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The Basic Harmonic forms on a Non-Harmonic Foliation

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Abstract. We study the basic harmonic forms on non-harmonic foliations and prove that on an isoparametric Riemannian foliation with transverse Killing tension field, (i) if the transversal Ricci curvature is quasi-positive, then $H_B^1(\mathcal{F}) = 0$, (ii) if the transversal curvature operator F is quasi-positive, then $H_B^r(\mathcal{F}) = 0$ for $0 < r < q$.

1 Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensitonal Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let ∇^M be the Levi-Civita connection with respect to g_M. Let TM be the tangent bundle of M and L the integrable subbundle of TM given by $\mathcal{F}$. The normal bundle Q of $\mathcal{F}$ is given by $Q = TM/L$. Then there exists an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0. \tag{1.1}$$

Let g_Q be the holonomy invariant metric on Q induced by g_M, that is,

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)) \quad \forall\ s,t \in \Gamma Q \tag{1.2}$$

This means that $\theta(X)g_Q = 0$ for $X \in \Gamma L$, where $\theta(X)$ is the transverse Lie derivative. The transverse Levi-Civita connection ∇ in Q is defined by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L \\ \pi(\nabla_X^M Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{1.3}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. Then we have the following.

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Proposition 1.1 ([3,6]) *The connection ∇ in Q is torsion-free and metrical with respect to g_Q.*

The curvature R_∇ of ∇ is defined by

$$R_\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$$

for any $X,\ Y \in \Gamma TM$ and $s \in \Gamma Q$. Since $i(X)R_\nabla = 0$ for any $X \in \Gamma L$([3]), we can define the transversal Ricci operator $\rho_\nabla : \Gamma Q \to \Gamma Q$ of $\mathcal{F}$ by

$$\rho_\nabla(s) = \sum_{a=1}^{q} R_\nabla(s, E_\alpha)E_\alpha, \tag{1.4}$$

where $\{E_a\}$ is a local orthonormal basic frame of Q. Let $\Omega_B^*(\mathcal{F})$ be the space of all *basic forms* on M, i.e.,

$$\Omega_B^*(\mathcal{F}) = \{\omega \in \Omega^*(M) | i(X)\omega = 0,\ \theta(X)\omega = 0, \quad \forall X \in \Gamma L\}. \tag{1.5}$$

The exterior differential on the de Rham complex $\Omega^*(M)$ restricts by the cartan formula $\theta(X) = di(X) + i(X)d$ to a differential $d_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F})$. The basic cohomology $H_B^*(\mathcal{F}) = H_B(\Omega_B^*(\mathcal{F}), d_B)$ plays the role of the De Rham cohomology of the leaf space $M/\mathcal{F}$ of the foliation. The *mean curvature vector field* τ of $\mathcal{F}$ is defined by

$$\tau = \sum_{i=1}^{p} \pi(\nabla_{E_i}^M E_i), \tag{1.6}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. The mean curvature form κ is defined by $\kappa(Z) = g_Q(\tau, Z)$ for all $Z \in \Gamma Q$.

From now on, let $\mathcal{F}$ be an *isoparametric* foliation, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$. It is well-known ([6]) that if $\kappa \in \Omega_B^1(\mathcal{F})$, it is closed, i.e., $d\kappa = 0$. We also need the star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ naturally associated to g_Q. The relations between $\bar{*}$ and $*$ are characterized by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}),$$
$$*\phi = \bar{*}\phi \wedge \chi_{\mathcal{F}}$$

for $\phi \in \Omega_B^r(\mathcal{F})$, where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator. So we can define a Riemannian metric $\langle\ ,\ \rangle_B$ on $\Omega_B^r(\mathcal{F})$ by

$$\langle \phi, \psi \rangle_B = \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}} \quad \forall \phi, \psi \in \Omega_B^r(\mathcal{F}). \tag{1.7}$$

Then the global inner product is given by

$$\ll \phi, \psi \gg_B = \int_M \langle \phi, \psi \rangle_B.$$

With respect to this scalar product, the adjoint $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ of d_B is given by

$$\delta_B \phi = (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa \wedge) \bar{*} \phi. \tag{1.8}$$

Then the basic Laplacian $\Delta_B = d_B \delta_B + \delta_B d_B$ explicitly involve the mean curvature. Let

$$\mathcal{H}_B^r(\mathcal{F}) = Ker \Delta_B \tag{1.9}$$

be the set of the *basic harmonic forms* of degree r. It is well known [2] that for $\kappa \in \Omega_B^1(\mathcal{F})$,

$$\Omega_B^r(\mathcal{F}) = im d_B \oplus im \delta_B \oplus \mathcal{H}_B^r(\mathcal{F}) \tag{1.10}$$

with finite dimensional $\mathcal{H}_B^r(\mathcal{F})$.

In this paper, we study the basic hrmonic forms under the curvature conditions on the non harmonic foliation.

2 The basic harmonic forms

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Let $\{E_a\}_{a=1,\cdots,q}$ be a local orthonormal basic frame with $(\nabla E_a)_x = 0$ for Q and $\{\theta^a\}$ its g_Q-dual. Then we have

Lemma 2.1 ([1]) *On the Riemannian foliation $\mathcal{F}$, we have*

$$d_B \phi = \sum_a \theta^a \wedge \nabla_{E_a} \phi, \quad \delta_B \phi = -\sum_a i(E_a) \nabla_{E_a} \phi + i(\tau) \phi.$$

Now, we introduce the operator $\nabla_{tr}^* \nabla_{tr} : \Omega_B^*(\mathcal{F}) \to \Omega_B^*(\mathcal{F})$ as

$$\nabla_{tr}^* \nabla_{tr} = -\sum_a \nabla_{E_a, E_a}^2 + \nabla_\tau,$$

where $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ for any $X, Y \in TM$. Then we have

Proposition 2.2 *The operator* $\nabla_{tr}^*\nabla_{tr}$ *satisfies*

$$\ll \nabla_{tr}^*\nabla_{tr}\phi_1, \phi_2 \gg_B = \ll \nabla\phi_1, \nabla\phi_2 \gg_B \tag{2.1}$$

for all $\phi_1,\ \phi_2 \in \Omega_B^*(\mathcal{F})$ *provided that one of* ϕ_1 *and* ϕ_2 *has compact support, where* $\langle\nabla\phi_1, \nabla\phi_2\rangle_B = \sum_a\langle\nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2\rangle$.

Proof. For any $\phi_1,\ \phi_2 \in \Omega_B^*(\mathcal{F})$, we have

$$\begin{aligned}
\langle\nabla_{tr}^*\nabla_{tr}\phi_1, \phi_2\rangle_B &= -\sum_a\langle\nabla_{E_a}\nabla_{E_a}\phi_1, \phi_2\rangle_B + \langle\nabla_\tau\phi_1, \phi_2\rangle_B \\
&= -\sum_a\{E_a\langle\nabla_{E_a}\phi_1, \phi_2\rangle_B - \langle\nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2\rangle_B\} \\
&\quad + \langle\nabla_\tau\phi_1, \phi_2\rangle_B \\
&= -\, div_\nabla(v) + \sum_a\langle\nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2\rangle_B + \langle\nabla_\tau\phi_1, \phi_2\rangle_B,
\end{aligned}$$

where $v \in \Gamma Q$ is defined by the condition that $g_Q(v, w) = \langle\nabla_w\phi_1, \phi_2\rangle_B$ for all $w \in \Gamma Q$. The last line is proved as follows: At $x \in M$,

$$div_\nabla(v) = \sum_a g_Q(\nabla_{E_a}v, E_a) = \sum_a E_a\langle\nabla_{E_a}\phi_1, \phi_2\rangle_B.$$

By the Green's theorem on a foliated Riemannian manifold([7]),

$$\int_M div_\nabla(v) = \ll \tau, v \gg_B = \ll \nabla_\tau\phi_1, \phi_2 \gg_B .$$

Hence the proof is completed. □

Now we define an operator $A_Y : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$ as

$$A_Y\phi = \theta(Y)\phi - \nabla_Y\phi, \tag{2.2}$$

where $\theta(Y)$ is the transverse Lie derivative. Now we define new operator $\tilde{\Delta}$ by

$$\tilde{\Delta} = \Delta_B - A_\tau. \tag{2.3}$$

Then $\tilde{\Delta}$ is a transversally elliptic but it is not self-adjoint. We call $\tilde{\Delta}$ as the generalized basic Laplacian. By a straight calculation, we have

Theorem 2.3 *On the Riemannian foliation $\mathcal{F}$, we have*

$$\tilde{\Delta}\phi = \nabla^*_{tr}\nabla_{tr}\phi + F(\phi) \quad \forall\phi \in \Omega^r_B(\mathcal{F}), \tag{2.4}$$

where $F(\phi) = \sum_{a,b}\theta_a \wedge i(E_b)R_\nabla(E_b, E_a)\phi$.

Proof. Let ϕ be a basic r-form. Let $\{E_a\}$ be a local orthonormal basic frame for Q with $\nabla E_a = 0$ and $\{\theta_a\}$ its g_Q-dual basis. Then we have

$$\begin{aligned}
d_B\delta_B\phi &= \sum_a \theta_a \wedge \nabla_{E_a}\{-\sum_b i(E_b)\nabla_{E_b}\phi + i(\tau)\phi\} \\
&= -\sum_{a,b}\theta_a \wedge \nabla_{E_a}\{i(E_b)\nabla_{E_b}\phi\} + \sum_a \theta_a \wedge \nabla_{E_a} i(\tau)\phi \\
&= -\sum_{a,b}\theta_a \wedge i(E_b)\nabla_{E_a}\nabla_{E_b}\phi + d_B i(\tau)\phi \\
\delta_B d_B\phi &= -\sum_{a,b} i(E_b)\nabla_{E_b}\{\theta^a \wedge \nabla_{E_a}\phi\} + i(\tau)d_B\phi \\
&= -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b}\theta_a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi + i(\tau)d_B\phi
\end{aligned}$$

Summing up the above two equations, we have

$$\begin{aligned}
\Delta_B\phi &= d_B i(\tau)\phi + i(\tau)d_B\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b}\theta_a \wedge i(E_b)R_\nabla(E_b, E_a)\phi \\
&= \theta(\tau)\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b}\theta_a \wedge i(E_b)R_\nabla(E_b, E_a)\phi.
\end{aligned}$$

Hence we have

$$\Delta_B\phi = -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + \nabla_\tau\phi + \sum_{a,b}\theta_a \wedge i(E_b)R_\nabla(E_b, E_a)\phi + A_\tau\phi,$$

which prove (2.4). □

From the Proposition 2.2 and Theorem 2.3, we have the following theorem.

Theorem 2.4 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$. If F is non-negative, $\tilde{\Delta}$-harmonic forms are parallel. If F is quasi-positive, then $Ker\tilde{\Delta} = \{0\}$.*

On the other hand, it is well-known ([7]) that if $\pi(Y)$ is a transverse Killing field, i.e., $\theta(Y)g_Q = 0$ if and only if

$$\langle A_Y\phi, \psi\rangle_B + \langle \phi, A_Y\psi\rangle_B = 0 \quad \forall\phi,\ \psi \in \Omega_B^r(\mathcal{F}). \tag{2.5}$$

From (2.5), if τ is a transverse Killing field, then for any $\phi \in \Omega_B^r(\mathcal{F})$

$$\langle A_\tau\phi, \phi\rangle_B = 0. \tag{2.6}$$

Hence we have from (2.3)

$$\langle \tilde{\Delta}\phi, \phi\rangle_B = \langle \Delta_B\phi, \phi\rangle_B \quad \forall\phi \in \Omega_B^r(\mathcal{F}).$$

By (2.4), if $\phi \in Ker\Delta_B$, then we have

$$0 = |\nabla_{tr}\phi|^2 + \langle F(\phi), \phi\rangle_B.$$

Hence we have the following theorem.

Theorem 2.5 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Assume that the tension field τ is a transverse Killing field. If F is quasi-positive, then every basic harmonic r-forms is zero. i.e., $\mathcal{H}_B^r(\mathcal{F}) = 0$.*

Remark. If $\mathcal{F}$ is minimal, $\Delta_B = \tilde{\Delta}$.

Let ϕ be a basic 1-form and $\phi^\sharp$ its g_Q-dual. Then we have

$$\begin{aligned}
\langle F(\phi), \phi\rangle &= \sum_{a,b}\langle \theta^a \wedge i(E_b)R_\nabla(E_b, E_a)\phi, \phi\rangle \\
&= \sum_{a,b} i(E_b)R_\nabla(E_b, E_a)\phi\langle \theta^a, \phi\rangle \\
&= \sum_{a,b} g_Q(R_\nabla(E_b, E_a)\phi^\sharp, E_b)\langle \theta^a, \phi\rangle \\
&= \sum_{a} g_Q(R_\nabla(\phi^\sharp, E_a)E_a), \phi^\sharp) = g_Q(\rho_\nabla(\phi^\sharp), \phi^*),
\end{aligned}$$

where ρ_∇ is the transversal Ricci curvature. From this equation, we have the following corollary.

Corollary 2.6 *Under the same assumptions as in Theorem 2.5, If the transversal Ricci curvature is non-negative, then every basic harmonic 1-form is parallel. If the transversal Ricci curvature is quasi positive, then every basic harmonic 1-form is zero, i.e., $\mathcal{H}_B^1(\mathcal{F}) = 0$.*

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