Semilinear Elliptic Singular Perturbation Problems with Nonuniform Interior Behavior

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반선형인 타원형 특이섭동문제에서 비일양 내부현상을 갖는 해들의 존재성에 관한 연구

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Summary

In this paper we investigate certain intermediate solutions for the Dirichlet problem $\varepsilon^2\Delta u+f(x,u)=0$ in Ω, $u=0$ on $\partial\Omega$, where ε is a small positive parameter and Ω is a bounded domain in R^n with $C^{2+\alpha}$ boundary for some $\alpha\in(0,1)$.

1. Introduction

In 1972, Sattinger used the method of sub and super solutions to study the stability of solutions of the elliptic boundary value problem

$$Lu+f(x,u)=0 \quad in\ \Omega,$$
$$Bu=h \quad on\ \partial\Omega,$$

where L is a uniformly elliptic second order operator and B is a linear boundary operator, as equilibrium solutions of the parabolic problem

$$Lv+f(x,v)=v_t \quad in\ (0,\infty)\times\Omega,$$
$$Bv=h \quad on\ (0,\infty)\times\partial\Omega,$$
$$u(x,0)=u_0(x).$$

Specifically, he showed that solutions of the elliptic problem which are obtained by monotone iteration from a sub or super solution have one-sided stability. If there is a unique solution between a sub-super solution pair, then it is stable.

Several years later, Matano established that an "intermediate solution" exists between any two stable solutions. Existence of these intermediate solutions has also been established by others (see Brown, Budin, Hess, DeFigueiredo) using degree theory, variational methods or some combination thereof, especially in the case that f is independent of x.

In this paper we investigate certain intermediate solutions for the Dirichlet problem

$$\epsilon^2\Delta u+f(x,u)=0 \quad in\ \Omega, \qquad (1)$$

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$$u = 0 \quad on\ \partial\Omega, \tag{2}$$

where ϵ is a small positive parameter and Ω is a bounded domain in R^n with a $C^{2+\alpha}$ boundary for some $\alpha \in (0,1)$. Howes has obtained solutions of (1), (2) between sub and super solutions which exhibit boundary layer behavior while converging uniformly to stable zeroes of f on compact subsets of Ω as $\epsilon \to 0$. The results of Sattinger and Matano lead us to anticipate that there may be one or more intermediate solutions of (1), (2), if f has at least two stable zeroes.

Under appropriate conditions on the nonlinearity, it will be shown that there is an intermediate solution of (1), (2) having more complicated limiting behavior than that of the maximum and minimum solutions. When f is independent of x, our results are closely related to the work of Clement and Sweers, who showed that positive solutions of the boundary layer type are locally unique. In order to establish the main result, we first show that in the case where f is independent of x and Ω is a ball centered at the origin, there is an intermediate solution with a "spike" at $x=0$, the width of which is $O(\epsilon)$.

In the final section, we specialize our results to the case of an ordinary differential equation. Here the nonlinearity need have only one stable zero and one unstable zero in order to generate a solution with one or more spikes. Moreover, the location of these spikes can often be determined by using a geometrical approach due to Kath and based on the Melnikov integral.

We conclude this section by noting some general properties of solutions of (1).

Lemma 1. Let $h: R^n \to [0,\infty)$ be a C^2 function, M, N be positive constants and $\beta \in [0,2)$. Then there is an $L>0$ so that: if u is a solution of (1) on Ω, $s_2 > s_1 \geq 0$, $h^{-1}([s_1, s_2]) \subset \bar{\Omega}$, and

$$|f(x, u(x,\epsilon))| > M\epsilon^{\beta},$$

$$|u(x,\epsilon)| < N,$$

for $s_1 < h(x) < s_2$, then $s_2 - s_1 < L\epsilon^{1-.5\beta}$ for all $\epsilon > 0$.

Proof. Consider the case that $f(x, u(x,\epsilon)) > M\epsilon^{\beta}$.

Define

$$v(x) = \frac{8N}{(s_2 - s_1)^2}(h(x) - s_1)(s_2 - h(x)) - N.$$

Then $v=-N$ where $h(x)=s_1$ or s_2, $v=N$ where $h=.5(S_1+S_2)$, and $\Delta v = O((s_2-s_1)^{-2})$.

Let $U=h^{-1}(S_1, S_2) \subset \Omega$ and $w=u-v$. Then $w \geq 0$ on ∂U and $w<0$ at some points in U. Now

$$\Delta w \leq -M\epsilon^{\beta-2} + O((s_2 - s_1)^{-2})$$

on U. By the maximum principle, Δw cannot be negative everywhere on U, so there is an L for which $s_2 - s_1 < L\epsilon^{1-.5\beta}$.

Lemma 2. Let N be a positive constant. There is an L so that if u satisfies (1) and $|u| \leq N$, the $|Du| \leq \frac{L}{\epsilon}$ on $\bar{\Omega}$, where D is any first order differential operator.

Proof. The conclusion follows from the standard Schauder estimates for linear elliptic boundary value problems.

Taken together, these lemmas give a fairly precise version of the well-known "folk theorem" that layer regions for solutions of (1) tend to be of thickness $O(\epsilon)$.

2. Solutions with radial symmetry

Consider the special problem

$$\epsilon^2 \Delta u + g(u) = 0 \quad in\ B, \tag{3}$$

$$u = 0 \quad on\ \partial B, \tag{4}$$

where B is the ball of radius R centered at the origin. Concerning g we assume:

A_1 $g \in C^1[0,\infty)$;

A_2 *there are numbers* $0 < z_1 < z_2 < z_3$ *so that* $g(z_i) = 0$ *for* $i = 1,2,3$, $g'(z_i) < 0$ *for* $i = 1,3$, *and* g *has no other zeroes between* z_1 *and* z_3;

A_3 $\int_\theta^{z_1} g(u)\,du > 0$ *for* $0 \le \theta < z_1$;

A_4 $\int_{z_1}^{z_3} g(u)\,du > 0$.

According to a theorem of Gidas, Ni, and Nirenberg, every positive solution u of (3), (4), is radially symmetric, and $u(r,\epsilon)$ satisfies

$$\epsilon^2\left(u'' + \frac{n-1}{r}u'\right) + g(u) = 0, \quad 0 < r < R,$$
$$u'(0) = u(R) = 0, \qquad (5)$$
$$u'(r) < 0, \quad 0 < r < R.$$

Let u_0 be the unique number in (z_1, z_3) so that

$$\int_{z_1}^{u_0} g(u)\,du = 0.$$

The next lemma yields a positive solution of (3), (4), with a narrow spike at the enter of the ball.

Lemma 3. Assume A_1–A_4 *and that* ϵ *is small. Then (3), (4) has a positive solution* $u(r,\epsilon)$ *so that* $u(0,\epsilon) \in (u_0, z_3)$ *and* $u(r,\epsilon) \to z_1$ *as* $\epsilon \to 0$ *uniformly on compact subsets of* $(0, R)$.

Proof. Although it is possible to give a more elementary proof of this lemma, we prefer to present a brief proof based on published results.

By a well-known construction (see Howes), there are subsolutions φ_1, φ_3, for (3), (4), so that $\varphi_i = 0$ on $\partial\Omega$, $0 < \varphi_i < z_i$ in Ω, and $\varphi_i \to z_i$ as $\epsilon \to 0$ uniformly on compact subsets of Ω for $i=1,3$. Now z_1 and z_3 are corresponding supersolutions for (3), (4), so there are positive solutions u_1 between φ and z_1 and u_3 between φ_3 and z_3. Furthermore, by Theorem 2′ of Clement and Sweers (see also Smoller and Wasserman), they are the only such solutions. Consequently, they are stable and Matano's result provides an intermediate solution $u(r,\epsilon)$.

Let w be a nonnegative C^∞ function in Ω so that $w < \varphi_1$ except in a small ball U about 0 and $w(0) \in (z_1, z_3)$. Again by [Clement, Sweers, Theorem 2′] we have $u < w$ somewhere in U. From Lemma 1, u is near the zeroes of g except on r intervals of length $O(\epsilon)$.

Using (5), we obtain that u satisfies

$$\frac{\epsilon^2}{2}\left(u'^2(r_2) - u'^2(r_1)\right) + \epsilon^2(n-1)\int_{r_1}^{r_2} \frac{u'^2}{r}\,dr$$
$$= \int_{u(r_2)}^{u(r_1)} g(s)\,ds \qquad (6)$$

for $0 \le r_1 \le r_2 \le R$. Let $M > 0$ be fixed and consider the points r_1 where $|u'(r_1)| < M\epsilon^{-.5}$. From (6)

$$\int_{u(r)}^{u(r_1)} g(s) > -O(\epsilon) \quad (0 \le r \le r_1)$$

at all such r_1. It follows that u is near z_1 or z_3 on (o,R) except for $O(\epsilon)$ intervals.

However, from the remarks in the previous paragraph, u can be near z_3 only for r near 0. Finally, the fact that $u(0) > u_0$ follows from (6) with $r_1 = 0$.

Actually, this lemma is true under slightly more general hypotheses. In A_2, it is enough to assume that z_1 an z_3 are of finite order and that g changes sign as it passes through each of them.

In section 4 we will show that for ordinary differential equations, the nonlinearity need have only two zeroes to produce a solution with one or more spikes. The following example shows that two zeroes may not suffice if $n>2$.

Example 1.

$$\epsilon^2\Delta u + (u-1)^p - u + 1 = 0 \quad in\ B,$$

$$u = 0 \quad on\ \partial B.$$

Recall that every positive solution of this problem is radial. If u is such a solution with $u(0,\epsilon)>1$, then $u(0,\epsilon)>2$ by the maximum principle. Let $v=u-1$. Then v satisfies

$$\epsilon^2\Delta v + v^p - v = 0,$$

$v'(0,\epsilon)=0$, and $v(0,\epsilon)>1$. However, a theorem due to Ni states that such a v must be positive for all r if $p\geq\frac{n+2}{n-2}$. We conclude that the Dirichlet problem does not have a positive solution u with $u(0,\epsilon)>1$ in this case for any value of ϵ.

3. Existence and behavior of an intermediate solution

Now Ω in (1), (2), will represent an arbitrary smoothly bounded domain in R^n. We make the following assumptions about f:

A_5 $f \in C^1(\overline{\Omega}\times R)$;

A_6 *there are* $C^2(\overline{\Omega})$ *functions* $z_i(x)$ $(i=1,2,3)$ *so that*

$z_1(x) < z_2(x) < z_3(x),\ z_2(x) > 0,\ f(x,z_i(x)) = 0\ (i=1,2,3),$

$f_u(x,z_i(x)) < 0\ (i=1,3)$, *and* $z_2(x)$ *is the unique zero of* f *between* z_1 *and* z_3 $(x\in\overline{\Omega})$;

A_7 $\int_\theta^{z_1(x)} f(x,u)\,du > 0$ *for* $x\in\partial\Omega$ *and* θ *between* 0 *and* z_1;

A_8 *there is a nonempty open set* $W\subset\Omega$ *so that*

$$\int_{z_1(x)}^{z_3(x)} f(x,u)\,du > 0 \text{ for } x\in W.$$

As in section 2, $u_0(x)$ is defined to be the unique solution of

$$\int_{z_1(x)}^{u_0(x)} f(x,u)\,du = 0$$

for $x\in W$.

Our fundamental result is contained in :

Theorem 1. Assume A_5–A_8. *Let* $\gamma>0$ *be a small constant and let* ϵ *be sufficiently small. Then for each ball* $B\subset W$, (1), (2), *has a solution* $u(x,\epsilon)$ *so that:*

(a) $z_1(x)-\gamma < u(x,\epsilon)$ *if* $dist(x,\partial\Omega) >> \epsilon$;

(b) $u(x,\epsilon) < z_3(x)+\gamma$ *for all* $x\in\overline{\Omega}$;

(c) $\exists x\in B$ *so that* $u(x,\epsilon) < u_0(x)+\gamma$;

Furthermore, there is a computable $\delta>0$, *independent of* ϵ, *so that*

(d) $max\{u(x,\epsilon): x\in\Omega\} \equiv u(x_0,\epsilon) > z_2(x_0)+\delta.$

Proof. We will construct two pairs of sub and super solutions φ_1,ψ_1, and φ_2,ψ_2.

Let $t(x)$ represent the distance of each x from $\partial\Omega$ and $s(x)$ the point on $\partial\Omega$ closest to x. Then t and s induce a coordinate system on a small neighborhood of $\partial\Omega$ (see Berger, Fraenkel).

Because of assumptions A_5–A_7, the problem has a subsolution of the form

$$\varphi_1(x,\epsilon) = z_1(x) - \gamma + \Gamma(s(x),\frac{t(x)}{\epsilon},\epsilon),$$

where Γ is a boundary layer correction (see Howes). Here Γ has the following properties : $\Gamma=0$ outside a small neighborhood of the boundary and $\Gamma=\gamma-z_1(x)$ for $x\in\partial\Omega$.

A corresponding supersolution is $\psi_1(x)=z_2(x)-C$, where C is a small positive constant chosen so that $z_2(x)-C>max\{0,z_1(x)\}$ for all $x\in\Omega$.

A larger supersolution is $\psi_2(x,\epsilon)=z_3(x)+D\epsilon^2$, for some sufficiently large positive constant D.

In order to define an associated subsolution φ_2, let an open ball $B_1(x_1,r_1)\subset B$ and a C^1 function $g(u)$ be chosen so that

$$g(u) < f(x,u),$$

$$g(u) = 0 \ at \ \overline{z_1} < \overline{z_2} < \overline{z_3},$$

$$g'(\overline{z_i}) < 0 \ for \ i = 1,3,$$

$$\int_{z_1}^{z_3} g(u)\, du > 0,$$

for $x\in B_1$ and appropriate u. The zeroes $\overline{z}_1(x)$, $\overline{z}_2(x)$ of g here are chosen to be close to the zeroes of f, while $\overline{z}_3(x)$ is slightly larger than $u_0(x)$ for $x\in B_1$. Let v be a solution of $\epsilon^2\triangle v+g(v)=0$ in B_1 given by Lemma 3 (with appropriate translation of the indepen- dent and dependent variables) so that $v\to\overline{z}_1\langle z_1$ as $\epsilon\to 0$ uniformly on compact subsets of $B_1\setminus\{x_1\}$ and $v(x_1)\in(u_0(x_1),\ u_0(x_1)+r)$. Let χ be a smooth cutoff function so that $\chi=1$ for $0\le |x-x_1|\le\frac{r_1}{2}$, $\chi=0$, for $|x-x_1|\ge r_1$, and $0\le\chi\le 1$. Then it is readily checked that $\varphi_2=\chi v+(1-\chi)\varphi_1$ is a subsolution for (1), (2).

Note that φ_2 has these properties: $\varphi_2\ge\varphi_1$, $\varphi_2=\varphi_1$ outside B_1, $\varphi_2\langle\psi_2$, and $\varphi_2(x_1)\in(u_0(x_1),\ u_0(x_1)+r)$.

By the statement and proof of Theorem 1.6 in Amann, there is a solution $u(x,\epsilon)$ of (1), (2), which satisfies (a), (b) and (c), and such that $u(x,\epsilon)\rangle z_2(x)$ for some $x\in\Omega$.

In order to prove (d), we first make a change of dependent variable. Let

$$w(x,\epsilon) = u(x,\epsilon) - z_2(x) + min\{z_2(x) : x \in \Omega\}.$$

Then $w(x,\epsilon)$ satisfies:

$$\epsilon^2\Delta w + F(x,w,\epsilon) = 0, \ \ in\ \Omega \qquad (7)$$

$$w \le 0, \ \ on\ \partial\Omega \qquad (8)$$

wher $F(x,w,\epsilon)=f(x, w+z_2(x)-min\ z_2)+\epsilon^2\triangle z_2$. Let the zeroes of F corresponding to z_1,z_2,z_3, be denoted $a_1 a_2, a_3$, respectively. Note that $a_2\rangle 0$ is essentially fixed with maximum variation $O(\epsilon)$.

Now we choose $\delta\rangle 0$ and a C^1 function $G(u)$ so that: $G(0)\ge 0$; G has exactly three zeroes $b_1\langle b_2\langle b_3$ so that $b_1\in(max\{a_1(x),0\},\ a_2(x))$, $b_2\langle a_2(x)$, and $b_3\in(a_2(x),\ a_3(x))$ for all $x\in\bar{\Omega}$; $G(w)\rangle F(x,w)$ for $0\le w\le a_2(x)+\delta\langle b_3$ and $x\in\Omega$; and $\int_{b_1}^{\theta} G(s)\ ds\langle 0$ for $b_1\langle\theta\le b_3$.

Suppose that $w(x,\epsilon)\le a_2(x)+\delta$ for all $x\in\bar{\Omega}$ and all sufficiently small ϵ. Replace F by a modified function $\bar{F}$ so that $F(w)=\bar{F}(w)$ for $0\le w\le a_2(x)+\delta$ and $G(w)\ge\bar{F}(w)$ for $a_2(x)+\delta\langle w\le b_3$ and $x\in\Omega$. Then w still satisfies (7) with F replaced by $\bar{F}$. Now consider

$$\epsilon^2\Delta u + G(u) = 0 \ \ in\ V, \qquad (9)$$

$$u = 0 \ \ on\ \partial V, \qquad (10)$$

where V is a ball centered at the origin which contains Ω. Let $\varphi=max\{w,0\}$. Then φ is a subsolution for (9), (10), so that $max\{\varphi(x) : x\in V\}\rangle b_2$; b_3 is a supersolution. Then (9), (10), has a positive, radially symmetric solution $\bar{u}(r,\epsilon)$ with $max\ \bar{u}=\bar{u}(0)\rangle b_2$.

On the one hand,

$$.5(\bar{u}'(r))^2 + \epsilon^{-2}\int_{\bar{u}(0)}^{\bar{u}(r)} G(s)\, ds = (1-n)\int_0^r \frac{(\bar{u}')^2}{s} ds,$$

so that $\int_{\bar{u}(0)}^{\bar{u}(r)} G(s)\ ds\langle 0$ for all $r\rangle 0$. On the other hand, since $\bar{u}(0)\in(b_2,b_3)$,

$$\int_{\bar{u}(0)}^{b_1} G(s)\, ds > 0,$$

so we have a contradiction. Consequently, for all small ϵ, $w(x, \epsilon) > a_2(x) + \delta$ for some x, so

$$u(x,\epsilon) > z_2(x) - min\{z_2\} + a_2(x) + \delta$$
$$> z_2(x) + \delta + O(\epsilon^2)$$

for some $x \in \Omega$.

Our proof of (d) is an adaptation of an argument due to Dancer and Schmitt in which they show for the case where f is independent of x that positive solutions of the Dirichlet problem have maximum values at least u_0. Consequently, in that case we can take $\delta = u_0 - z_2$.

Theorem 1 gives us an intermediate solution of (1), (2), which has in addition to boundary layer behavior of the monotone type some more complicated limiting behavior as $\epsilon \to 0$. It appears to be difficult to give more precise information about this solution in this generality. However, the next theorem rules out the possibility of classical shock layer behavior in W.

Theorem 2 Assume A_5–A_8. Let $r > 0$ be independent of ϵ, let $B(y, r)$ be a ball with closure in W, and let u be the solution of (1), (2) *given by Theorem* 1. *If $u(x, \epsilon) \to z_2(x)$ as $\epsilon \to 0$ for $x \in B(y, r)$, then there is a $\lambda > 0$ independent of ϵ so that $u(x, \epsilon) > z_2(x) + \lambda$ on $B(y, r + \lambda)$ for all small ϵ.*

Proof. Suppose no such λ exists. Then there are sequences $\epsilon_n \to 0$ and x_n so that $u(x_n, \epsilon_n) < z_2(x_n) + \frac{1}{n}$ and $dist(x_n, \partial B(y, r)) < \frac{1}{n}$.

Consider $B_n \equiv B(x_n, \frac{2}{n}) \subset W$. For n sufficiently large, we can choose a nonnegative constant C so that $C > -z_1(x)$ for $x \in B_n$ and $g(u + C) < f(x, u)$ for $x \in B_n$ *and appropriate* u so that g satisfies A_1–A_4. (We take $C = 0$ if $z_1 > 0$.) Then $u + C$ is a supersolution for

$$\epsilon^2 \Delta v + g(v) = 0 \ \ in \ B_n, \qquad (11)$$

$$v = 0 \ \ on \ \partial B_n, \qquad (12)$$

and we can construct a positive subsolution φ for (11), (12) much as in Theorem 1 with maximum in $B(y, r)$ so that $max\varphi > u_0(x_n) + C - \frac{1}{n}$ and $\varphi < u + C$. Then (11), (12), has a positive solution v between the sub and supersolutions with its maximum at x_n. However, we then have both $u(x_n, \epsilon_n) + C < z_2(x_n) + C < z_2(x_n) + \frac{1}{n} + C$ and $v(x_n) > u_0(x_n) + C - \frac{1}{n}$, a contradiction of $v < u + C$.

The local uniqueness result of Clement and Sweers gives a stronger conclusion if f is independent of x and $u > 0$. Namely, for every $C > 0$ there is no ϵ-independent ball B so that $u(x, \epsilon) > z_2(x) + C$ for $x \in B$ and all small ϵ.

We conclude this section with an example which illustrates these results and which also provides a counter-example to a theorem of De Santi dealing with the existence of solutions to (1), (2), with spike layers.

Example 2.

$$\epsilon^2 \Delta u + f(r, u) = 0 \ \ in \ B(0,1)$$
$$u = 0 \ \ on \ \partial B(0,1),$$

where $B(0,1)$ is the unit ball in R^n.

Assume f satisfies A_5–A_8, with $z_1 > 0$ and W a ball centered at the origin. Also assume $f_r(r, u) < 0$ for $0 < r < 1$ and all u. Then Theorem 1′ of Gidas, Ni, Nirenberg implies that all positive solutions of the Dirichlet problem are radially symmetric and are decreasing as functions or r.

Let $B_1 \subset W$ be a small ball centered at the origin and let $u(r, \epsilon)$ be the positive solution of Theorem 1 corresponding to B_1. Now $max\{u(r, \epsilon) : 0 \le r \le 1\} = u(0, \epsilon) > z_2(0) + \delta$ for some positive δ independent of ϵ, and $u(r,\epsilon) < u_0(r) + \gamma$ somewhere in B_1. By Lemma 1 and Theorem 2, $u(r,\epsilon)$ must descend from its maximum at $r=0$ to a value near z_1 or z_2 when $r \ll 1$.

Now consider the identity for $r_1 < r_2$:

$$.5\epsilon^2(u'^2(r_2) - u'^2(r_1)) + \epsilon^2(n-1)\int_{r_1}^{r_2} \frac{u'^2}{r}\, dr$$
$$= \int_{r_1}^{r_2} f(r,u)u'(r)\, dr$$
$$= \int_{u(r_1)}^{u(r_2)} f(r_1,u)\, du + \int_{r_1}^{r_2} (f(r,u) - f(r_1,u))u'(r)$$
$$dr \le \int_{u(r_1)}^{u(r_2)} f(r_1,u)\, du + O(r_2 - r_1).$$

It the length of the interval on which u is near z_2 is bounded away from 0 as $\epsilon \to 0$, then the identity gives a contradiction of the assumed properties of f when u descends from z_2 to z_1 or 0. We conclude, as in Lemma 3, that $u(r,\epsilon)$ has a narrow spike at 0 and converges to z_1 as $\epsilon \to 0$ uniformly on compact subsets of $(0,1)$.

To obtain a counter-example to De Santi's result, assume $W = B(0,1)$ and $f(r,0) \ge 0$ for $0 \le r \ge 1$. Recall that $\int_{z_1(r)}^{u_0(r)} f(r,u)\, du = 0$ for $0 \le r \le 1$. Then $u_0(r)$ is strictly inceasing since

$$0 = f(r,u_0(r))u_0'(r) + \int_{z_1(r)}^{u_0(r)} f_r(r,u)\, du.$$

for $0 < r < 1$. Fix $r_1 \in (0,1)$ so that $f(r, u_0(r_1)) > 0$ for $0 \le r \le 1$. Let

$$I(r) = \int_{z_1(r)}^{u_0(r_1)} f(r,u)\, du.$$

We have $I(r)(r_1 - r)$ is positive for $0 \le r \le 1, r \ne r_1$ and $\nabla I(r) \ne 0$ at $r = r_1$.

The hypotheses of Theorem 4.2 in De Santi are satisfied here; the conclusion of the theorem is that the boundary value problem has a solution which converges to z_1 in $B \setminus \{r = r_1\}$ and to $u_0(r_1)$ on $\{r = r_1\}$ as $\epsilon \to 0$. Futhermore, the solution would be positive in this case since it obtained in the proof by the method of sub and super solutions and since 0 is a subsolution. However, the existence of such a solution would contradict the fact that all positive solutions are decreasing as functions of r.

4. Ordinary differential equations

We now specialize our results to the two point boundary value problem

$$\epsilon^2 u'' + f(x,u) = 0, \quad (0 < x < 1) \tag{13}$$

$$u(0) = u(1) = 0. \tag{14}$$

The assumptions on f will be similar to those of Theorem 1, but in this case f need have only two zeroes. For simplicity, we take $W = (0,1)$, but the more general situation is amenable to the same techniques. In addition to the previous hypotheses A_5 and A_7, we assume:

A'_6 *there are* $C^2([0,1])$ *functions* $z_1(x) < z_2(x)$ *so that* $z_2(x) > 0$, $f(x, z_i(x)) = 0$ $(i = 1,2)$, *and* $f_u(x, z_1(x)) < 0$ $(x \in [0,1])$;

A'_8 *there is a* $u_0(x) > z_2(x)$ *so that* $f(x, u_0(x)) > 0$, $z_2(x)$ *is the only zero of* f *between* $z_1(x)$ *and* $u_0(x)$, *and* $\int_{z_1(x)}^{u_0(x)} f(x,u)\, du = 0$ $(x \in [0,1])$.

The next theorem show that some solution of (13), (14), has a narrow spike in the interval $[0,1]$.

Theorem 3 *Assume* A_5, A'_6, A_7, *and* A'_8. *For* ϵ *sufficiently small,* (13), (14), *has a solution* $u(x,\epsilon)$ *satisfying (a), (c), and (d) of Theorem* 1. *Furthermore,* $\max\{u(x,\epsilon) : 0 < x < 1\} \equiv u(x_0,\epsilon) < u_0(x_0) + 2\gamma$, *and there is an interval* $[a(\epsilon), b(\epsilon)]$ *containing* x_0 *so that*

$b(\epsilon)-a(\epsilon)=O(\epsilon)$, $u(a(\epsilon),\epsilon)=z_1(a(\epsilon))$, *and* $u(b(\epsilon),\epsilon)=z_1(b(\epsilon))$.

Proof. The first step is to modify f for $u>u_0(x)+2\gamma$, where γ is the small positive constant in the statement of Theorem 1. Let $z_3(x)>u_0(x)+2\gamma$ be a $C^2([0,1])$ function and let $f_1(x,u)$ be a C^1 modification of f having the following properties : $f_1=f$ for $u\le u_0(x)+2\gamma$, $f_1>0$ for $z_2(x)<u<z_3(x)$, and $f_1(x,z_3(x))=0$ $(0\le x\le 1)$. We apply Theorem 1 to f_1 to obtain a solution $u(x,\epsilon)$ for sufficiently small ϵ of

$$\epsilon^2 u'' + f_1(x,u) = 0, \quad (0 < x < 1) \qquad (15)$$

$$u(0) = u(1) = 0, \qquad (16)$$

which has the properties listed in that theorem.

On any interval $[x_1,x_2]$ where u' does not change sign, we claim :

$$.5\epsilon^2[u'^2(x_2) - u'^2(x_1)] + \int_{u(x_1)}^{u(x_2)} f_1(x_1,u)\,du = O(x_2 - x_1). \qquad (17)$$

To establish (17), write (15) in the form

$$\epsilon^2 u''u' + f_1(x_1,u)u' = [f_1(x_1,u) - f(x,u)]u'.$$

Then

$$.5\epsilon^2[u'^2(x_1) - u'^2(x_2)] + \int_{u(x_1)}^{u(x_2)} f_1(x_1,u)\,du = \int_{x_1}^{x_2}[f_1(x_1,u) - f_1(x,u)]u'\,dx,$$

and (17) follows from the mean value theorem for integrals.

We want to show that $u(x)\le u_0(x)+2\gamma$ for $x\in[0,1]$ and small ϵ so that u is a solution of the original problem (13), (14). Suppose that

$$u(x_0) = max\{u(x) : 0 \le x \le 1\} > u_0(x_0) + 2\gamma,$$

for some arbitrarily small values of ϵ. By Theorem 1(c) there is an $x\in(0,1)$ so that $u(x)<u_0(x)+\gamma$. We consider only the case that $x<x_0$. Let

$$x_2 = inf\{t \in (x,x_0) : u(t) = u_0(t) + 2\gamma\},$$

$$\eta = sup\{t \in (x,x_2) : u(t) = u_0(t) + \gamma\}.$$

Note that x_2 and η are bounded away from the endpoints as $\epsilon\to 0$. By Lemma 1, $x_2-\eta=O(\epsilon)$. We also have for $x\in[\eta,x_2]$ that $u_0(x)+\gamma\le u(x)\le u_0(x)+2\gamma$, so $(u-u_0)'(\eta)\ge 0$. If $(u-u_0)'(\eta)=0$, then (17) with $x_1=\eta$ immediately gives a contradiction.

If $(u-u_0)'(\eta)>0$, then $u(x)<u_0(x)+\gamma$ on some interval to the left of η. From (17),

$$\int_{u(x)}^{u_0(x_2)+2\gamma} f_1(x,u)\,du = .5\epsilon^2[u'^2(x) - u'^2(x_2)] + O(x_2 - x_1).$$

Choose C independent of ϵ so that

$$\int_{\theta}^{u_0(x_2)+2\gamma} f_1(x,u)\,du > C > 0,$$

for $z_1(x)-\gamma\le\theta\le u_0(x)+\gamma$ and x near x_2. If $u(x)<u_0(x)+\gamma$ and x is near x_2, then $|u'(x)|>\frac{C}{\epsilon}$ for x in a small ϵ-independent interval, so we have a contradiction. It follows that u is a solution of the original problem (13), (14).

The remaining properties of u now follow from Lemma 1.

The quesion of possible locations of spikes for solutions of (13), (14), has been studied by a number of authors. For the case that f is independent of x and $z_1<0<z_2$, O'Malley has used phase plane analysis to show that solutions exist with increasing numbers of spikes as $\epsilon\to 0$, but that the spikes have to occur at equally spaced points in the interval. On the other hand, if $0<z_1$, then Lemma 3 is applicable, and there is a single spike at the center of the interval. It is readily shown that the maximum value of u approaches u_0 as $\epsilon\to 0$.

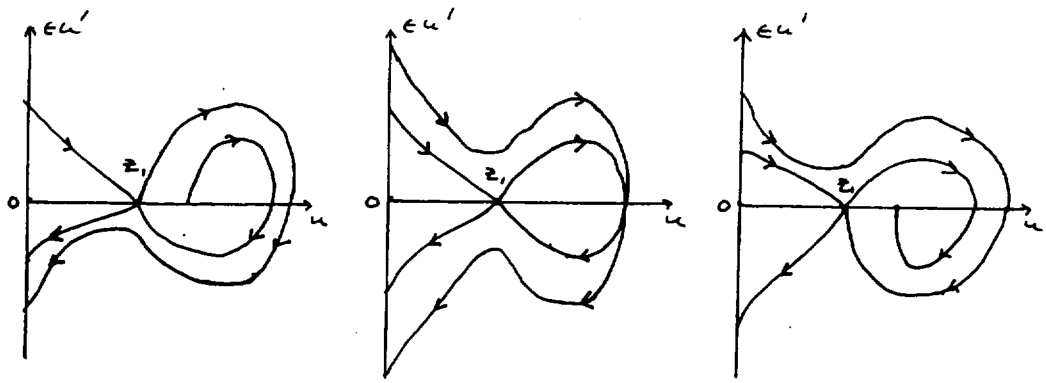
In case f does vary with x, Kath has used the Melnikov integral

$$M(x_0) = \int_{-\infty}^{\infty}\int_{z_1(t)}^{v(t)} f_x(x_0,u)\,du\,dt,$$

where $t=\frac{x-x_0}{\epsilon}$ and $v(t)$ solves $v_{tt}+f(x_0,v)=0$, $v\to 0$ as $t\to\pm\infty$, and $v_t(0)=0$, to fine where spikes can occur. We refer the reader to his paper for a thorough discussion. Our final example treats only the simplest case.

Example 3.

For (13), (14), assume $z_1(x)>0$ on [0,1] and $M(x_0)$ is strictly decreasing as function of $x_0 \in [0,1]$. Now $M(x_0)$ measures the difference in energy at x_0 between the solution of (13) which approaches z_1 as $t\to-\infty$ and the solution which approaches z_1 as $t\to\infty$. (see the figure below.)



There are three possibilities. If $M(x_0)=0$ at some $x_0\in(0,1)$, then there is a solution u with a single spike near x_0 since this is the only location where a trajectory can make a complete circuit from a position near z_1, around z_1, and back near z_1. If $M>0$ on [0, 1], then the spike must occur at the right endpoint. If $M<0$ on [0,1], then the spike occurs at the left endpoint. In all cases, u follows z_1 on the remainder of the interval except at the endpoints.

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〈國文抄錄〉

반선형인 타원형 특이섭동문제에서 비일양 내부현상을 갖는 해들의 존재성에 관한 연구

반선형인 타원형 특이섭동 경계치 문제 $\varepsilon^2\Delta u+f(x,u)=0$ in Ω, $u=0$ on $\partial\Omega$,에서 해들의 형태에 관한 연구로서, 만약 Ω의 경계가 미분가능하고, 양의 매개변수 ε이 충분히 작으면, 소위 중간해라고 불리우면 Ω의 내부에서 비일양적 현상을 나타내는 해가 존재한다는 사실을 증명한다.