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# ON THE WEYL'S SPECTRUM OF WEIGHT II

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ABSTRACT. We study the properties of $\alpha-$Fredholm operators and the Weyl's spectrum of weight $\alpha$, $\omega_\alpha(T)$, of an operator. We show that similarity preserves $\alpha-$Weyl's theorem and give a condition for an operator to be of the form unitary+$\alpha-$compact. We also introduce the class $W_\alpha$ for any cardinal $\alpha$ and study its properties.

## 0. Introduction

Throughout the paper, $H$ denotes a fixed (complex) Hilbert space of dimension $h \geq \aleph_0$, the cardinality of the set of natural numbers and we write $B(H)$ for the set of all bounded linear operators on $H$. For each cardinal $\alpha$ with $\aleph_0 \leq \alpha \leq h$, let $I_\alpha$ denote the two-sided ideal in $B(H)$ of all bounded operators of rank less than $\alpha$ and let $\mathfrak{I}_\alpha$ denote the uniform closure of $I_\alpha$. Then the $\mathfrak{I}_\alpha$ are precisely the proper closed two-sided ideals of $B(H)$. Of course, $\mathfrak{I}_{\aleph_0}$ is the ideal of compact operators and $\mathfrak{I}_h$ is the maximal closed two-sided ideal of $B(H)$. If $\aleph_0 \leq \alpha < \beta \leq h$, then $\mathfrak{I}_\alpha \subseteq \mathfrak{I}_\beta$ and $\mathfrak{I}_\alpha \neq \mathfrak{I}_\beta$. For each operator $T$, $\hat{T}$ denotes the coset $T + \mathfrak{I}_\alpha$ in the $C^*-$algebra $B(H)/\mathfrak{I}_\alpha$. The ordinary spectrum of the canonical image $\hat{T}$ of $T$ in the quotient $C^*-$algebra $B(H)/\mathfrak{I}_\alpha$ is called the spectrum of $T$ of weight $\alpha$ and denoted by $\sigma_\alpha(T)$. That is, $\sigma_\alpha(T)$ is the collection of all complex numbers $\lambda$ such that $T - \lambda I$ is not invertible modulo $\mathfrak{I}_\alpha$. Hence $\sigma_\alpha(T)$ is nonempty and compact [3]. $\pi_\alpha(T)$ is used to denote the approximate point spectrum of $\hat{T}$. If $T$ is $\alpha-$compact, i.e., $T \in \mathfrak{I}_\alpha$, then $\sigma_\alpha(T) = \sigma(\hat{T}) = \{0\}$. Since $\mathfrak{I}_\alpha$ are self-adjoint ideals, Re $\sigma_\alpha(T) = \{0\} = \sigma_\alpha(\text{Re } T)$.

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In [7], Yadav and Arora defined the Weyl's spectrum of weight $\alpha$, $\omega_\alpha(T)$, of an operator $T$ on $H$ by

$$\omega_\alpha(T) = \cap_{K \in \mathfrak{I}_\alpha} \sigma(T+K).$$

We say [7] that *$\alpha-$ Weyl's theorem holds for $T$* if

$$\sigma(T) - \omega_\alpha(T) = \pi_{0\alpha}(T)$$

where $\pi_{0\alpha}(T)$ denotes the set of all isolated eigenvalues of multiplicity less than $\alpha$. For each operator $T$, $\omega_\alpha(T)$ is a nonempty compact subset of $\sigma(T)$ [See Theorem 2.1], and if $T$ is normal then $\sigma_\alpha(T) = \omega_\alpha(T) = \pi_\alpha(T)$ [3, Corollary 4.7.1]. $0 \notin \omega_\alpha(T)$ if and only if $T$ is of the form $S+K$, where $S$ is invertible and $K \in \mathfrak{I}_\alpha$. Again it follows from the selfadjointness of the ideal $\mathfrak{I}_\alpha$ that $\overline{\omega_\alpha(T)} = \omega_\alpha(T^*)$ for any operator $T$.

In [8], Yang introduced the class $W$ of operators as follows: A bounded linear operator $T$ in $B(H)$ is said to belong to class $W$ if

$$\sigma_e(T) = \omega(T) \ ,$$

where $\sigma_e(T)$ denote the essential spectrum of $T$. Motivated by this we say that a bounded linear operator $T$ in $B(H)$ belongs to class $W_\alpha$ if

$$\sigma_\alpha(T) = \omega_\alpha(T) \ .$$

If $T$ is a normal operator then $\sigma_\alpha(T) = \omega_\alpha(T)$ [3].

In this paper, we study the properties of $\alpha-$Fredholm operators and the Weyl's spectrum of weight $\alpha$, $\omega_\alpha(T)$, of an operator. We show that similarity preserves $\alpha-$Weyl's theorem and give a condition for an operator to be of the form unitary+$\alpha-$compact. We also introduce the class $W_\alpha$ for any cardinal $\alpha$ and study its properties.

## 1. $\alpha$-Fredholm Operators

We recall ([3]) that a subspace $K$ of a Hilbert space $H$ is called $\alpha$-closed if there is a closed subspace $L$ of $H$ such that $L \subset K$ and $\dim(K \cap L^\perp) < \alpha$ and an operator $T$ on $H$ is an $\alpha$-Fredholm operator if $v(T) < \alpha, \rho'(T) < \alpha$ and range of $T$ is $\alpha$-closed, where $v(T)$ is nullity of $T$ and $\rho'(T)$ is corank of $T$.

**Theorem 1.1.** *Let $T$ and $S$ be commuting operators in $B(H)$. Then $TS$ is an $\alpha$-Fredholm operator if and only if $T$ and $S$ both are $\alpha$-Fredholm operators.*

*Proof.* Let $T$ and $S$ be $\alpha$-Fredholm operators. Since an $\alpha$-Fredholm operator is invertible modulo $\mathfrak{J}_\alpha$ [3], there exists an operator $T_1$ such that

$$F_1 = I - TT_1 \in \mathfrak{J}_\alpha, \quad \text{and} \quad F_2 = I - T_1T \in \mathfrak{J}_\alpha.$$

Also there exists an operator $S_1$ such that

$$F_3 = I - SS_1 \in \mathfrak{J}_\alpha, \quad \text{and} \quad F_4 = I - S_1S \in \mathfrak{J}_\alpha.$$

Then

$$\begin{aligned} T_1S_1ST &= T_1(I - F_4)T = T_1T - T_1F_4T \\ &= I - F_2 - T_1F_4T = I - F_5 \\ STT_1S_1 &= S(I - F_1)S_1 \\ &= I - F_3 - SF_1S_1T = I - F_6 \ , \end{aligned}$$

where $F_5$ and $F_6$ are in $\mathfrak{J}_\alpha$. Hence by [3] $ST$ is an $\alpha$-Fredholm operator.

Conversely, let $ST$ be $\alpha$-Fredholm operator. Since $ST = TS$,

$$N(S) \cup N(T) \subseteq N(ST), \text{ and } N(S^*) \cup N(T^*) \subseteq N(ST)^*.$$

Thus dim $N(S) \leq$ dim $N(ST) < \alpha$ and similarly

$$\dim N(T) < \alpha, \dim N(S^*) < \alpha, \quad \text{and} \quad \dim N(T^*) < \alpha.$$

Since $TS$ is $\alpha$-Fredholm operator, by [3] $TS$ is bounded below on some closed subspace $K$ of codimension less than $\alpha$. This means that

$$||TSx|| \geq \varepsilon ||x||, \ \ x \in K$$

where dim $K^\perp < \alpha$. Since $||TSx|| \leq ||T|| \, ||Sx||$, for each $x$ in $K$

$$||Sx|| \geq \varepsilon ||T||^{-1} ||x|| \ .$$

Hence $S$ is bounded below on a closed subspace $K$ of codimension less than $\alpha$. Therefore by [3] $R(S)$ is $\alpha$-closed. Similarly we can prove that $R(T)$ is $\alpha$-closed. Hence $T$ and $S$ are $\alpha$-Fredholm operators. □

**Theorem 1.2.** *Assume that $S$ and $T$ in $B(H)$ are such that $TS$ is an $\alpha$-Fredholm operator. Then $S$ is $\alpha$-Fredholm operator if and only if $T$ is $\alpha$-Fredholm.*

*Proof.* First, we assume that $S$ is $\alpha$-Fredholm. Then by [3] there exists an operator $S_1$ such that

$$I - S_1 S = F_1, \quad \text{and} \quad I - SS_1 = F_2$$

where $F_1, F_2 \in \mathfrak{J}_\alpha$. Now $SS_1 = I - F_2$. Therefore $TSS_1 = T - TF_2$.

Now $S_1$ is $\alpha$-Fredholm operator as $S_1$ is invertible modulo $\mathfrak{J}_\alpha$. $TS$ is $\alpha$-Fredholm operator by hypothesis. By necessary part of Theorem 1.1, $TSS_1$ is $\alpha$-Fredholm operator. Since $TF_2$ is in $\mathfrak{J}_\alpha$, $T$ is $\alpha$-Fredholm operator. By the same argument, if we assume that $T$ is an $\alpha$-Fredholm operator, then $S$ is $\alpha$-Fredholm. □

**Theorem 1.3.** *Let $S$ be an $\alpha$-Fredholm operator. Then there is an $\varepsilon > 0$ such that for any $T$ in $B(H)$ satisfying $\|T\| < \varepsilon$, $S + T$ is also $\alpha$-Fredholm.*

*Proof.* By [3], there exists an operator $S_1$ such that

$$I - SS_1 = F_1, \quad \text{and} \quad I - S_1 S = F_2$$

where $F_1$, $F_2$ are $\mathfrak{J}_\alpha$. We note that $S_1 \neq 0$. Also

$$S_1(S+T) = S_1 S + S_1 T = I - F_2 + S_1 T,$$
$$(S+T)S_1 = SS_1 + TS_1 = I - F_1 + TS_1.$$

Take $\varepsilon = \| S_1 \|^{-1}$. Then for $T$ satisfying $\|T\| < \varepsilon$,

$$\| S_1 T \| \leq \| S_1 \| \, \| T \| < 1.$$

Similarly $\| TS_1 \| < 1$. Thus the operator $I + TS_1$ and $I + S_1 T$ have bounded inverses. Consequently

$$(I + S_1 T)^{-1} S_1 (S+T) = I - (I + S_1 T)^{-1} F_2,$$
$$(S+T) S_1 (I + TS_1)^{-1} = I - F_1 (I + TS_1)^{-1}.$$

Therefore by [3] $S + T$ is $\alpha$-Fredholm. □

## 2. $\alpha$-Weyl's spectrum

**Theorem 2.1.** *([7]) For any operator $T$, $\omega_\alpha(T)$ is a nonempty compact subset of $\sigma(T)$.*

*Proof.* That $\omega_\alpha(T)$ is a compact subset of $\sigma(T)$ follows from the definition. We claim that $\sigma_\alpha(T) \subseteq \omega_\alpha(T)$. Let $\lambda \in \sigma_\alpha(T)$. Then $\hat{T} - \lambda\hat{I}$ is not invertible in $B(H)/\mathfrak{I}_\alpha$. Let $\lambda \notin \omega_\alpha(T)$. Then $T - \lambda I = S + K$, where $S$ is invertible and $K \in \mathfrak{I}_\alpha$. Hence $\hat{T} - \lambda\hat{I} = \hat{S}$, where $\hat{S}$ is invertible in $B(H)/\mathfrak{I}_\alpha$, a contradiction. Hence $\lambda \in \omega_\alpha(T)$. Thus $\omega_\alpha(T)$ is a nonempty compact subset of $\sigma(T)$. □

**Lemma 2.2.** *([7]) For an arbitrary operator $T$ and a polynomial $p$,*

$$\omega_\alpha(p(T)) \subseteq p(\omega_\alpha(T)).$$

*However, if $T$ is normal then for any continuous function $f$ on $\sigma(T)$,*

$$\omega_\alpha(f(T)) = f(\omega_\alpha(T)).$$

*Proof.* Suppose $\mu \notin p(\omega_\alpha(T))$. Write

$$p(\lambda) - \mu = a(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$

For each $j$, $p(\lambda_j) = \mu \notin p(\omega_\alpha(T))$. Then $\lambda_j \notin \omega_\alpha(T)$ and therefore $T - \lambda_j = S_j + F_j$ for each $j$, where $S_j$ is invertible and $F_j \in \mathfrak{I}_\alpha$. Hence

$$\begin{aligned} p(T) - \mu I &= a(S_1 + F_1)(S_2 + F_2)\cdots(S_n + F_n) \\ &= S + F, \text{ say} \end{aligned}$$

where $S$ is invertible and $F \in \mathfrak{I}_\alpha$. Hence $\mu \notin \omega_\alpha(p(T))$ and so $\omega_\alpha(p(T)) \subseteq p(\omega_\alpha(T))$.

Now if $T$ is normal, then $\omega_\alpha(T) = \sigma_\alpha(T)$([3], Corollary 4.7). Also $\hat{T}$ is normal in $B(H)/\mathfrak{I}_\alpha$. Hence by $C^*$–algebra theory, $f(\hat{T})$ exists and $f(\hat{T}) = \widehat{f(T)}$ [Dixmier, Proposition 1.5.3, p.11]. We have

$$\omega_\alpha(f(T)) = \sigma(\widehat{f(T)}) = \sigma(f(\hat{T})) = f(\sigma(\hat{T})) = f(\omega_\alpha(T)).$$

□

**Theorem 2.3.** *If $T$ is any operator, then $\omega_\alpha(T+K) = \omega_\alpha(T)$ holds if and only if $K$ is in $\mathfrak{J}_\alpha$.*

*Proof.* If $K \in \mathfrak{J}_\alpha$, then it follows right from the definition that $\omega_\alpha(T+K) = \omega_\alpha(T)$.

Conversely suppose that $\omega_\alpha(T+K) = \omega_\alpha(T)$. If $T=0$, we get $\omega_\alpha(K) = \{0\}$. Thus $\omega_\alpha(K^*) = \overline{\omega_\alpha(K)} = \{0\}$ and hence

$$\omega_\alpha(K+K^*) = \omega_\alpha(K^*) = \{0\}$$

and

$$\omega_\alpha(K-K^*) = \omega_\alpha(K^*) = \{0\}.$$

However $K+K^*$ and $K-K^*$ are both normal operators, and so are in $\mathfrak{J}_\alpha$. Hence $K = [(K+K^*)+(K-K^*)]/2 \in \mathfrak{J}_\alpha$. □

**Theorem 2.4.** *If $\sigma(T+K) = \sigma_\alpha(T)$ for some $K \in \mathfrak{J}_\alpha$, then $\omega_\alpha(T) = \sigma_\alpha(T)$.*

*Proof.* By hypothesis $\omega_\alpha(T) = \cap_{C\in\mathfrak{J}_\alpha}\sigma(T+C) \subseteq \sigma(T+K) = \sigma_\alpha(T)$ for some $K \in \mathfrak{J}_\alpha$. Hence $\omega_\alpha(T) = \sigma_\alpha(T)$. □

**Theorem 2.5.** $\omega_\alpha\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} \subseteq \omega_\alpha(A)\cup\omega_\alpha(B)$.

*Proof.* Let $\lambda \notin \omega_\alpha(A)\cup\omega_\alpha(B)$. Then $\lambda\notin\omega_\alpha(A)$ and $\lambda\notin\omega_\alpha(B)$ and hence

$$A-\lambda I = S_1+K_1 \quad\text{and}\quad B-\lambda I = S_2+K_2$$

where $S_1$ and $S_2$ are invertible and $K_1$ and $K_2$ are in $\mathfrak{J}_\alpha$. Consider

$$\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} - \lambda I = \begin{pmatrix} S_1+K_1 & 0 \\ 0 & S_2+K_2\end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2\end{pmatrix} + \begin{pmatrix} K_1 & 0 \\ 0 & K_2\end{pmatrix}$$

where $\begin{pmatrix} S_1 & 0 \\ 0 & S_2\end{pmatrix}$ is invertible and $\begin{pmatrix} K_1 & 0 \\ 0 & K_2\end{pmatrix} \in \mathfrak{J}_\alpha$. Therefore $\lambda\notin\omega_\alpha\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}$ and thus

$$\omega_\alpha\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} \subset \omega_\alpha(A)\cup\omega_\alpha(B)\ .$$

□

K. K Oberai [5] has proved that if $T_n \to T$, $\lim\sigma(\hat{T}_n) = \sigma(\hat{T})$ then $\lim\omega(T_n) = \omega(T)$. We however prove the following :

**Theorem 2.6.** *Let $T_n \to T$. If* $\lim \sigma_\alpha(T_n) = \sigma_\alpha(T)$ *then*

$$\omega_\alpha(T) \subset \liminf \omega_\alpha(T_n) \ ,$$

*that is $T \to \omega_\alpha(T)$ is lower semi-continuous at $T$.*

*Proof.* Suppose $\lambda \notin \liminf \omega_\alpha(T_n)$. This means that there exists a neighborhood $V$ of $\lambda$ that does not intersect infinitely many $\omega_\alpha(T_n)$. Since $\sigma_\alpha(T_n) \subset \omega_\alpha(T_n)$ for each $n$, $V$ does not intersect infinitely many $\sigma_\alpha(T_n)$. Hence $\lambda \notin \lim \sigma_\alpha(T_n) = \sigma_\alpha(T) \subset \omega_\alpha(T)$. Therefore $\lambda \notin \omega_\alpha(T)$. Thus $\omega_\alpha(T) \subset \liminf \omega_\alpha(T_n)$. Hence $T \to \omega_\alpha(T)$ is lower semi-continuous □

**Theorem 2.7.** *Let $T \in B(H)$ be similar to an operator $S$. If $\alpha-$Weyl's theorem holds for $T$, then $\alpha-$Weyl's theorem holds for $S$.*

*Proof.* Let $S$ be similar to $T$. Then there exists an invertible operator $P$ such that $P^{-1}TP = S$. Note [2] that $T$ is of the form invertible $+\alpha-$compact if and only if $P^{-1}TP = S$ is of that form. Thus

$$\omega_\alpha(S) = \omega_\alpha(P^{-1}TP) = \omega_\alpha(T). \tag{1}$$

By [4, Problem 75]

$$\sigma(S) = \sigma(P^{-1}TP) = \sigma(T) \quad \text{and} \quad \sigma_p(S) = \sigma_p(P^{-1}TP) = \sigma_p(T). \tag{2}$$

It suffice to show that $\ker(T-\lambda) = P(\ker(S-\lambda))$ and so $\dim\ker(T-\lambda) = \dim P(\ker(S-\lambda))$. If $x \in \ker(T-\lambda)$, then

$$\begin{aligned} S(P^{-1}x) &= (P^{-1}TP)(P^{-1}x) = P^{-1}T(PP^{-1}x) \\ &= P^{-1}Tx = P^{-1}(\lambda x) = \lambda P^{-1}x. \end{aligned}$$

Thus $P^{-1}x \in \ker(S-\lambda)$ and so $x \in P(\ker(S-\lambda))$.

Conversely if $x \in P(\ker(S-\lambda))$, then $x = Py$ for some $y \in \ker(S-\lambda)$ and so $x = Py$ and $P^{-1}TPy = \lambda y$. Hence $TPy = P(\lambda y) = \lambda Py$, i.e., $Tx = \lambda x$, and so $x \in \ker(T-\lambda)$. Therefore $\ker(T-\lambda) = P(\ker(S-\lambda))$ and so $\dim\ker(T-\lambda) = \dim P(\ker(S-\lambda)) = \dim\ker(S-\lambda)$ since $P$ is invertible.

From this it is obvious that $\pi_{0\alpha}(T) = \pi_{0\alpha}(P^{-1}TP) = \pi_{0\alpha}(S)$, where $\pi_{0\alpha}(T)$ denotes the isolated points of $\sigma(T)$ that are eigenvalues of multiplicity less than $\alpha$. Since $\alpha-$Weyl's theorem holds for $T$, $\omega_\alpha(T) = \sigma(T) - \pi_{0\alpha}(T)$. From (1) and (2), $\omega_\alpha(S) = \omega_\alpha(P^{-1}TP) = \omega_\alpha(T) = \sigma(T) - \pi_{0\alpha}(T) = \sigma(S) - \pi_{0\alpha}(S)$. Hence $\alpha-$Weyl's theorem holds for $S$. □

**Corollary 2.8.** *Let $T \in B(H)$ be unitarily equivalent to an operator $S$. If $\alpha-$ Weyl's theorem holds for $T$, then $\alpha-$ Weyl's theorem holds for $S$.*

We say that $T$ in $B(H)$ is *$\alpha-$ Weyl* if $T$ is of the form $S+K$, where $S$ is invertible and $K \in \mathfrak{J}_\alpha$. In this case, if $\alpha = \aleph_0$, $T$ is said to be Weyl.

**Theorem 2.9.** *If $T$ in $B(H)$ is $\alpha-$ Weyl and if $S$ in $B(H)$ is such that $\pi(S) = \pi(T)^{-1}$, then $S$ is $\alpha-$ Weyl.*

*Proof.* Since $T$ is $\alpha-$Weyl, $T = U + K$, where $U$ is invertible and $K \in \mathfrak{J}_\alpha$, and this clearly implies that $S$ is of the form invertible + $\alpha-$compact, i.e., $S$ is $\alpha-$Weyl. □

**Theorem 2.10.** *If $\pi(T)$ is seminormal in $B(H)/\mathfrak{J}_\alpha$ and if $\omega_\alpha(T) \subseteq \{\lambda : |\lambda| = 1\}$, then $T$ is of form unitary + $\alpha-$ compact.*

*Proof.* By hypothesis, 0 is not in $\omega_\alpha(T)$ and so $T = S + K$, where $S$ is invertible and $K$ is $\alpha-$compact. Hence $\pi(T) = \pi(S)$. Since $\sigma(\hat{T}) = \sigma_\alpha(T) \subseteq \omega_\alpha(T) \subseteq \{\lambda : |\lambda| = 1\}$ and $\pi(T)$ is seminormal, $\pi(T)$ is unitary in $B(H)/\mathfrak{J}_\alpha$ and so $\pi(S^*S) = \pi(I)$. But square roots of a positive element of a $C^*-$algebra are unique, so $\pi((S^*S)^{1/2}) = \pi(I)$. Let the polar decomposition of $S$ be given by $S = U(S^*S)^{1/2}$, where $U$ is unitary. Then

$$\begin{aligned}\pi(T) = \pi(S) = \pi(U(S^*S)^{1/2}) &= \pi(U)\pi((S^*S)^{1/2}) \\ &= \pi(U)\pi(I) = \pi(U),\end{aligned}$$

so that $T - U$ is $\alpha-$compact. □

For an example, consider $T = U \oplus U^*$, where $U$ is the unilateral shift. In this case, $\omega(T) = \{\lambda : |\lambda| = 1\} = \sigma_e(T)$. But $T$ is not a normal operator. Since $I - UU^*$ and $UU^* - I$ are rank one operators, $\pi(T)$ is normal. By Theorem 2.10, $T = U \oplus U^*$ is of the form unitary + compact.

A bounded linear operator $T$ in $B(H)$ is said to belong to class $W$ ([8]) if

$$\sigma_e(T) = \omega(T) \ ,$$

where $\sigma_e(T)$ denote the essential spectrum of $T$. For example, define an operator $T$ on $l_2$ by

$$T(x_1, x_2, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots).$$

Then $\sigma(T) = \{0, \frac{1}{2}, \frac{1}{3}, \cdots\}$, and $\omega(T) = \sigma_e(T) = \{0\}$ since $T$ is compact. Hence $T$ is of class $W$. However, consider the weighted shift $U$ on $l_2$ given by

$$U(x_1, x_2, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Then $U$ is hyponormal, $\omega(U) = \sigma(U) = D(=$ the closed unit disc) and $\sigma_e(U) = C(=$ the unit circle). Hence $U$ is not of class $W$ and so we note that $T$ is not of class $W$, even if $T$ is hyponormal.

Motivated by this we say that a bounded linear operator $T$ in $B(H)$ belongs to class $W_\alpha$ if

$$\sigma_\alpha(T) = \omega_\alpha(T) \ .$$

If $T$ is a normal operator then $\sigma_\alpha(T) = \omega_\alpha(T)$ [3].

**Theorem 2.11.** *Let $T$ be an invertible operator in class $W_\alpha$ then $T^{-1}$ is also in class $W_\alpha$.*

*Proof.* Let $0 \neq \lambda \notin \omega_\alpha(T) = \cap_{K \in \mathfrak{J}_\alpha} \sigma(T + K)$. Then for some $K \in \mathfrak{J}_\alpha$, $\lambda \notin \sigma(T + K)$. Therefore $T + K - \lambda I$ is invertible modulo $\mathfrak{J}_\alpha$. This means $T + \mathfrak{J}_\alpha - \lambda I$ is invertible in $B(H)/\mathfrak{J}_\alpha$. Hence $\lambda \notin \sigma(T + \mathfrak{J}_\alpha)$. This gives that

$$\frac{1}{\lambda} \notin \sigma[(T + \mathfrak{J}_\alpha)^{-1}] = \sigma(T^{-1} + \mathfrak{J}_\alpha)$$

and so $\frac{1}{\lambda} \notin \cap_{K \in \mathfrak{J}_\alpha} \sigma(T^{-1} + K)$. Thus $\frac{1}{\lambda} \notin \omega_\alpha(T^{-1})$. Therefore $\frac{1}{\omega_\alpha(T^{-1})} \subseteq \omega_\alpha(T)$. Hence $\omega_\alpha(T^{-1}) \subseteq \frac{1}{\omega_\alpha(T)}$. Replacing $T$ by $T^{-1}$ we get $\omega_\alpha(T) \subseteq \frac{1}{\omega_\alpha(T^{-1})}$. Therefore $\omega_\alpha(T^{-1}) = \frac{1}{\omega_\alpha(T)}$. Now

$$\omega_\alpha(T^{-1}) = \frac{1}{\omega_\alpha(T)} = \frac{1}{\sigma_\alpha(T)} = \sigma_\alpha(T).$$

Hence $T^{-1}$ is also in class $W_\alpha$. □

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