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ON THE LIMIT OF SOLUTIONS FOR SDI ON FINITE DIMENSIONAL SPACE

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ABSTRACT. For the stochastic differential inclusion of the form $dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt$, where σ, b are set-valued maps, B is a standard Brownian motion, we study the limit of solutions.

1. INTRODUCTION

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a right-continuous increasing family $(\mathfrak{F}_t)_{t\geq 0}$ of sub σ-fields of $\mathfrak{F}$ each containing all P-null sets. Let $B = (B_t)_{t\geq 0}$ be an r-dimensional $(\mathfrak{F}_t)$-Brownian motion. We consider the following stochastic differential inclusion.

$$dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt, \tag{1.1}$$

where $\sigma : [0,T] \times \mathbf{R}^d \to \mathcal{P}(\mathbf{R}^d \otimes \mathbf{R}^r)$, $b : [0,T] \times \mathbf{R}^d \to \mathcal{P}(\mathbf{R}^d)$ are set-valued maps. In recent years the study of the existence and properties of solution for these stochastic differential inclusions have been developed by many authors ([4]). Furthermore the results for the viable solutions have been made ([2], [6]). For the stochastic differential equation associated with (1.1), many results for the existence, uniqueness and properties of solutions have been done under various conditions that σ and b are continuous and bounded or Lipschitzean or Hölder continuous ([3]). We proved the existence of solution for stochastic differential inclusion (1.1) under the condition that σ and b satisfy the local Lipschitz property and linear growth ([7]). Furthermore we proved any solution for stochastic differential inclusion (1.1) is bounded ([9]).

In this paper, we study the limit of solutions for stochastic differential inclusion (1.1).

2. Preliminaries

We prepare the definition of solution for stochastic differential inclusion and some results for the stochastic differential equation and selection theorems.

Definition 2.1. An r-dimensional continuous process $B = (B_t)_{t\in[0,\infty)}$ is called an r-dimensional $(\mathfrak{F}_t)$-Brownian motion if it is $(\mathfrak{F}_t)$-adapted and satisfies

$$E[\exp[i < \xi, B_t - B_s >] \mid \mathfrak{F}_s] = \exp[-(t-s)|\xi|^2/2], \quad \text{a.s.}$$
$$\text{for every } \xi \in \mathbf{R}^r \text{ and } 0 \le s < t.$$

Let us consider the stochastic differential inclusion

$$dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt \tag{1.1}$$

with the initial value $X_0 = x_0$, where $\sigma : [0,T]\times\mathbf{R}^d \to \mathbf{R}^d\otimes\mathbf{R}^r$, $b : [0,T]\times\mathbf{R}^d \to \mathbf{R}^d$ are set-valued maps and x_0 is a $\mathbf{R}^d$-valued $\mathfrak{F}_0$-measurable function.

Definition 2.2. A predictable continuous stochastic process $X = \{X_t,\ t \in [0,T]\}$ is said to be a solution of (1.1) on $[0,T]$ with the initial condition x_0 if there are predictable random processes $f : \Omega \times [0,T] \to \mathbf{R}^d \otimes \mathbf{R}^r$, $g : \Omega \times [0,T] \to \mathbf{R}^d$ such that $f(t) \in \sigma(t, X_t)$, $g(t) \in b(t, X_t)$ a.s. on $[0,T]$ and for every $t \in [0,T]$,

$$X_t = x_0 + \int_0^t f(s)\, dB_s + \int_0^t g(s)\, ds \quad \text{a.s..}$$

For the stochastic differential equation

$$X_t = \xi + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds, \tag{2.1}$$

where $\sigma : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^r$, $b : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$ are $\mathfrak{B}([0,T]) \otimes \mathfrak{B}(\mathbf{R}^d) \otimes \mathfrak{F}_T$-measurable and $\mathfrak{F}_t$-progressively measurable for each $x \in \mathbf{R}^d$, ξ is $\mathfrak{F}_0$-measurable, the following theorems are well known.

Theorem 2.3. ([5]) We assume the followings.

(i) For each $N > 0$, there exists a constant $C_N > 0$ such that

$$\begin{cases} \|\sigma(t,x) - \sigma(t,y)\| \le C_N \cdot |x-y|, & x, y \in B_N \\ |b(t,x) - b(t,y)| \le C_N \cdot |x-y|, & x, y \in B_N, \end{cases}$$

where $B_N = \{x \in \mathbf{R}^d, |x| \le N\}$ and $\|\sigma\|^2 = \sum_{j=1}^r \sum_{i=1}^d |\sigma_j^i|^2 \equiv \mathrm{tr}(\sigma\sigma^*)$.

(ii) There exists a constant $K > 0$ such that

$$\frac{1}{2}||\sigma(t,x)||^2 + x^* \cdot b(t,x) \leq K(r(t)^2 + |x|^2),$$

where $r(t)$ is a progressively measurable such that

$$E\left[\ |\xi|^2 + \int_0^T \{|b(s,0)|^2 + r(s)^2\}ds\ \right] < \infty.$$

Then (2.1) has unique solution X_t and

$$E[|X_t|^2] \leq E\left[\ |\xi|^2 + 2K\int_0^t r(s)^2 ds\ \right] e^{2Kt}, \quad \forall t \leq T.$$

For a Banach space X with the norm $||\cdot||$ and for non-empty sets A, A' in X, we denote $||A|| = \sup\{||a|| \mid a \in A\}$, $d(a, A') = \inf\{d(a, a') \mid a' \in A'\}$, $d(A, A') = \sup\{d(a, A') \mid a \in A\}$ and $d_H(A, A') = \max\{d(A, A'), d(A', A)\}$, a Hausdorff metric. Given a family of sets $\{F_\alpha \mid \alpha \in A\}$, a selection is a map $\alpha \to f_\alpha$ in F_α. The most famous continuous selection theorem is the following result by Michael.

Theorem 2.4. ([1]) Let X be a metric space, Y a Banach space. Let F from X into the closed convex subsets of Y be lower semi-continuous. Then there exists $f : X \to Y$, a continuous selection from F.

Proof. Step 1. Let us given by proving the following claim : given any convex (not necessarily closed) valued lower semi-continuous map Φ and every $\varepsilon > 0$, there exists a continuous $\phi : X \to Y$ such that for ξ in $X, d(\phi(\xi), \Phi(\xi)) \leq \varepsilon$.

In fact, for every $x \in X$, let $y_x \in \Phi(x)$ and let $\delta_x > 0$ be such that $(y_x + \varepsilon \mathring{A}) \cap \Phi(x') \neq \emptyset$ for x' in $B(x, \delta_x)$, where $\mathring{A}$ denotes the open unit ball. Since X is metric, it is paracompact. Hence there exists a locally finite refinement $\{\mathfrak{U}_x\}_x \in X$ of $\{B(x, \delta_x)\}_x$. Let $\{\pi_x(\cdot)\}_x$ be a partition of unity subordinate to it. The mapping $\varphi : X \to Y$ given by $\varphi(\xi) = \sum \pi_x(\xi) y_x$ is continuous since it is locally a finite sum of continuous functions. Fix ξ. Whenever $\pi_x(\xi) > 0$, $\xi \in \mathfrak{U}_x \subset B(x, \delta_x)$, hence $y_x \in \Phi(\xi) + \varepsilon \mathring{A}$. Since this latter set is convex, any convex combination of such y's (in particular, $\varphi(\xi)$) belongs to it.

Step 2. Next we claim that we can define a sequence $\{f_n\}$ of continuous mappings from X into Y with the following properties

i) for each $\xi \in X, d(f_n(\xi), F(\xi)) \leq \frac{1}{2^n}, \quad n = 1, 2, \cdots,$

ii) for each $\xi \in X, ||f_n(\xi) - f_{n-1}(\xi)|| \leq \frac{1}{2^{n-2}}, \quad n = 2, \cdots.$

For $n = 1$ it is enough to take in the claim of part Step 1, $\Phi = F$ and $\varepsilon = 1/2$. Assume we have defined mappings f_n satisfying i) up to $n = \nu$. We shall define $f_{\nu+1}$ satisfying i) and ii) as follows. Consider the set $\Phi(\xi) \doteq (f_\nu(\xi) + \frac{1}{2^\nu}\mathring{A}) \cap F(\xi)$. By i) it is not empty, and it is a convex set. The map $\xi \to \Phi(\xi)$ is lower semicontinuous and by the claim of Step 1, there exists a continuous φ such that $d(\varphi(x), \Phi(x)) \leq \frac{1}{2^{\nu+1}}$. Set $f_{\nu+1}(\xi) \doteq \varphi(\xi)$. A fortiori $d(f_{\nu+1}(\xi), F(\xi)) \leq \frac{1}{2^{\nu+1}}$, proving i). Also $f_{\nu+1}(\xi) \in \Phi(\xi) + \frac{1}{2^{\nu+1}}\mathring{A} \subset f_\nu(\xi) + (\frac{1}{2^\nu} + \frac{1}{2^{\nu+1}})\mathring{A}$ i.e.,

$$||f_{\nu+1}(\xi) - f_\nu(\xi)|| \leq \frac{1}{2^{\nu-1}}$$

proving ii).

Step 3. Since the series $\sum \frac{1}{2^n}$ converges, $\{f_n(\cdot)\}$ is a Cauchy sequence, uniformly converging to a continuous $f(\cdot)$. Since the values of F are closed, by i) of part Step 2, f is a selection from F. □

Let $A \subset \mathbf{R}^n$ be a compact convex body, i.e., a compact set with nonempty interior, and let m_n be the n-dimensional Lebesgue measure. Since $m_n(A)$ is positive, we can define the barycenter of A as

$$b(A) = \frac{1}{m_n(A)} \int_A x \, dm_n.$$

Lemma 2.5. ([1]) The barycenter of A, $b(A)$, belongs to A.

Proof. Assume the contrary: $d(b(A), A)$ is positive. Set a to be $\pi_A(b(A))$, b to be $b(A)$ and $p \doteq b - a$.

By the characterization of the best approximation we have that for all x in A, $< x - a, p > \leq 0$. However from

$$p = b - a = \frac{1}{m_n(A)} \int_A (x - a) dm_n$$

we have

$$\begin{aligned} ||p||^2 &= < \frac{1}{m_n(A)} \int_A (x - a) dm_n, p > \\ &= \frac{1}{m_n(A)} \int_A < x - a, p > dm_n \leq 0, \end{aligned}$$

a contradiction; hence $b(A)$ belongs to A. □

Lemma 2.6. ([1]) Let $A \subset \mathbf{R}^n$ be compact and convex and consider $A^1 \doteq A + B$, where B is the closed unit ball. Then $b(A^1)$ belongs to A.

Proof. As above assume it is not so. Set a to be $\pi_A(b(A^1))$, the point of A nearest to $b = b(A^1)$, set $p \doteq b - a$ and $\hat{p} = p/||p||$. Then

$$||p||^2 = \frac{1}{m_n(A^1)} \int_{A^1} < x - a, p > dm_n \tag{2.2}$$

and as, before, to reach a contradiction it is enough to show that the right hand side is non positive. It is convenient to consider S_P, the linear transformation mapping x into its symmetric with respect to the hyperplane orthogonal to p through a:

$$S_P(x) = a + (x - a) - 2 < x - a, \hat{p} > \hat{p}$$

Set $A^1_+ \doteq \{a \in A^1 | < x - a, p > > 0\}$, $A^1_- \doteq \{x \in A^1 | < x - a, p > \leq 0\}$. We remark that $S_P(A^1_+) \subset A^1$. In fact fix x in A^1_+ and consider $S_P(x)$:

Set x' to be the projection of $\pi_A(x)$ on the line through x and $S_P(x)$. By the Pythagorean theorem to show that $||x - \pi_A(x)|| \geq ||S_P(x) - \pi_A(x)||$ it is enough to show that $||x - x'|| \geq ||S_P(x) - x'||$. We have that

$$||x - x'|| = < x - x', \hat{p} > = < x - a, \hat{p} > - < x' - a, \hat{p} >$$

and

$$\begin{aligned} ||S_P(x) - x'|| &= - < S_P(x) - x', \hat{p} > = - < S_P(x) - a, \hat{p} > + < x' - a, \hat{p} > \\ &= < x - a, \hat{p} > + < x' - a, \hat{p} > . \end{aligned}$$

Since, again by the characterization of the best approximation, x' belongs to A^1_-,

$$d(S_P(x), A) \leq ||S_P(x) - \pi_A(x)|| \leq ||x - \pi_A(x)|| = d(x, A) \leq 1.$$

Then $S_P(x)$ belongs to A^1. Write A^1 as $(A^1_+ \cup S_P(A^1_+)) \cup (A^1 \backslash (A^1_+ \cup S_P(A^1_+)))$ and consider the integral in (2.2) separately on these two subsets. Remark that the first is invariant with respect to the transformation S_P, that the determinant of the Jacobian of the transformation S_P is one and that the map $x \to < x - a, \hat{p} >$ is antisymmetric with respect to S_P. The change of variables formula hence yields

$$\begin{aligned} \int_{S_P(A^1_+ \cup S_P(A^1_+))} < x - a, p > &= \int_{(A^1_+ \cup S_P(A^1_+))} < S_P(x) - a, p > \\ &= - \int_{S_P(A^1_+ \cup S_P(A^1_+))} < x - a, p > . \end{aligned}$$

Hence this integral is zero. Since $A^1 \backslash (A^1_+ \cup S_P(A^1_+))$ is contained in A^1_-,

$$\int_{A^1} < x - a, p > \leq 0$$

the desired contradiction. □

Using Lemma 2.5 and 2.6, we have the following local Lipschitz barycentric selection theorem.

Theorem 2.7. ([8]) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a local Lipschitz set-valued map with compact convex images, i.e., there exists a constant $K_N > 0$ such that

$$d_H(F(x), F(y)) \leq K_N \cdot |x-y|, \quad \forall x, y \in B_N = \{x \in \mathbb{R}^n, |x| \leq N\}.$$

Assume moreover that there exists a constant $C > 0$ such that $||F(x)|| \leq C \cdot (1+|x|)$, for every $x \in \mathbb{R}^n$. Then there exist a constant $\hat{C}_N > 0$ and a single valued map $f : \mathbb{R}^n \to \mathbb{R}^n$, local Lipschitzean with constant $\hat{C}_N$, a selection from F.

Proof. By Lemma 2.5 and 2.6, the single valued map $b^1 = x \to b(F(x) + B)$ is a selection from F. We have to prove that it is a local Lipschitzean selection.

Fix $x, x' \in B_N$. Call $\Phi(x) \doteq F(x) + B, \Phi'(x') \doteq F(x') + B$. Since $||\Phi(x)|| \leq ||F(x) + B|| \leq ||F(x)|| + 1 \leq C \cdot (1 + |x|) + 1 \leq C \cdot (1 + N) + 1 = C_{N'}$ and $m_n(\Phi(x)) \leq C_{N''}$, we have

$$\begin{aligned}
&\frac{1}{m_n(\Phi(x))} \int_{\Phi(x)} x \, dm_n - \frac{1}{m_n(\Phi'(x'))} \int_{\Phi'(x')} x \, dm_n \\
&\leq \Big| \Big(\frac{1}{m_n(\Phi(x))} - \frac{1}{m_n(\Phi'(x'))} \Big) \int_{\Phi(x) \cap \Phi'(x')} x \, dm_n \Big| \\
&\quad + \Big| \frac{1}{m_n(\Phi(x))} \int_{\Phi(x) \setminus \Phi'(x')} x \, dm_n - \frac{1}{m_n(\Phi'(x'))} \int_{\Phi'(x') \setminus \Phi(x)} x \, dm_n \Big| \\
&\leq |m_n(\Phi(x)) - m_n(\Phi'(x'))| \cdot C_{N'} \cdot C_{N''} / (m_n(B))^2 \\
&\quad + \{ m_n(\Phi(x) \setminus \Phi'(x')) + m_n(\Phi'(x') \setminus \Phi(x)) \} \cdot C_{N'} \cdot C_{N''} / m_n(B).
\end{aligned} \tag{2.3}$$

We wish to express the above estimate in terms of $d_H(\Phi, \Phi')$. For this purpose, we begin to compare $m_n(\Phi + \delta B)$, $\delta > 0$, and $m_n(\Phi)$. Since the unit ball of $\mathbb{R}^n$ is contained in the unit cube $\{|x_i| \leq 1, i = 1, \cdots, n\}$, we can as well estimate

$$m_n\{\varphi + \sum \delta_i e_i \mid \varphi \in \Phi, |\delta_i| \leq \delta\}$$

where $\{e_i\}$ is an orthonormal basis. From elementary calculus we have that when S is a convex set and ν a unit vector, the measure of $\{S + \delta_x \nu \mid |\delta_x| \leq \delta\}$ is $m_n(S) + |\delta| m_{n-1}(P_\nu(S))$ where P_ν is the projection of S into the hyperplane normal to ν through the origin ($P_\nu(S)$ is the "shadow" of S). Denote by

$$\Phi_\nu \doteq \{\varphi + \sum_{i=1}^{\nu} \delta_i e_i | \varphi \in \Phi, \delta_i \leq \delta\}$$

and by P_i the projection along the direction e_i. Recursively we obtain

$$m_n(\Phi_n) \leq m_n(\Phi) + \delta \sum_{j=0}^{n-1} m_{n-1}(P_{n-j}(\Phi_{n-j})).$$

Since Φ is contained in $(M+1)B$, each element of each $P_j(\Phi_j)$ has a distance from the origin of at most $(M+1)+\delta\sqrt{n}$, so that, setting B_{n-1} the unit ball in $\mathbb{R}^{n-1}$,

$$\begin{aligned} m_n(\Phi+\delta B) &\le m_n(\Phi_n) \\ &\le m_n(\Phi)+\delta n m_{n-1}((M+1+\delta\sqrt{n})B_{n-1}) \\ &\le m_n(\Phi)+\delta K \end{aligned}$$

for some constant K. Set δ to be $d_H(\Phi,\Phi')$. Then $\Phi'\subset\Phi+\delta B$ and $\Phi\subset\Phi'+\delta B$, hence $m_n(\Phi\backslash\Phi')\le m_n(\Phi'+\delta B)-m_n(\Phi')$, and $m_n(\Phi'\backslash\Phi)\le m_n(\Phi+\delta B)-m_n(\Phi)$. Analogously, $|m_n(\Phi)-m_n(\Phi')|\le K\delta$. Hence by (2.3), we obtain

$$|b(F(x)+B)-b(F(x')+B)|\le C'_N\cdot d_H(F(x)+B,F(x')+B)$$

for a suitable C'_N. Finally, since K_N is the local Lipschitz constant of F and set $\hat{C}_N$ to be $K_N\cdot K$. We have

$$\begin{aligned} |b^1(x)-b^1(x')| &\le K\cdot d_H(F(x)+B,F(x')+B) \\ &\le K\cdot d_H(F(x),F(x'))\le \hat{C}_N\cdot d(x,x'), \end{aligned}$$

i.e. $f=b^1$ is the required Lipschitzean selection. □

Thus we have the following another main theorem by the above lemmas and Theorem 2.7.

Theorem 2.8. ([7]) Assume that

(i) for each $N>0$, there exist constants $C>0$ and $C_N>0$ such that

$$\begin{cases} d_H(\sigma(t,x)-\sigma(t,y))\le C_N\cdot|x-y|, & x,y\in B_N,\\ d_H(b(t,x)-b(t,y))\le C_N\cdot|x-y|, & x,y\in B_N,\\ ||\sigma(t,x)||+|b(t,x)|\le C\cdot(1+|x|), & x\in\mathbb{R}^n, \end{cases}$$

where $B_N=\{x\in\mathbb{R}^d,|x|\le N\}$,

(ii) there exists a constant $K>0$ such that

$$\frac{1}{2}||\sigma(t,x)||^2+|x|\cdot|b(t,x)|\le K(r(t)^2+|x|^2),$$

where $r(t)$ is a progressively measurable such that

$$E\left[\ |x_0|^2+\int_0^T\{|b(s,0)|^2+r(s)^2\}\ \right]ds<\infty.$$

Then (1.1) has a solution X_t and

$$E[|X_t|^2]\le E\left[\ |x_0|^2+2K\int_0^t r(s)^2ds\ \right]e^{2Kt},\ \ \forall t\le T.$$

Proof. By the hypothesis (i) and Theorem 2.7, σ and b have local Lipschitzean selection. Thus the proof is complete by Theorem 2.3. □

3. Main Results

Theorem 3.1. ([8]) Assume that $\sigma : [0,T] \times \mathbf{R}^d \to \mathcal{P}(\mathbf{R}^d \otimes \mathbf{R}^r)$, $b : [0,T] \times \mathbf{R}^d \to \mathcal{P}(\mathbb{R}^d)$ are closed convex set-valued which are Lipschitz, i.e., there exists a constant $L > 0$ such that

$$\begin{cases} d_H(\sigma(t,x), \sigma(t,y)) \le L \cdot |x-y|, \\ d_H(b(t,x), b(t,y)) \le L \cdot |x-y|. \end{cases}$$

Then there exists a solution $X \in \Lambda^q = L^q(\Omega \to C([0,T] \to \mathbf{R}^d))$ for the stochastic differential inclusion (1.1).

Proof. For $X \in \Lambda^q$, let

$$S(X) = \Big\{ \theta \in \Lambda^q \mid \theta_t = x_0 + \int_0^t f(s)dB_s + \int_0^t g(s)ds, \\ f(s) \in \sigma(s, X_s),\ g(s) \in b(s, X_s),\ f, g : \text{predictable} \Big\}.$$

The proof is complete if we prove that there exists a fixed point for the map $S : \Lambda^q \to \mathfrak{P}(\Lambda^q)$, where $\mathfrak{P}(\Lambda^q) = \{A \subset \Lambda^q \mid A \text{ is bounded and closed in } C([0,T] \to \mathbf{R}^d) \text{ a.s.}\}$.

For the existence of fixed point, we have to prove that S is a contraction map for sufficiently small T, i.e., there exists $\rho \in (0,1)$ such that $d_H(S(X), S(Y)) \le \rho ||X - Y||_{\Lambda^q}$.

For closed convex set $C \subset \mathbf{R}^d$, define $P_C(x) \in \mathbf{R}^d$ by

$$||x - P_C(x)|| = d(x, C).$$

Then $P_C(x)$ exists uniquely. Let $Z \in S(X)$ and $Y \in \Lambda^q$. Then there exist $f_s \in \sigma(s, X_s)$ and $g_s \in b(s, X_s)$ such that

$$Z_t = x_0 + \int_0^t f_s dB_s + \int_0^t g_s ds.$$

Define $\hat{f}_s$, $\hat{g}_s$ by

$$\hat{f}_s = P_{\sigma(s,Y_s)}(f_s) \text{ and } \hat{g}_s = P_{b(s,Y_s)}(g_s).$$

By hypothesis,

$$|f_s - \hat{f}_s| \le d_H(\sigma(s, X_s), \sigma(s, Y_s)) \le L \cdot |X_s - Y_s| \quad \text{and}$$
$$|g_s - \hat{g}_s| \le d_H(b(s, X_s), b(s, Y_s)) \le L \cdot |X_s - Y_s|.$$

Letting $\hat{Z}_t = x_0 + \int_0^t \hat{f}_s dB_s + \int_0^t \hat{g}_s ds$, we have $\hat{Z} \in S(Y)$. Note that

$$\begin{aligned} E\left[\sup_{0\le t\le T} |Z_t - \hat{Z}_t|^q\right] &\le C_T \cdot E\left[\int_0^T |f_s - \hat{f}_s|^q ds\right] + C_T \cdot E\left[\int_0^T |g_s - \hat{g}_s|^q ds\right] \\ &\le L \cdot C_T \cdot \int_0^T E[|X_s - Y_s|^q] ds + L \cdot C_T \cdot \int_0^T E[|X_s - Y_s|^q] ds \\ &\le 2L \cdot C_T \cdot T \cdot E\left[\sup_{0\le s\le T} |X_s - Y_s|^q\right] \\ &= 2L \cdot C_T \cdot T \cdot \|X - Y\|_{\Lambda^q}^q. \end{aligned}$$

Thus for every $Z \in S(X)$, there exists $\hat{Z} \in S(Y)$ such that $\|Z - \hat{Z}\|_{\Lambda^q}^q \le 2L \cdot C_T \cdot T\|X - Y\|_{\Lambda^q}^q$. Therefore $d_H(S(X), S(Y)) \le (2L \cdot C_T \cdot T)^{1/q} \cdot \|X - Y\|_{\Lambda^q}$. Taking T sufficiently small, it can be that $2L \cdot C_T \cdot T < 1$. Hence S is a contraction map. Connecting the solutions, we can prove the existence of the solution X_t of (1.1) on $[0, T]$. □

Theorem 3.2. ([9]) Let X_t be any solution of (1.1). Then X_t is bounded, i.e., for $p \ge 2$,

$$E[\sup_{0\le s\le t} |X_s|^p] < \infty.$$

Proof. Let X_t be a solution. Then there exist $f_s \in \sigma(X_s)$ and $g_s \in b(X_s)$ such that

$$X_t = x + \int_0^t f_s dB_s + \int_0^t g_s ds.$$

Since

$$\begin{aligned} E[\sup_{0\le s\le t} |X_s|^p] &\le 3^{p-1}|x|^p + 3^{p-1}C_1 E\left[\{\int_0^t |f_s|^2 ds\}^{p/2}\right] \\ &\quad + 3^{p-1}E\left[\{\int_0^t |g_s|^2 ds\}^p\right] \\ &\le 3^{p-1}|x|^p + 3^{p-1}C_1 E\left[\{\int_0^t |f_s|^p ds\}\{\int_0^t 1 ds\}^{\frac{p-2}{2}}\right] \\ &\quad + 3^{p-1}E\left[\int_0^t |g_s|^p ds\{\int_0^t 1 ds\}^{p-1}\right] \end{aligned}$$

$$\leq 3^{p-1}|x|^p + 3^{p-1}C_1T^{\frac{p-2}{2}} \int_0^t E[|f_s|^p]ds$$
$$+ 3^{p-1}T^{p-1} \int_0^t E[|g_s|^p]ds$$
$$\leq 3^{p-1}|x|^p + 3^{p-1}C_1T^{\frac{p-2}{2}} \int_0^t E[|\sigma(X_s)|^p]ds$$
$$+ 3^{p-1}T^{p-1} \int_0^t E[|b(X_s)|^p]ds$$
$$\leq 3^{p-1}|x|^p + 3^{p-1}C_1T^{\frac{p-2}{2}} \int_0^t K^p(1 + E[|X_s|^p])2^{p-1}ds$$
$$+ 3^{p-1}T^{p-1} \int_0^t K^p(1 + E[|X_s|^p])2^{p-1}ds,$$

if we put $\varphi(t) = E[\sup_{0\leq s\leq t} |X_s|^p]$,

$$\varphi(t) \leq 3^{p-1}|x|^p + 6^{p-1}K^pT^{\frac{p}{2}}C_1 + 6^{p-1}K^pT^{\frac{p-2}{2}}C_1 \int_0^t \varphi(s)ds$$
$$+ 6^{p-1}K^pT^p + 6^{p-1}K^pT^{p-1} \int_0^t \varphi(s)ds$$
$$= 3^{p-1}|x|^p + 6^{p-1}K^pT^{\frac{p}{2}}(C_1 + 1)$$
$$+ 6^{p-1}K^p(T^{\frac{p-2}{2}}C_1 + T^{p-1}) \int_0^t \varphi(s)ds.$$

By Gronwall's inequality,

$$\varphi(t) \leq (3^{p-1}|x|^p + 6^{p-1}K^pT^{\frac{p}{2}}(C_1 + 1)) \cdot \exp(6^{p-1}K^p(T^{\frac{p-2}{2}}C_1 + T^{p-1})t).$$

Hence X_t is bounded. □

Let $\{X_t^n\}_{n=1,2,\cdots}$ be a sequence of solutions of (1.1) converging to X_t, i.e.,

$$\lim_{n\to\infty} E[\sup_{0\leq t\leq T} |X_t^n - X_t|^p] = 0.$$

Since $\{X_t^n\}$ are solutions of (1.1), there exist sequences $\{\xi_t^n\}$ and $\{\eta_t^n\}$ such that

$$X_t^n = x + \int_0^t \xi_s^n dB_s + \int_0^t \eta_s^n ds.$$

Put $\hat{\xi}_t^n = P_{\sigma(X_t)}(\xi_t^n)$ and $\hat{\eta}_t^n = P_{b(X_t)}(\eta_t^n)$. Then by hypothesis,

$$|\hat{\xi}_t^n - \xi_t^n| \leq d_H(\sigma(X_t), \sigma(X_t^n)) \leq L|X_t - X_t^n|$$
$$|\hat{\eta}_t^n - \eta_t^n| \leq d_H(b(X_t), b(X_t^n)) \leq L|X_t - X_t^n|.$$

Note that $\{\hat{\xi}_t^n\}$ and $\{\hat{\eta}_t^n\}$ are L^p-bounded. In fact,

$$E[\int_0^T |\hat{\xi}_t^n|^p dt] \leq E[\int_0^T |\sigma(X_t)|^p dt] \leq E[2^p K^p \int_0^T (1 + |X_t|^p) dt].$$

Thus there exists a weak convergent subsequence. For simplicity, assume that $\{\hat{\xi}_t^n\}$ converges to $\hat{\xi}_t^\infty$ weakly. Since the sequence does not converge strongly, we have to take some subsequence which converges strongly. Let K_n be the set of all convex combinations of $\hat{\xi}_t^n, \hat{\xi}_t^{n+1}, \hat{\xi}_t^{n+2}, \cdots$. Then since $\bar{K}_n$ is closed convex and weak-closed, $\hat{\xi}_t^\infty \in \bar{K}_n$. Thus $\hat{\xi}_t^\infty \in \bigcap_n \bar{K}_n$.

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유한차원 공간에서의 확률포함방정식의 해의 극한에 관하여

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집합치 함수들로 주어지는 다음과 같은 확률포함방정식의 해로서 주어지는 함수열의 극한에 관하여 연구하였다.

$$dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt$$

여기서 σ와 b는 집합치 함수이고 B_t는 standard Brownian motion이다. 위와 같이 주어진 확률포함방정식의 해의 존재성은 저자가 이미 증명하였고 존재하는 해들이 어떤 의미로 유계라는 것도 증명하였다. 본 논문에서는 해들로 이루어진 함수열의 극한에 관한 성질을 연구하였는데 이는 앞으로 해들의 연속성을 증명하는데 이용되어질 것이라고 사료된다.