On Some Properties of B_1–Proximity

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B_1–Proximity의 몇가지 성질에 關하여

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Introduction

The theory of proximity spaces was essentially discovered in 1950 by Efemovič when he axiomatically characterized the proximity relation "A is near B", which is denoted by $A\delta B$, for subsets A and B of a set X. Efremovič's axioms for this nearness relation δ are as follows:

(E1) $A\delta B$ implies $B\delta A$.

(E2) $A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$.

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(E3) $A\delta B$ implies $A\neq\phi$

(E4) $A\cap B\neq\phi$ implies $A\delta B$.

(E5) $A\bar{\delta}B$ implies there exists a subset E such that $A\bar{\delta}E$ and $(X-E)\bar{\delta}B$ ($\bar{\delta}$ means the negation of δ).

A binary relation δ satisfying axioms (E1)–(E5) on the power set of X is called the *Efremovič's proximity* on X.

Hayashi introduced the notion of 'paraproximity' by replacing the word 'finite' by 'arbitrary' and thereby strengthening Efremovičs's 'union' axim to read: for an arbitrary index set I, $A\delta(\bigcup_{i\in I} B_i)$ iff $A\delta B_i$ for some $i\in I$ (Hayashi, E., 1964).

A binary realtion δ between X and subsets of X is called the *K–proximity* on X if δ satisfies the following: (Kim, et al, 1973)

(K1) $x\delta A\cup B$ iff $x\delta A$ or $x\delta B$.

(K2) $x\bar{\delta}\phi$ for all $x\in X$.

(K3) $x\in A$ implies $x\delta A$.

(K4) $x\bar{\delta}A$ implies there is a subset E such that $x\bar{\delta}E$ and $y\bar{\delta}A$ for all $y\in X-E$.

In this note we neglect the axiom (K4) and replace (K1) by a stronger axiom, which we call

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a "B_1-proximity" and examine some of its properties.

I. B_0-Proximity and B_1-Proximity

1.1. Definition. Let ξ be a relation between a set X and its power set PX. Consider the following axioms:

(A0) $x\xi(A\cup B)$ if and only if $x\xi A$ or $x\xi B$.

(A1) For any non-void index set I, $A\xi\bigcup_{i\in I}B_i$ if and only if there exists an index $j\in I$ such that $A\xi B_j$.

(A2) $x\bar{\xi}\phi$ for all $x\in X$ ($\bar{\xi}$ means the negation of ξ).

(A3) $x\in A$ implies $x\xi A$.

ξ is called a B_0-*proximity* on X iff ξ satisfies (A0), (A2) and (A3). ξ is called a B_1-*proximity* on X iff ξ satisfies (A1), (A2) and (A3).

In such a case, (X, ξ) is called a (B_0-*proximity*, B_1-*proximity*) space iff ξ is a (B_0-*proximity*, B_1-*proximity*, resp.) on X.

1.2. Remark. Every K-proximity on X is also a B_0-proximity on X.

1.3. Definition. Let (X, ξ_1) and (Y, ξ_2) be two B_0-proximity spaces (or B_1-proximity spaces). A function $f: X\rightarrow Y$ is said to be a proximal map iff $x\xi_1 A$ implies $f(x)\xi_2 f(A)$. The category of B_0-proximity spaces and proximal maps is denoted by <u>B_0-*Prox*</u>. Its full subcategory whose objects are the B_1-proximity spaces is denoted by <u>B_1-*Prox*</u>.

1.4. Proposition. Let (X, ξ) be a B_1-proximity space. Define an operator α on the power set PX by $\alpha A=\{x: x\xi A\}$. Then α satisfies following properties:

(1) $\alpha\phi=\phi$.

(2) $A\subset\alpha A$ for each $A\subset X$.

(3) $\alpha(A\cup B)=\alpha A\cup\alpha B$.

(4) $A\subset B$ implies $\alpha A\subset\alpha B$.

Proof. (1) It follows from (A2).

(2) By (A3), if $x\in A$ then $x\xi A$ or $x\in\alpha A$. Therefore $A\subset\alpha A$.

(3) It is clear from (A1).

(4) If $x\in\alpha A$, the $x\xi A$ iff $x\xi B$ by (A1).

1.5. Remark. Since the operator α dosen't satisfy $\alpha\alpha A=\alpha A$ for each $A\subset X$, α is not a Kuratowski's closure operator.

1.6. Proposition. Let (X, ξ) be a B_1-proximity space. Then there exists a topology $\tau(\xi)$ on X such that each closed set in $\tau(\xi)$ is precisely the fixed set under the operator α.

Proof. Consider a family $F=\{A: \alpha A=A\}$ of subsets of X.

i) By (1), (2) in 1.4, we have $\phi\in F$, $X\in F$, resp.

ii) Let $\{A_i: i\in I\}$ be an arbitrary collection of members of F. If $x\xi\bigcap_{i\in I}A_i$ then $x\xi A_i$ for each $i\in I$, and so $x\in\alpha A_i=A_i$ for each $i\in I$. Hence $x\in\bigcap_{i\in I}A_i$.

iii) Let A, B be elements of F. Then from (3) in 1.4, $A\cup B\in F$. Therefore the family $\{X-A: \alpha A=A\}$ forms a topolgy $\tau(\xi)$ on X.

1.7. Propersition. In a B_1-proximity space (X, ξ), the following statements are equivalent:

(1) $x\xi A$.

(2) $x\xi\{y\}$ for some $y\in A$.

(3) $\{x\}\cap\alpha A\neq\phi$

Proof. (1)⇒(2). Since $x\xi A$, i.e. $x\xi\bigcup_{i\in A}\{y\}$, from (A1), there is $y\in A$ with $x\xi\{y\}$.

(2)⇒(3). If $x\xi\{y\}$, then $x\in\alpha\{y\}$ or $x\in\alpha A$, and so $\{x\}\cap\alpha A\neq\phi$.

(3)⇒(1). It is clear.

1.8. Theorem. Let (X, ξ) be a B_1-proximity space. Suppose that ξ satisfies the following condition: $x \xi \{y\}$ implies $y \xi \{x\}$. Then the followings are equivalent:

(1) $x \xi A$.

(2) $x \xi \{y\}$ for some $y \in A$.

(3) $\{x\} \cap \alpha A \neq \phi$.

(4) $\alpha \{x\} \cap A \neq \phi$.

Proof. It is suffice to show that (3) iff (4). Since $\{x\} \cap \alpha A \neq \phi$, $x \in \alpha A$ or $x \xi A$, so $x \xi \{y\}$ for some $y \in A$. But $x \xi \{y\}$ implies $y \xi \{x\}$, hence $y \in \alpha \{x\} \cap A$; $\alpha \{x\} \cap A \neq \phi$. Suppose that $\alpha \{x\} \cap A \neq \phi$. Then there is $y \in X$ such that $y \in \alpha \{x\} \cap A$. That is $y \in \alpha \{x\}$ and $y \in A$. Therefore $x \in \alpha \{y\}$ and $\alpha\{y\} \subset \alpha A$. This implies $\{x\} \cap \alpha A \neq \phi$.

Ⅱ. Main Results

The following theorem is an analogous concept in (Kong, 1980).

2.1 Theorem. $\underline{B_1 - Prox}$ is a bicoreflective subcategory of $\underline{B_0 - Prox}$.

Proof. Take any object (X, ξ) in $\underline{B_0 - Prox}$. Define the relation ξ_1 on the power set of X as follows: $x \xi_1 A$ if and only if there is $y \in A$ such that $x \xi \{y\}$. Then it is clear that ξ_1 satisfies the axiom (A2) and (A3). For any non-void index set I, spppose $x \xi_1 \bigcup_{i \in I} A_i$.

Then there is $y \in \bigcup_{i \in I} A_i$ with $x \xi \{y\}$. This imlies $x \xi_1 A_j$ for $y \in A_j$. Conversely, if $x \xi_1 A_j$ for some $j \in I$, it is obvious that $x \xi_1 \bigcup_{i \in I} A_i$. Thus $(X, \xi_1) \in \underline{B_1 - Prox}$.

Let $1_x : (X, \xi_1) \to (X, \xi)$ be the identity map. Then by the defnition of ξ_1 it is clear that 1_x is a proximal map. Take any object $(Y, \xi') \in \underline{B_1 - Prox}$ and take any proximal map $f : (Y, \xi') \to (X, \xi)$. It remains to show $f : (Y, \xi') \to (X, \xi_1)$ is a proximal map. Suppose that $x \xi' A$. Then by 1.7, there is $y \in A$ with $x \xi' \{y\}$, so that $f(x) \xi \{f(y)\}$ and $f(y) \in f(A)$. Thus $f(x) \xi_1 f(A)$. This completes the proof.

2.2. Corollary. (Herrlich & Strecker, 1973)

$\underline{B_1 - Prox}$ is coproductive and cohereditary in $\underline{B_0 - Prox}$.

2.3. Definition. (Naimpally & Warrack, 1971) A subset A of a B_1-proximity space (X, ξ) is a *ξ-neighborhod of a point x* in X (in symbols $x \langle A$) iff $x \bar{\xi} (X-A)$.

2.4. Proposition. Given a B_1-proximity (X, ξ) the relation $\langle$ satisfies the following properties:

(1) $x \langle X$ for every x in X.

(2) $x \langle A$ implies $x \in A$.

(3) If $x \langle A$ and $A \subset B$ then $x \langle B$.

(4) If $x \langle A_i$ for $i = 1, 2, \cdots, n$ iff $x \langle \bigcap_{i=1}^{n} A_i$.

(5) For any index set I, $x \langle \bigcup_{i \in I} A_i$ iff $x \langle A_i$ for every $i \in I$.

(6) If $x \langle A$ then $\{x\} \subset A \subset \alpha A$.

Proof. (1) Since $x \bar{\xi} \phi$, $x \langle X$.

(2) Since $x \langle A$, $x \bar{\xi} (X-A)$, which implies $x \notin (X-A)$, so $x \in A$.

(3) If $A \subset B$, then $X-B \subset X-A$. Thus $x \langle A$ implies $x \bar{\xi} (X-B)$ or $x \langle B$.

(4) For any $i = 1, 2 \cdots, n$, $x \bar{\xi} (X-A_i)$ iff $x \bar{\xi} \bigcup_{i=1}^{n} (X-A_i)$ iff $x \bar{\xi} (X - \bigcap_{i=1}^{n} A_i)$ iff $x \langle \bigcap_{i=1}^{n} A_i$.

(5) For any index set I, $x \langle \bigcup_{i \in I} A_i$

iff $x \bar{\xi} (X - \bigcup_{i \in I} A_i)$

iff $x \bar{\xi} \bigcap_{i \in I} (X-A_i)$

iff $x \bar{\xi} (X-A_i)$ for every $i \in I$

iff $x \langle A_i$ for every $i \in I$

(6) From (2), $x \langle A$ implies $x \xi A$. Therefore $x \in \alpha A$.

2.5. Theorem. If $\langle$ is a binary relation between X and PX satisfylng the properties (1)–(5) in the proposition 2.4 and ξ is defined by $x \bar{\xi} A$ iff $x \ll X-A$. then ξ is a B_1 -proximity on X. A is a ξ-neighborhol of x iff $x \ll A$.

Proof. (A1) For any non-void index set I. $x \bar{\xi} A_i$ for each $i \in I$ iff $x \ll X-A_i$ for each $i \in I$ iff $x \ll \bigcap_{i \in I}(X-A_i)$ iff $x \bar{\xi} \bigcup_{i \in I} A_i$.

(A2) If $x \in X$ the $x \ll X$. which implies $x \bar{\xi} \phi$.

(A3) If $x \bar{\xi} A$ the $x \ll X-A$. so $x \in X-A$ or $x \notin A$.

2.6. Lemma. Let (X, ξ) be a B_1-proximity space. Then the followings are equivalent:

(1) $x \ll A$ implies there exists a subset B of X such that $x \ll B$ and $y \ll A$ for every y in B.

(2) If $x \bar{\xi} A$ then there exists a subset E of X such that $x \ll E$ and $y \bar{\xi} A$ for every y in E.

Proof. It is immediate from 2.3.

The condition in 2.6 will ensure that α is a topological closure operator.

2.7. Theorem. If a B_1-proximity space (X, ξ) satisfies the condition in 2.6, the operator α is a topological closure operator.

Proof. By Proposition 1.6. it remains to show that $\alpha\alpha A = \alpha A$ for each $A \subset X$. To do so. we must show that $\alpha\alpha A \subset \alpha A$. Suppose that $x \notin \alpha A$. Then we have $x \bar{\xi} A$. i.e. $x \ll X-A$. Thus there exists a set $E \subset X$ such that $x \ll E$ and $y \bar{\xi} A$ for every $y \in E$. From 2.4(2) and 1.4(2). $X \in E$ and $y \in X - \alpha A$. Therefore $x \in E \subset X - \alpha A \subset X-A$. Consequently $x \bar{\xi} \alpha\alpha A$. that is $x \notin \alpha\alpha A$.

2.8. Remark. The operator α is the closure operator of the topology that it induces: the closed sets are precisely the set of the form αA for each $A \subset X$

Literature Cited

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國 文 抄 錄

본 논문에서는 Kim C. Y. 가 소개한 K−Proximity 공간의 공리를 수정하여, 좀 더 일반화된 Proximity인 B_1 −Proximity를 정의하여 이것에 관한 몇 가지 성질들을 조사하였다.