) \in \pi_p$, $(x,f^*) \in \pi_p$, $(x,f_1) \in \pi_p$, where $f_1=(f+f^*)/2$ is self-adjoint, and

$$\pi_p\subseteq\sigma(X)=\{(x,f) \in S(X)\times S(B') : f(\langle x,x\rangle)=1\}.$$

DEFINITION 2.1. *The numerical range* W(T) *of an operator* T *in* B(X) *is defined by*

$$W(T)=\{f(\langle Tx,x\rangle) : (x,f) \in \pi_p\},$$

and the numerical radius $\omega(T)$ *of* T *is the number*

$$\omega(T)=\sup\{|\lambda| : \lambda \in W(T)\}.$$

This generalizes the classical concept of numerical range on a Hilbert space, since in case $B=\mathbf{C}$ a Hilbert B-module is a Hilbert space. It is obvious that for any T, S in B(X) and $\alpha, \beta \in \mathbf{C}$, $W(\alpha T+\beta S)\subseteq\alpha W(T)+\beta W(S)$, W(T) includes the point spectrum of T, $W(T^*)=\overline{W(T)}$ for each $T \in A(X)$, and for each T in B(X), $W(T)\subseteq W_B(T)\subseteq V(T)\subseteq V(B(X), T)$ where $W_B(T)$, V(T), and V(B(X), T) denote the B-spatial numerical range, the spatial numerical range, and the algebea numerical range of T respectively (see Bonsall and Duncan (1971), Yang(1984)). From these facts it is obvious that $\omega(\cdot)$ is a seminorm on B(X), and $\omega(T)\leqq\|T\|$ for each T in B(X).

LEMMA 2.2 *Let* Γ *be a subset of* π_P *such that its natural projection* $\pi(\Gamma)$ *is norm dense in* S(X). *Then for each* $T \in B(X)$.

(a) $\inf\{(\|I+\alpha T\|-1)/\alpha : \alpha>0\}$
$= \sup\{\mathrm{Re}\ f(\langle Tx,x\rangle) : (x,f)\in\Gamma\}$,

(b) $\sup\{\frac{1}{\alpha}\log\|\exp(\alpha T)\| : \alpha>0\}$
$= \sup\{\mathrm{Re}\ f(\langle Tx,x\rangle):(x,f)\in\Gamma\}$.

PROOF. (a) Let $\mu=\sup\{\mathrm{Re}\ f(\langle Tx,x\rangle) : (x,f)\in\Gamma\}$. By Bonsall and Duncan(1971), we have $\mu\leqq\max\{\mathrm{Re}\ \lambda : \lambda \in V(B(X), T)\} =$

$$\inf\{\frac{1}{\alpha}(\|I+\alpha T\|-1) : \alpha>0\} = \lim_{\alpha\to 0+}\frac{1}{\alpha}(\|I+\alpha T\|-1). \quad (*)$$

It is obvious when $T=0$, so we assume that $T\neq0$. Choose α such that $0<\alpha<\|T\|^{-1}$. Let $x\in S(X)$ and $\varepsilon>0$. Since $\pi(\Gamma)$ is dense in S(X), there exists $(y,g)\in\Gamma$ such that $\|x-y\|_x<\varepsilon$. We have $\mathrm{Re}\ g(\langle Ty, y\rangle)\leqq\mu\leqq\|T\|$ and so $\|(I-\alpha T)y\|_X\geqq\mathrm{Re}\ g(\langle(I-\alpha T)y,y\rangle)=1-\alpha\,\mathrm{Re}\ g(\langle Ty,y\rangle)\geqq1-\alpha\mu>0$. Therefore $\|(I-\alpha T)x\|_X\geqq1-\alpha\mu-\|I-\alpha T\|\varepsilon$. Since ε is arbitrary, this gives $\|(I-\alpha T)x\|_x\geqq1-\alpha\mu$, and therefore $\|(I-\alpha T)x\|_X\geqq(1-\alpha\mu)\|x\|_X$ ($x\in X$). If we replace x by $(I+\alpha T)x$, this gives

$$\|(I+\alpha T)x\|_X\leqq\frac{1}{1-\alpha\mu}\|(I-\alpha^2T^2)x\|_X$$

($x\in X$), and so

$$\|I+\alpha T\|\leqq(1+\alpha^2\|T^2\|)/(1-\alpha\mu).$$

Therefore

$(\|I+\alpha T\|-1)/\alpha\leqq(\mu+\alpha\|T^2\|)/(1-\alpha\mu)$ and this with (*) completes the proof.

(b) Let $\mu = \sup\{\text{Re } f(\langle Tx, x\rangle) : (x,f) \in \Gamma\}$. Then by Theorem 3.4 (Bonsall and Duncan, 1971), $\mu \leqq \sup\{\text{Re }\lambda : \lambda \in (B(X),T)\} =$

$$\sup\{\frac{1}{\alpha}\log\|\exp(\alpha T)\| : \alpha > 0\} = \lim_{\alpha\to 0+}\frac{1}{\alpha}\log\|\exp(\alpha T)\|.$$

The case $T=0$ is obvious, so assume that $T \neq 0$. Let $0 < \alpha < \|T\|^{-1}$, $\varepsilon > 0$ and $x \in S(X)$. Since $\pi(\Gamma)$ is dense in $S(X)$, there exists $(y,g) \in \Gamma$ such that $\|x-y\|_X < \varepsilon$. We have $\text{Re } g(\langle Ty, y\rangle) \leqq \mu \leqq \|T\|$ and so $\|(I-\alpha T)y\|_X \geqq \text{Re } g(\langle I-\alpha T)y, y\rangle) = 1-\alpha \text{ Re } g(\langle Ty,y\rangle) \geqq 1-\alpha\mu \geqq 1-\alpha\|T\| > 0$. Therefore $\|(I-\alpha T)x\|_X \geqq 1-\alpha\mu - \|I-\alpha T\|\varepsilon$. Since ε is arbitrary this gives $\|(I-\alpha T)x\|_X \geqq 1-\alpha\mu$ and therefore $\|(I-\alpha T)x\|_X \geqq (1-\alpha\mu)\|x\|_X$ $(x \in X)$. By induction we have (*) $\|(I-\alpha T)^n x\|_X \geqq (1-\alpha\mu)^n\|x\|_X (x \in X, n=1,2,\cdots\cdots)$. We have $1-\alpha\mu/n > 0$ for all sufficiently large n. Therefore replacing α by α/n in (*) and letting $n \to \infty$, we obtain $\|\exp(-\alpha T)x\|_X \geqq \exp(-\alpha\mu)\|x\|_X$. Taking $x=\exp(\alpha T)x$, we get $\|\exp(\alpha T)\| \leqq \exp(\alpha\mu)$ and so $\sup\{\frac{1}{a}\log\|\exp(\alpha T)\| : \alpha > 0\} \lneqq \mu$. Hence our conclusion holds.

THEOREM 2.3. *Let Γ be a subset of π_P such that its natural projection $\pi(\Gamma)$ is norm dense in* $S(X)$. *Then for each* $T \in B(X)$, $\overline{co}\{f(\langle Tx,x\rangle) : (x,f) \in \Gamma\} = V(B(X),T)$ *where* $\overline{co}$ E *denotes the closed convex hull of a set* E.

PROOF. By Lemma 2.2 and Theorem 2.5 (Bonsall and Duncan, 1971), we have $\sup\{\text{Re } f(\langle Tx,x\rangle) : (x,f) \in \Gamma\} = \sup\{\text{Re }\lambda : \lambda \in V(B(X),T)\}$. By replacing T by appropriate scalar mulitiple of T, using the fact that $V(B(X), T)$ is a closed convex set, and Lemma 4 (Niestegge, 1983), our proof is complete.

COROLLARY 2.4 (Bonsall and Duncan, 1971). *Let Γ be a a subset of* $\{(x,x^*) \in S(X) \times S(X') : x^*(x)=1\}$ *such that its natural projection $\pi(\Gamma)$ is norm dense in* $S(X)$. *Then for each* $T \in B(X)$, $\overline{co}\{x^*(Tx) : (x,x^*) \in \Gamma\} = V(B(X),T)$.

COROLLARY 22.5. *For each* $T \in B(X)$, *we have*

(a) $\overline{co}\ W(T) = V(B(X),T) = \overline{co}\ V(T)$,

(b) $\omega(T) = \sup\{|\lambda| : \lambda \in V(B(X),T)\} = \nu(T)$.

COROLLARY 2.6. *If* X *is a Hilbert space, then* $V(B(X), T) = W(T)$ *is the closure of the usual numerical range of* T.

We denote the spectrum, the approximate point spectrum, and the compression spectrum of an operator $T \in B(X)$ by $\sigma(T)$, $\sigma_a(T)$, and $\sigma_c(T)$ respectively. We note that for each $T \in B(X)$, $\sigma_a(T) \subseteq \overline{W(T)}$, and in particular, the boundary in the complex plane of the spectrum of T, $\partial\sigma(T)$ is contained in $\overline{W(T)}$.

By Bonsall and Duncan (1971), and Corollary 2.5, we have $\sigma(T) \subseteq V(B(X), T) = \overline{co}\ W(T)$. However we have the following stronger statement.

THEOREM 2.7. *For each* T *in* A(X), *we have* $\sigma(T) \subseteq \overline{W(T)}$.

PROOF. For each T in B(X) we have $\sigma(T) = \sigma_a(T) \cup \sigma_c(T)$ by Berberian (1974). If $\lambda \in \sigma_c(T)$, then the range of $T-\lambda I$ is not dense so since $R(\lambda I-T)^{\perp} = N(\bar{\lambda} I-T^*)$, the range of $T-\lambda I$ has a nonzero orthogonal complement. Hence $\bar{\lambda}$ is an eigenvalue of T^* so that $\bar{\lambda} \in W(T^*)$, and therefore $\lambda \in W(T)$. On the other hand, $\sigma_a(T) \subseteq \overline{W(T)}$. Therefore $\sigma(T) \subseteq \overline{W(T)}$.

COROLLARY 2.8 (Williams, 1967). *For each* T *in* A(X) *we have* $\sigma(T) \subseteq \overline{V(T)}$.

Now we present an extension of Theorem 2.7.

THEOREM 2.9. *Let* S, T *be operators in* A(X) *and* $0 \notin \overline{W(T)}$. *Then* $\sigma(T^{-1}S) \subseteq \overline{W(S)}/\overline{W(T)} = \{\lambda/\mu : \lambda \in \overline{W(S)},\ \mu \in \overline{W(T)}\}$.

PROOF. Let z be a complex number not belonging to the above set on the right. Then there exists $d > 0$ such that $|z\mu - \lambda| \geq d$ ($\lambda \in \overline{W(S)}$, $\mu \in \overline{W(T)}$). Given $(x,f) \in \pi_P$, we have $\|(zT-S)x\|_X \geq |f(\langle(zT-S)x,x\rangle)| \geq d$ since $f(\langle Tx,x\rangle) \in W(T)$ and $f(\langle Sx,x\rangle) \in W(S)$. Similarly, $\|(zT-S)^*x\|_X \geq d$. By Berberian (1974), we conclude that $zT-S$ is invertible. Since $0 \notin \overline{W(T)}$ and $\sigma(T) \subseteq \overline{W(T)}$, T is invertible. Therefore $zI-T^{-1}S$ is invertible i.e., $z \notin \sigma(T^{-1}S)$.

COROLLARY 2.10 (Williams, 1967). *Let* S, T *be operators in* A(X) *and* $0 \notin \overline{V(T)}$. *Then* $\sigma(T^{-1}S) \subseteq \overline{V(S)}/\overline{V(T)}$.

In order to apply the numerical range to unital C^*-algebra A, we consider the isometirc left regular representation $T_a : A \to B(A)$, where $T_a(b) = ab$ ($a,b \in A$).

THEOREM 2.11. *Let* A *be a* C^**-algebra with unity* e. *Then for all* $a \in A$, $W(T_a) = \{f(a) : f \in P\}$.

PROOF. Since any C^*-algebra A is a Hilbert A-module with $\langle a,b\rangle = b^*a$ ($a,b \in A$), and $W(T_a) = \{f(\langle T_a b,b\rangle) : (b,f) \in \pi_P\} = \{f(b^*ab) : (b,f) \in \pi_P\}$, for each $f \in P$, $(e,f) \in \pi_P$, and $f(a) \in W(T_a)$. Thus $\{f(a) : f \in P\} \subseteq W(T_a)$. On the other hand, for each $(b,f) \in \pi_P$, we define $f_b(c) = f(b^*cb)$ for all $c \in A$. Then f_b is a linear functional on A, $f_b(e) = 1$, and $\|f_b\| = 1$, so that $f_b \in P$. Hence $f(\langle T_a b,b\rangle) = f(b^*ab) = f_b(a) \in \{f(a) : f \in P\}$. Therefore $W(T_a) = \{f(a) : f \in P\}$.

From Stampfli and Williams (1968), we note that $\{f(a) : f \in P\}$ is a convex closed and contains the spectrum $\sigma(a)$ for each a in a unital C^*-algebra A.

Ⅲ. Topological properties

We turn now to topological properties of our concept of numerical range. We recall that *the norm×weak* topology in* $X \times B'$ is the product topology in $X \times B'$ given by the norm topology on X and the weak* topology on B^* (Yang, 1984).

The following two results are essentially due to Bonsall et al (1968).

LEMMA 3.1. *Let* E *be a subset of* π_P *such that is relatively closed in* π_P *with respect to the norm× weak* topology. Then* $\pi(E)$ *is a norm closed subset of* X.

PROOF. The proof is similar to that of Lemma 3.2 (Yang, 1984).

THEOREM 3.2. π_P *is a connected subset of* $X \times B'$ *with the norm×weak* topology, unless* X *has dimension one over* R.

PROOF. The proof is similar to that of Theorem 3.3 (Yang, 1984).

COROLLARY 3.3. W(T) *is connected.*

PROOF. We have $|f(\langle Tx,x\rangle) - g(\langle Ty,y\rangle)| \leq \|Tx-Ty\|_X + \|Ty\|_X \|x-y\|_X + |(f-g)(\langle Ty,y\rangle)|$ ($(x,f), (x,g) \in \pi_P$). Therefore the mapping $(x,f) \to f(\langle Tx,x\rangle)$ is a continuous mapping of π_P with the relative norm×weak* topology onto W(T). Therefore by Theorem 3.2, W(T) is connected, unless X has dimension one over R. In case X has dimension one over R, $W(T) = \{\lambda\}$ since the unit sphere consists of just two vectors $\pm u$ and $Tu = \lambda u$ for some real λ.

By Theorem 4.1 (Bonsall and Duncan, 1971) and Corollary 2.5, $\omega(T) \leq \|T\| \leq e\,\omega(T)$ ($e = \exp 1$) for T in B(X). Let B(X) be the set of all bounded linear operators, endowed with the uniform operator topology induced by the norm on X. Then $\omega(.)$ is a norm on B(X), inducing the same topology on B(X) as that induced by the original norm. *The numerical index of* X is the real number n(X) defined by

$n(X)=\inf\{\nu(T): T \in B(X), \|T\|=1\}$.

By Corollary 2.5, $n(X)=\inf\{\omega(T): T \in B(X), \|T\|=1\}$. It is obvious that $1/e \leqq n(X) \leqq 1$. It has long been known that for a complex Hilbert space X of dimension greater than one, $n(X)=\frac{1}{2}$ (Halmos, 1982).

Consider a pair of compact subsets of the complex plane, M and N, and write $M+(\epsilon)=\{z+\alpha : z \in M, |\alpha| < \epsilon\}$ for $\epsilon > 0$. Then we define $d(M,N)=\inf\{\epsilon : M \subseteq N+(\epsilon)$ and $N \subseteq M+(\epsilon)\}$. Thus we can consider $d(\overline{W(T)}, \overline{W(S)})$ as a metric, the "Hausdorff metric" on sets associated with T and S. Using this notation, we have the following theorem which is taken from Halmos (1982)

THEOREM 3.4. $\overline{W}(\cdot)$ *is a continous function from* B(X), *endowed with the uniform operator topology to the set of compact subsets of* **C**, *endowed with the Hausdorff metric topology. Also* ω *is a real-valued function on* B(X).

PROOF. Let T, S be any operators. If $\|S-T\| < \epsilon$, and $(x,f) \in \pi_P$, then $|f(\langle (S-T)x,x\rangle)| \leqq \|S-T\| < \epsilon$, and so $f(\langle Sx,x\rangle)=f(\langle Tx,x\rangle)+f(\langle (S-T)x,x\rangle) \in W(T)+(\epsilon)$. It follows that $W(S) \subseteq W(T)+(\epsilon)$. Thus $\overline{W(S)} \subseteq \overline{W(T)}+(\epsilon)$ By symmetry, $\overline{W(T)} \leqq \overline{W(S)}+(\epsilon)$. Thus $\|S-T\| < \epsilon$ implies $d(\overline{W(S)}, \overline{W(T)}) \leqq \epsilon$, and $W(\cdot)$ is a continuous function from B(X) to the set of compact subsets of **C**, endowed with the Hausdorff metric topology.

Also $\omega(S) \leqq \omega(T)+\omega$ and $\omega(T) \leqq \omega(S)+\epsilon$ imply $|\omega(S)-\omega(T)| \leqq \epsilon$. So $\omega(.)$ is a continuous real-valued function of its argument.

Given $x \in S(X)$, let $W(T,x)=\{f(\langle Tx,x\rangle): f \in P, f(\langle x,x\rangle)=1\}$, $(T \in B(X))$. Then it is obvious that

$W(T)=\bigcup\{W(T,x): x \in S(X)\}$.

LEMMA 3.5. (Bonsall and Duncan, 1973). *Let* X, Y *be metric spaces with* Y *compact, let* ϕ *be a mapping of* X *into* 2^Y *such that* $\phi(x)$ *is closed for each* $x \in X$. *Then* ϕ *is upper semicontinuous of and only if* $x_n \in X$, $y_n \in \phi(x_n)$ $(n=1,2,\cdots)$, $x=\lim x_n$, $y=\lim y_n$ *imply* $y \in \phi(x)$.

THEOREM 3.6. *The mapping* $x \rightarrow W(T,x)$ *is an upper semicontinuous mapping of* S(X) *with the norm topology into the novoid compact convex subsets of* **C**.

PROOF. The sets W(T,x) are nonvoid compact convex subsets of a compact disc in **C**. Let $x_n \in S(X)$, $\lambda_n \in W(T,x_n)$, $\lim \|x_n-x\|_X=0$, and $\lim |\lambda_n-\lambda|=0$. Then there exists $f_n \in P$ such that $f_n(\langle x_n,x_n\rangle)=1$ and $\lambda_n=f_n(\langle Tx_n,x_n\rangle)$. By the weak* compactness of the unit ball in B', there exists a weak* cluster point f of $\{f_n\}$ with $\|f_n\| \leqq 1$. Also

$$|1-f(\langle x,x\rangle)| \leqq |f_n(\langle x_n-x,x_n\rangle)| + |f_n(\langle x,x_n-x\rangle)| + |(f_n-f)(\langle x,x\rangle)| \leqq 2\|x_n-x\|_X + |(f_n-f)(\langle x,x\rangle)|,$$

from which $f(\langle x,x\rangle)=1$, and $f(e)=1$. Finally

$$|\lambda-f(\langle Tx,x\rangle)| \leqq |\lambda-\lambda_n|+|f_n(\langle Tx_n-Tx,x_n\rangle)|+|f_n(\langle Tx,x_n-x\rangle)|+|(f_n-f)(\langle Tx,x\rangle)| \leqq |\lambda-\lambda_n|+\|Tx_n-Tx\|_X+\|x_n-x\|_X\|Tx\|_X+|(f_n-f)(\langle Tx,x\rangle)|,$$

which gives $\lambda=f(\langle Tx,x\rangle)$. By Lemma 3.5, $x \rightarrow W(T,x)$ is an upper semicontiuous mapping.

Since the numerical range of a bounded operator T on a Hilbert space need not be a closed set, the same is true of W(T). For we let B be a C^*-algebra and we denote by $\ell^2(B)$ the space of sequences $x=(x_1,x_2,\cdots)$, $x_k \in B$, $1 \leqq k < \infty$, which satisfy the condition that $\Sigma\ x_k{}^*x_k$ converges. The space ℓ^2 (B) becomes a right B–module when we define $xb=(x_1b, x_2b,\cdots)$ for $x=(x_1,x_2,\cdots)\ \epsilon\ \ell^2(B)$, $b\ \epsilon\ B$ and a pre–Hilbert B–module when we set $(x,y)=\Sigma y_k{}^*x_k$ for $x=(x_1,x_2,\cdots)$, $y=(y_1,y_2,\cdots)\ \epsilon\ \ell^2(B)$. We introduce a norm in $\ell^2(B)$ by the formula $\| x \|^2 = \| \Sigma x_k{}^* x_k \|$. By the Cauchy–Schwarz–Bunjakovkii inequality, $\ell^2(B)$ is a Hilbert B–module. We define a bounded linear operator T on $\ell^2(B)$ by $Tx=(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3,\cdots)$ $(x=(x_1,x_2,\cdots)\ \epsilon\ \ell^2(B))$. Then $T=T^*$, $\frac{1}{n}\ \epsilon\ W(T)$ for all $n=1,2,\cdots$, but $0 \not\epsilon\ W(T)$. Since W(T) is connected, $W(T)=(0,1]$. Hence W(T) is not closed.

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國 文 抄 錄

Hilbert C*–加群上의 C*–代數値 內積과 單位的 C*–代數上에서 定義된 連續 陽의 線型 汎函數에 의하여 이 加群上의 有界 線型 作用素의 數域을 새로이 定義한 후 Toeplitz, Bauer, Yang 등의 立場에서 본 數域 개념과의 포함관계를 밝히고, Hilbert 空間上에서 스펙트럼과 數域과의 관계를 밝히는 Williams의 定理를 확장했다. 한편 새로 定義된 數域은 복소수 C의 連結部分集台이 되지만 閉部分集台은 되지 않는 位相性質과 Lumer의 公式이 성립함을 밝혔다.