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FIBRATIONS IN CONVERGENCE SPACES

Jin Won Park
Dept. of Mathematics Education
Cheju National University

In the theory of fibrations, the covering homotopy property, the Hurewicz fibration, the Dold fibration and the fibre homotopy equivalence are baisically important. In this case, the notion of exponential laws play central role. Many researchers have been studied these properties in compactly generated spaces and quasi-topological spaces. However, in a structural point of view it has not been completely successful to find a convenient category of fibred spaces. The main reason was that the category of compactly generated spaces is not a quasitopos and quasi-topological spaces do not form a category, but a quasi-category. So, it is natural to study these properties in convergence spaces which is a catesian closed.

On the other hand, the theory of fibration have been developed in the situation of having variable base spaces. In particular, given fibrations $p : X \to A, q : Y \to B$ the construction and properties of a function space $C_{AB}(X, Y)$ and an associated fibration $p \cdot q : C_{AB}(X, Y) \to A \times B$ are mainly concerned. In this case also fibrewise exponential laws play crucial role.

In this paper, we introduce a function space structure which will allow us fibrewise exponential laws in convergence spaces over variable base spaces. And using these exponential laws, we obtain some important results on fibration comparing with known results.

FIBRATIONS IN CONVERGENCE SPACES

JIN WON PARK

CHEJU NATIONAL UNIVERSITY
DEPARTMENT OF MATHEMATICS EDUCATION

1. Preliminaries

In the theory of fibrations, the covering homotopy property, the Hurewicz fibration, the Dold fibration and the fibre homotopy equivalence are baisically important. In this case, the notion of exponential laws play central role. So far, compactly generated spaces and quasi-topological spaces have been main objectives for these studies. However, in a structural point of view it has not been completely successful to find a convenient category of fibred spaces. The main reason was that the category of compactly generated spaces is not a quasitopos and quasi-topological spaces do not form a category, but a quasi-category. So, it is natural to consider the category of convergence spaces which is a catesian closed. With this consideration, in 1992, Min and Lee [17] obtained natural exponential laws in the category of convergence spaces over a base B.

On the other hand, the theory of fibration have been developed in the situation of having variable base spaces. In particular, given fibrations $p : X \to A, q : Y \to B$ the construction and properties of a function space $C_{AB}(X, Y)$ and an associated fibration $p \cdot q : C_{AB}(X, Y) \to A \times B$ are mainly concerned. In this case also fibrewise exponential laws play crucial role [3-9,18,19].

In this paper, we introduce a function space structure which will allow us fibrewise exponential laws in convergence spaces over variable base spaces. And using these exponential laws, we obtain some important results on fibration comparing with known results.

Let $p : X \to A$ and $q : Y \to B$ be continuous maps where X, Y, A and B are convergence spaces. A *fibre preserving map* from p to q is a pair (f_1, f_0) of continuous maps $f_1 : X \to Y$ and $f_0 : A \to B$ such that $q \circ f_1 = f_0 \circ p$, i.e., the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f_0]{} & B \end{array}$$

commutes. We write this map by $(f_1, f_0) : p \to q$.

For given $p : X \to A$ and $q : Y \to B$, let

$$C_{AB}(X, Y) = \bigcup_{a \in A, b \in B} C(X_a, Y_b)$$

as a set, where $C(X_a, Y_b)$ is the set of all continuous maps from X_a to Y_b. Define a convergence structure on $C_{AB}(X, Y)$ as follows. A filter $\mathcal{F}$ converges to f in $C_{AB}(X, Y)$, where $f \in C(X_a, Y_b)$ if and only if
(1) for any filter $\mathcal{A}$ in X which converges to $x \in X_a$, $(\mathcal{F} \bigcap \dot{f})(\mathcal{A} \bigcap \dot{x})$ converges to $f(x)$ in Y and
(2) $p \cdot q(\mathcal{F})$ converges to $p \cdot q(f)$ in $A \times B$, where $p \cdot q : C_{AB}(X, Y) \to A \times B$ is defined by $p \cdot q(g) = (a, b)$ if $g \in C(X_a, Y_b)$.
medskip

Proposition 1.1. *$C_{AB}(X, Y)$ equipped with this structure is a convergence space over $A \times B$ with projection $p \cdot q$.*

Remark. If A and B are one-point spaces, then this structure is the same as the continuous convergence structure on $C(X, Y)$ [2].

For given $p : X \to A$ and $q : Y \to B$, we can consider $X \times B$ and $A \times Y$ as a convergence spaces over $A \times B$ with projections $p \times 1_B$ and $1_A \times q$, respectively. With this consideration, we have the following proposition.

Theorem 1.2. *Let $p : X \to A$ and $q : Y \to B$ be given. Then $C_{AB}(X, Y)$ is homeomorphic to $C_{A\times B}(X \times B, A \times Y)$ with the fibrewise continuous convergence structure.*

Proof. Define $\phi : C_{AB}(X, Y) \to C_{A\times B}(X \times B, A \times Y)$ as follows. For $f \in C(X_a, Y_b)$, $\phi(f)$ is a function from $X_a \times \{b\}$ to $\{a\} \times Y_b$ which is defined by $\phi(f)(x, b) = (a, f(x))$. Note that this definition is well defined and ϕ is a bijection. Suppose $\mathcal{F} \to f$ in $C_{AB}(X, Y)$. We want to show that $\phi(\mathcal{F}) \to \phi(f)$ in $C_{A\times B}(X \times B, A \times Y)$. Let $\mathcal{A} \to (x, b) \in (X \times B)_{(a,b)} = X_a \times \{b\}$. Then $\pi_1(\mathcal{A}) \to x$ in X. Hence $(\mathcal{F} \bigcap \dot{f})(\pi_1(\mathcal{A}) \bigcap \dot{x}) \to f(x)$.

It is enough to show that $(\phi(\mathcal{F}) \bigcap \dot{\widehat{\phi(f)}})(\mathcal{A} \bigcap \dot{\widehat{(x, b)}}) \to \phi(f)(x, b) = (a, f(x))$. But, $pi_2(\phi(\mathcal{F}) \bigcap \dot{\widehat{\phi(f)}})(\mathcal{A} \bigcap \dot{\widehat{(x, b)}}) = (\mathcal{F} \bigcap \dot{f})(\pi_1(\mathcal{A}) \bigcap \dot{x}) \to f(x)$ and $\pi_1(\phi(\mathcal{F}) \bigcap \dot{\widehat{\phi(f)}})(\mathcal{A} \bigcap \dot{\widehat{(x, b)}}) = \pi_1(p \cdot q)(\mathcal{F}) \to a$.

In all, $(\phi(\mathcal{F}) \bigcap \dot{\widehat{\phi(f)}})(\mathcal{A} \bigcap \dot{\widehat{(x, b)}}) \to \phi(f)(x, b)$. Moreover, let the projection of $C_{A\times B}(X \times B, A \times Y)$ be denoted by r. Then it is easy to see that $(p \cdot q)(f) = r(\phi(f))$. Hence $r(\phi(\mathcal{F})) \to r(\phi(f))$, since $(p \cdot q)(\mathcal{F}) \to (p \cdot q)(f)$. Therefore $\phi(\mathcal{F}) \to \phi(f)$ in $C_{A\times B}(X \times B, A \times Y)$. Therefore ϕ is continuous.

Conversely, suppose $\mathcal{F} \to f$ in $C_{A\times B}(X \times B, A \times Y)$ where $f : (X \times B)_{(a,b)} \to (A \times Y)_{(a,b)}$, i.e., $f : X_a \times \{b\} \to \{a\} \times Y_b$. Denote $\phi^{-1} = \varphi$. Note that $\varphi(f)(x) = \pi_2 \circ f(x, b)$. We want to show that $\varphi(\mathcal{F}) \to \varphi(f)$ in $C_{AB}(X, Y)$. Let $\mathcal{A} \to x \in X_a$. Then $\mathcal{A} \times \dot{b} \to (x, b)$ in $X \times B$. Thus $(\mathcal{F} \bigcap \dot{f})((\mathcal{A} \times \dot{b}) \bigcap \dot{\widehat{(x, b)}}) \to f(x, b)$. Hence $(\varphi(\mathcal{F}) \bigcap \dot{\widehat{\varphi(f)}})(\mathcal{A} \bigcap \dot{x}) = \pi_2((\mathcal{F} \bigcap \dot{f})((\mathcal{A} \times \dot{b}) \bigcap \dot{\widehat{(x, b)}})) \to \varphi(f)(x)$. Note that $r(f) = (p \cdot q)(\varphi(f))$. Therefore $\varphi(\mathcal{F}) \to \varphi(f)$ in $C_{AB}(X, Y)$, i.e., φ is continuous.

Proposition 1.3. *Let $p : X \to A, q : Y \to B$ and $r : Z \to B$ be given. If $f : Y \to Z$ is a morphism over B then $f_\sharp : C_{AB}(X, Y) \to C_{AB}(X, Z)$, defined by $f_\sharp(g) = (f_b)g : X_a \to Z_b$, is a morphism over $A \times B$ where $g : X_a \to Y_b$ with $a \in A$ and $b \in B$.*

Proof. Let $\mathcal{F} \to g$ in $C_{AB}(X,Y)$, where $g : X_a \to Y_b$. Let $\mathcal{A} \to \dot{x}$ and $x \in X_a$. We need to show that $(f_\sharp(\mathcal{F}) \cap f_\sharp(\dot{g}))(\mathcal{A} \cap \dot{x})$ converges to $f_\sharp(g)(x)$ in Z. But, since $\mathcal{F}$ converges to g in $C_{AB}(X,Y)$, $(\mathcal{F} \cap \dot{g})(\mathcal{A} \cap \dot{x})$ converges to $g(x)$ in Y, and hence $f_b((\mathcal{F} \cap \dot{g})(\mathcal{A} \cap \dot{x}))$ converges to $f_b((g)(x)) = f_\sharp(g)(x)$ in Z. Therefore $(f_\sharp(\mathcal{F}) \cap f_\sharp(\dot{g}))(\mathcal{A} \cap \dot{x})$ converges to $f_\sharp(g)(x)$. Moreover, since $p \cdot q(\mathcal{F})$ converges to $p \cdot q(g)$, $p \cdot r(f_\sharp(\mathcal{F}))$ converges to $p \cdot r(f_\sharp(g))$. In all, $f_\sharp(\mathcal{F})$ converges to $f_\sharp(g)$.

Now, we obtain some important exponential laws.

Theorem 1.4. *Let $p : X \to A, q : Y \to B$ and $r : Z \to D$ be given. Then*

$$\phi : C_{ABD}(Y \times X, Z) \to C_{ABD}(X, C_{BD}(Y,Z))$$

which is defined by $\phi(f)(x)(y) = f(y,x)$ is an isomrphism.

Let $q : Y \to A$ and $r : Z \to B$ be given. Let $M_{AB}(Y,Z) = \{(f_1, f_0) | (f_1, f_0) : q \to r\}$. We consider $M_{AB}(Y,Z)$ as a subspace of $C(Y,Z) \times C(A,B)$. For given convergence spaces Y and Z over B with improjections q and r, respectively, let $M_B(Y,Z) = \{f : Y \to Z | f$ is continuous and $r \circ f = q\}$. Note that $M_B(Y,Z) \subseteq C(Y,Z)$. Let $M_B(Y,Z)$ have the subspace structure with respect to $C(Y,Z)$. And. consider $M_{XB}(Y \times_A X, Z)$ and $M_A(X, C_{AB}(Y,Z))$. In this case, $Y \times_A X$ is considered as a space over X with natural projection and $C_{AB}(Y,Z)$ as a space over A with projection $q \cdot_1 r$. Under these considerations, we have another type of exponential law.

Theorem 1.5. *Let $p : X \to A, q : Y \to A$ and $r : Z \to B$ be given. Then*

$$\phi : M_{XB}(Y \times_A X, Z) \to M_A(X, C_{AB}(Y,Z))$$

which is defined by $\phi(f_1, f_0)(x)(y) = f_1(y,x)$ is an isomorphism.

Remark. If A and B are one-point spaces, then we have the well-known exponential law $C(Y \times X, Z) \simeq C(X, C(Y,Z))$, where the morphisms $f : Y \times X \to Z$ and $f^0 : X \to C(Y,Z)$ are related by $f(y,x) = f^0(x)(y)$ where $y \in Y$ and $x \in X$. Moreover, by the direc calculation, we have $C(X \times Y, C(I,Z)) \simeq C(Y \times I, C(X,Z))$.

Corollary 1.6. *Let $q : Y \to B$ and $r : Z \to B$ be given. Then there is an isomorphism between the space of homotopies $H : Y \times I \to Z$ such that $r \circ H = \pi_1 \circ (q \times 1_I)$ and the space of homotopies $H^0 : B \times I \to C_{BB}(Y, Z)$ over $B \times B$, that lift $(\pi_1, \pi_1) : B \times I \to B \times B$ over $q \cdot r$, defined by $H(y, t) = H^0(b, t)(y)$ where $q(y) = b$ and $t \in I$.*

Corollary 1.7. *There is an isomorphism*

$$M_{AB}(Y, Z) \cong M_A(A, C_{AB}(Y, Z))$$

i.e., there is an isomorphism between the space of fibre preserving maps $(f_1, f_0) : q \to r$ and the space of cross-sections to $q \cdot_1 r$.

2. Fibrations

In this section, we study the covering homotopy property.

Definition 2.1. (1) A morphism $p : X \to B$ is said to *have covering homotopy property* (*CHP*) with respect to a homotopy $H : A \times I \to B$ if for every morphism $h : A \to X$ such that $p \circ h(a) = H(a, 0)$ for all $a \in A$, there is a homotopy $G : A \times I \to X$ such that $p \circ G = H$ and $G(a, 0) = h(a)$ for all $a \in A$, i.e.,

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ {\scriptstyle \sigma_0}\downarrow & \overset{G}{\nearrow} & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow[H]{} & B \end{array}$$

(2) p is called a *Hurewicz fibration* if it has CHP with respect to all homotopies $H : A \times I \to B$.

(3) p is called a *Dold fibration* if it has CHP with respect to all homotopies $H : A \times I \to B$ such that $H(a, t) = H(a, 0)$ for all $t \in [0, 1/2]$.

Remark. [19] The condition in (3) is equivalent to the following statement : for any morphism $h : A \to X$ and homotopy $H : A \times I \to B$

such that $p \circ h(\_) = H(\_,0)$, there is a homotopy $G : A \times I \to X$ such that $p \circ G = H$ and $G(\_,0) \simeq h$.

Examples (1) Let $p : X \to B$ be a fibration and $q : Y \to B$ be any morphism. Then $\pi_1 : X \times_B Y \to X$ is a fibration.

(2) Let $p : X \to B$ be a Hurewicz fibration. Then, for any morphism $\xi : B' \to B$, the pull-back $p' : \xi^* X \to B'$ is a Hurewicz fibration. In fact, consider the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{h} & \xi^* X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \sigma_0}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow[H]{} & B' & \xrightarrow[\xi]{} & B \end{array}$$

By the definition of the pull-back and the fact that p is a Hurewicz fibration, the result follows.

Proposition 2.2. *Let $q : Y \to B$ and $r : Z \to B$ be given and suppose $f : Y \to Z$ is a Hurewicz fibration. Then, for any $p : X \to B$, $f_\sharp : C_B(X,Y) \to C_B(X,Z)$, defined by $f_\sharp(g) = f_b \circ g$ where $g : X_b \to Y_b$, is a Hurewicz fibration.*

Proof. For any convergence space A over B, consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & C_B(X,Y) \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle f_\sharp} \\ A \times I & \xrightarrow[H]{} & C_B(X,Z) \end{array}$$

By the exponential law, we have another commutative diagram

$$\begin{array}{ccc} A \times_B X & \xrightarrow{h^\circ} & Y \\ {\scriptstyle \sigma_0 \times 1_X}\downarrow & & \downarrow{\scriptstyle f} \\ (A \times I) \times_B X & \xrightarrow[H^\circ]{} & Z \end{array}$$

Since there is a homeomorphism $k : (A \times I) \times_B X \to (A \times_B X) \times I$, in this case $A \times I$ has the projection $\overline{p}$, defined by $\overline{p}(a, t) = p(a)$, consider the following diagram

$$\begin{array}{ccc} A \times_B X & \xrightarrow{h^\circ} & Y \\ {\scriptstyle k\circ(\sigma_0 \times 1_X)}\downarrow & & \downarrow{\scriptstyle f} \\ (A \times_B X) \times I & \xrightarrow[k\circ H^\circ]{} & Z \end{array}$$

Since f is a Hurewicz fibration, there is a homotopy $G' : (A \times_B X) \times I \to Y$ such that $G' \circ k \circ (\sigma_0 \times 1_X) = h^\circ$ and $f \circ G' = k \circ H^\circ$. Define $G^\circ : (A \times I) \times_B X \to Y$ by $G^\circ = G' \circ k$ and let G be the adjiont of G°. Then $G \circ \sigma_0 = h$ and $f_\sharp \circ G = H$. Hence $f_\sharp$ is a Hurewicz fibration.

Corollary 2.3. *Let $p : X \to B$ be a Hurewicz fibration. Then, for any space Y, $p^* : C(Y, X) \to C(Y, B)$, where $p^*(f) = p \circ f$, is a Hurewicz fibration.*

For any convergence space B, let $C(I, B)$ have the continuous convergence structure and e_0^B as a projection, where $e_0^B(\lambda) = \lambda(0)$. For given $p : X \to B$, consider the following pullback diagram

$$\begin{array}{ccc} C(I,B) \times_B X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ C(I,B) & \xrightarrow[e_0^B]{} & B \end{array}$$

Then for given $e_0^X : C(I, X) \to X$ and $C(I, p) : C(I, X) \to C(I, B)$ where $C(I, p)(\lambda) = p \circ \lambda$, there exists a unique map $\pi : C(I, X) \to C(I, B) \times_B X$ such that $\pi_2 \circ \pi = e_0^X$ and $\pi_1 \circ \pi = C(I, p)$. In fact, π is defined by $\pi(\lambda) = (C(I, p)(\lambda), e_0^X(\lambda))$. If there exists a map $\Gamma : C(I, B) \times_B X \to C(I, X)$ such that $\pi \circ \Gamma = 1_{C(I,B)\times_B X}$, we call Γ a lifting function for $p : X \to B$.

Proposition 2.4. *Let $p : X \to B$ be given. Then the following statements are equivalent.*

(1) *p is a Hurewicz fibration.*

(2) *For given $q : D \to A$ and $(g, h) : q \to p$ and any homotopy $H : A \times I \to B$ of h, there exists a homotopy $G : D \times I \to X$ of g such that $p \circ G = H \circ (q \times 1_I)$.*

(3) *there exists a lifting function Γ for $p : X \to B$.*

Proof. (1)⇒(2) Given a fibre preserving map $(g, h) : q \to p$ and a homotopy $H : A \times I \to B$ of h, consider the following commutative diagram

$$\begin{array}{ccccc} D & & \xrightarrow{g} & & X \\ {\scriptstyle \sigma_0}\downarrow & & & & \downarrow{\scriptstyle p} \\ D \times I & \xrightarrow[q \times 1_I]{} & A \times I & \xrightarrow[H]{} & B \end{array}$$

Since p is a Hurewicz fibration, there exists a homotopy $G : D \times I \to X$ such that $G \circ \sigma_0 = g$ and $p \circ G = H \circ (q \times 1_I)$.

(2)⇒(3) Let $g : C(I, B) \times_B X \to X$ be π_2 and $h : C(I, B) \times_B X \to B$ be $p \circ \pi_2$. Then we have the following commutative diagram

$$\begin{array}{ccc} C(I, B) \times_B X & \xrightarrow{g} & X \\ {\scriptstyle 1_{C(I,B) \times_B X}}\downarrow & & \downarrow{\scriptstyle p} \\ C(I, B) \times_B X & \xrightarrow[h]{} & B \end{array}$$

Let $H : (C(I, B) \times_B X) \times I \to B$ be a homotopy of h defined by $H((\lambda, x), t) = \lambda(t)$. By (2), there exists a homotopy $G : (C(I, B) \times_B X) \times I \to X$ of g such that $p \circ G = H \circ (1_{C(I,B) \times_B X} \times 1_I)$. By the exponential law $C((C(I, B) \times_B X) \times I, X) \cong C(C(I, B) \times_B X, C(I, X))$, there exists a map $\Gamma : C(I, B) \times_B X \to C(I, X)$ which is defined by $\Gamma(\lambda, x)(t) = G((\lambda, x), t)$. This Γ is a lifting function for $p : X \to B$.

(3)⇒(1) Consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow[H]{} & B \end{array}$$

Let $H' : A \to C(I,B)$ be defined by $H'(a)(t) = H(a,t)$. Note that $(H'(a), h(a)) \in C(I,B) \times_B X$. Define $G' : A \to C(I,X)$ by $G'(a) = \Gamma(H'(a), h(a))$. By the exponential law $C(A, C(I,X)) \cong C(A \times I, X)$, there exists a map $G : A \times I \to X$ which is defined by $G(a,t) = G'(a)(t)$. Then $p \circ G = H$ and $G \circ \sigma_0 = h$. Hence p is a Hurewicz fibration.

Proposition 2.5. *For any morphism $p : X \to B$, $p' : C(I,B) \times_B X \to B$, where $p'(\lambda, x) = \lambda(1)$, is a Hurewicz fibration.*

Proof. Consider the morphism $\pi : C(I, C(I,B) \times_B X) \to C(I,B) \times_B (C(I,B) \times_B X)$, where $\pi(\lambda) = (p' \circ \lambda, \lambda(0))$. Define $\Gamma : C(I,B) \times_B (C(I,B) \times_B X) \to C(I, C(I,B) \times_B X)$ as follows: for $(\omega, \tau, x) \in C(I,B) \times_B (C(I,B) \times_B X)$, $\Gamma(\omega, \tau, x)(t) = (q, x)$, where q is the path from $p(x)$ to $\omega(t)$ along τ and ω. If $(\omega, \tau, x) \in C(I,B) \times_B (C(I,B) \times_B X)$, then $\tau(0) = p(x)$ and $\omega(0) = p'(\tau, x) = \tau(1)$. Hence this definition is well defined. Moreover, $\Gamma(\omega, \tau, x)(0) = (q, x)$, where q is the path from $p(x)$ to $\omega(0) = \tau(1)$ along τ, i.e., $\Gamma(\omega, \tau, x)(0) = (\tau, x)$. Therefore, $\pi(\Gamma(\omega, \tau, x)) = (p' \circ \Gamma(\omega, \tau, x), \Gamma(\omega, \tau, x)(0)) = (p' \circ \Gamma(\omega, \tau, x), (\tau, x))$. And $p' \circ \Gamma(\omega, \tau, x)(t) = p'(q, x) = q(1) = \omega(t)$, i.e., $p' \circ \Gamma(\omega, \tau, x) = \omega$. So, $\pi \circ \Gamma(\omega, \tau, x) = (\omega, \tau, x)$. This means that Γ is a lifting function for p'. By Proposition 2.4, p' is a Hurewicz fibration.

Remark. The fibration in the above proposition is called the mapping track fibration.

Proposition 2.6. *$p : X \to B$ is a Hurewicz fibration if and only if the statement (2) in proposition 3.3 holds for the induced space, i.e., for* *given diagrams*

$$\begin{array}{ccc} A \times_B X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[h]{} & B \end{array} \qquad \begin{array}{ccc} (A \times_B X) \times I & & X \\ {\scriptstyle \pi_1 \times 1_I}\downarrow & & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow[H]{} & B \end{array}$$

there exists a homotopy $G : (A \times_B X) \times I \to X$ *such that* $p \circ G = H \circ (\pi_1 \times 1_I)$.

Proof. The only if part is trivial. Let $q : D \to A$ and $g : D \to X$ be given and $p \circ g = f \circ q$. Then there exists a unique map $k : D \to A \times_B X$ such that $\pi_2 \circ k = g$ and $\pi_1 \circ k = q$. Define the homotopy $K : D \times I \to X$ by $K = G \circ (k \times 1_I)$. Then this homotopy satisfies the conditions in (2).

Proposition 2.7. *Let* $q : Y \to A$ *and* $r : Z \to B$ *be given.*

(1) *If* q *and* r *are Hurewicz fibrations, then so is* $q \cdot r : C_{AB}(Y, Z) \to A \times B$.

(2) *If* q *and* r *are Dold fibrations, then so is* $q \cdot r : C_{AB}(Y, Z) \to A \times B$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & C_{AB}(Y, Z) \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle q \cdot r} \\ X \times I & \xrightarrow[H]{} & A \times B \end{array}$$

We consider X as a space over A with projection $p : X \xrightarrow{\sigma_0} X \times I \xrightarrow{H} A \times B \xrightarrow{\pi_1} A$. By the exponential law, there is a continuous map $h^\circ : Y \times_A X \to Z$ defined by $h^\circ(y, x) = h(x)(y)$.

Define $G : (Y \times_A X) \times I \xrightarrow{\pi_2 \times 1_I} X \times I \xrightarrow{H} A \times B \xrightarrow{\pi_2} B$ and consider the following diagram

$$\begin{array}{ccc} Y \times_A X & \xrightarrow{h^\circ} & Z \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle r} \\ (Y \times_A X) \times I & \xrightarrow[G]{} & B \end{array}$$

Then $r \circ h^\circ(y, x) = r(h(x)(y))$ and $G \circ \sigma_0(y, x) = G(y, x, 0) = \pi_2 \circ H \circ (\pi_2 \times 1_I)(y, x, 0) = \pi_2(H(x, 0)) = \pi_2(q \cdot r(h(x)))$. But, $r(h(x)(y)) = \pi_2(q \cdot r(h(x)))$, and hence the diagram commutes. Since r is a Hurewicz fibration, there exists a homotopy $K : (Y \times_A X) \times I \to Z$ such that $K \circ \sigma_0 = h^\circ$ and $r \circ K = G$.

Consider the pull-back diagram

$$\begin{array}{ccccc} Y \times_A (X \times I) & & \xrightarrow{\pi_1} & & Y \\ {\scriptstyle \pi}\downarrow & & & & \downarrow{\scriptstyle q} \\ X \times I & \xrightarrow[H]{} & A \times B & \xrightarrow[\pi_1]{} & A \end{array}$$

Define $H^* : (X \times I) \times I \to A \times B$ by

$$H^*(x, s, t) = \begin{cases} H(x, t - s) & \text{if} \quad t - s \geq 0 \\ H(x, 0) & \text{if} \quad t - s \leq 0 \end{cases}$$

For these $\overline{H}$ and H^*, consider

$$\begin{array}{ccccc} Y \times_A (X \times I) & & \xrightarrow{\pi_1} & & Y \\ {\scriptstyle \sigma_0}\downarrow & & & & \downarrow{\scriptstyle q} \\ (Y \times_A (X \times I)) \times I & \xrightarrow[\pi \times 1_I]{} & (X \times I) \times I & \xrightarrow[\pi_1 \circ H^*]{} & A \end{array}$$

Then, for $x \in X_a$ and $y \in Y_a$, $q \circ \overline{H}(y, x, t) = q(y) = a$. And $\pi_1 \circ H^* \circ (\pi \times 1_I) \circ \sigma_0(y, x, t) = \pi_1 \circ H^* \circ (\pi \times 1_I)(y, x, t, 0) = \pi_1 \circ H^*(x, t, 0) =$

$\pi_1(H(x,0)) = \pi_1(q \cdot r(h(x))) = a$. Hence the diagram commutes. Since q is a Hurewicz fibration, there is a homotopy $L : (Y \times_A (X \times I)) \times I \to Y$ such that $L \circ \sigma_0 = \overline{H}$ and $q \circ L = \pi_1 \circ H^* \circ (\pi \times 1_I)$. Define $M : Y \times_A (X \times I) \to Y$ by $M(y,x,t) = L(y,x,t,t)$. Then M is continuous. Define $\alpha : Y \times_A (X \times I) \to (Y \times_A X) \times I$ by $\alpha(y,x,t) = (M(y,x,t),x,t)$. Then α is also continuous. For $K \circ \alpha : Y \times_A (X \times I) \to (Y \times_A X) \times I \to Z$, by the exponential law, there is a continuous map $N : X \times I \to C_{AB}(Y,Z)$ defined by $N(x,t)(y) = K \circ \alpha(y,x,t)$. Then this N is a lift of the given H. Indeed, $N \circ \sigma_0(x)(y) = N(x,0)(y) = (K \circ \alpha)(y,x,0) = K(M(x,0,y),x,0) = K(L(y,x,0,0),x,0) = h'(L(y,x,0,0),x) = h(x)$ $(L(y,x,0,0)) = h(x)(\overline{H}(y,x,0)) = h(x)(y)$. Hence $N \circ \sigma_0(x) = h(x)$. And, for $x \in X_a$, let $N(x,t) : Y_a \to Z_b$. Then $q \cdot r(N(x,t)) = (a,b)$. By the above pullback diagram, $\pi_1 \circ H(x,t) = a$. And, $b = r \circ N(x,t)(y) = r \circ K \circ \alpha(y,x,t) = r \circ K(M(y,x,t),x,t) = G(M(y,x,t),x,t) = \pi_2 \circ H \circ (\pi_2 \times 1_I)(M(y,x,t),x,t) = \pi_2 \circ H(x,t)$. Hence $H(x,t) = (a,b)$. In all, $(q \cdot r) \circ N = H$. This completes the proof.

(2) Note that if H has the property concerning $[0,1/2]$ that occurs in the Dold fibration, then so do the other homotopies involved.

We have the following results as corollaries of Proposition 2.7.

Corollary 2.8. *Let $q : Y \to B$ and $r : Z \to B$ be given.*

(1) *If q and r are Hurewicz fibrations, then so is $C_B(Y,Z) \to B$.*

(2) *If q and r are Dold fibrations, then so is $C_B(Y,Z) \to B$.*

Proof. Suppose $A = B$ in Proposition 2.7 and consider the following commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{h} & C_B(Y,Z) & \xrightarrow{inc} & C_{BB}(Y,Z) \\
\sigma_0 \downarrow & & q\cdot r \downarrow & & \downarrow q\cdot r \\
X \times I & \xrightarrow[H]{} & B & \xrightarrow[\Delta]{} & B \times B
\end{array}$$

Since $q \cdot r : C_{BB}(Y,Z) \to B \times B$ is a fibration, there exists a homotopy $K : X \times I \to C_{BB}(Y,Z)$ such that $K \circ \sigma_0 = inc oh$ and $(q \cdot r) \circ K = \Delta \circ H$. Let G be the coresriction of K. Then $G \circ \sigma_0 = h$ and $(q \cdot r) \circ G = H$.

Corollary 2.9. *$p : X \to B$ is a Hurewicz fibration if and only if $p \cdot p : C_{BB}(X, X) \to B \times B$ is a Hurewicz fibration.*

Proof. By Proposition 2.6, given a space A and map $f : A \to B$, it is enough to show that for given diagrams

$$\begin{array}{ccc} A \times_B X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} (A \times_B X) \times I & & X \\ {\scriptstyle \pi_1 \times 1_I}\downarrow & & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow[F]{} & B \end{array}$$

where F is a Homotopy of f, there exists a homotopy $G : (A \times_B X) \times I \to X$ of π_2 such that $p \circ G = F \circ (\pi_1 \times 1_I)$. Consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C_{BB}(X, X) \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle p \cdot p} \\ A \times I & \xrightarrow[(f \circ \pi_1, F)]{} & B \times B \end{array}$$

where $g(a) : X_{f(a)} \to X_{K(a,0)}$ is defined by $g(a)(x) = \pi_2(a, x) = x$. Since $p \cdot p$ is a Hurewicz fibration, there exists a homotopy $G' : A \times I \to C_{BB}(X, X)$ such that $(p \cdot p) \circ G' = (f \circ \pi_1, F)$ and $G' \circ \sigma_0 = g$. By the exponential law, there exists $G : (A \times I) \times_B X = (A \times_B X) \times I \to X$ which is defined by $G(a, x, t) = G'(a, t)(x)$. Then $p(G(a, x, t)) = p(G'(a, t)(x)) = F(a, t) = F(\pi_1 \times 1_I((a, x), t))$.

Examples (1) Let $q : Y \to B$ be any map. Then, for any $1_D : D \to D$, $q \cdot 1_D : C_{BD}(Y, D) \to B \times D$ is a fibration. In fact, $C_{BD}(Y, D) = \bigcup_{b \in B, d \in D} C(Y_b, D_d) = \bigcup_{b \in B, d \in D} C(Y_b, \{d\})$, i.e., $C_{BD}(Y, D)$ is the union of constant maps from each fibres on Y. Hence, for a given diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & C_{BD}(Y, D) \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle q \cdot 1_D} \\ A \times I & \xrightarrow[H]{} & B \times D \end{array}$$

the lift of H is defined by $G : A \times I \to C_{BD}(Y, D), G(a,t) : Y_b \to \{d\}$ if $H(a,t) = (b,d)$.

(2) Let $r : Z \to D$ be any map. Then, for any $1_B : B \to B$, $1_B \cdot r : C_{BD}(B,Z) \to B \times D$ need not be a fibration. In fact, $C_{BD}(B,Z) = \bigcup_{b \in B, d \in D} C(B_b, Z_d) = \bigcup_{b \in B, d \in D} C(\{b\}, Z_d)$, i.e., $C_{BD}(B,Z)$ is the union of maps with singleton domain. If $Z_d = \emptyset$ for some $d \in D$, then $1_B \cdot r : C_{BD}(B,Z) \to B \times D$ has not the covering homotopy property.

Now, we define the dual notion of the fibrations.

Definition 2.10. The morphism $u : B \to X$ is called a cofibration if for any space Y and for any morphism $f : X \to Y$ and $G : B \to C(I,Y)$ such that $\rho_0 \circ g = f \circ u$, there exists a morphism $H : X \to C(I,Y)$ such that $\rho_0 \circ H = f$ and $H \circ u = G$, i.e.

$$\begin{array}{ccc} B & \xrightarrow{G} & C(I,Y) \\ {\scriptstyle u}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle \rho_0} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Example Let $u : B \to X$ be a cofibration. Then, for any morphism $\xi : B \to B'$, the push-out $u' : B' \to \xi_* X$ is a cofibration. In fact, consider the following commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\xi} & B' \times_B X & \xrightarrow{G} & C(I,Y) \\ {\scriptstyle u}\downarrow & & {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ X \times I & \xrightarrow[H]{} & \xi_* X & \xrightarrow[f]{} & Y \end{array}$$

By the definition of the push-out and the fact that u is a cofibration, the result follows.

Proposition 2.11. *Let $u : B \to X$ be a cofibration. Then for any space Y, $u \times 1_Y : B \times Y \to X \times Y$ is also a cofibration.*

Proof. Consider thr following commutative diagram

$$\begin{array}{ccc} B\times Y & \xrightarrow{G} & C(I,Z) \\ {\scriptstyle u\times 1_Y}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ X\times Y & \xrightarrow[f]{} & Z \end{array}$$

Then by the exponential law, we have the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{G^\circ} & C(I,C(Y,Z)) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ X & \xrightarrow[f^\circ]{} & C(Y,Z) \end{array}$$

Since u is a cofibration, there exists $H^\circ : X \to C(I, C(Y, Z))$ such that $h^\circ \circ u = G^\circ$ and $\rho_0 \circ G^\circ = f^\circ$. Let $H : X \times Y \to C(I, Z)$ be the adjoint of H^0. This completes the proof.

Using above result and an exponential law, we have the followin result.

Proposition 2.12. *Let $u : B \to X$ be a cofibration. Then for any space Y, $u^* : C(X,Y) \to C(B,Y)$, which is defined by $u^*(f) = f \circ u$ is a Hurewicz fibration.*

Proof. Let $u : A \to X$ be a cofibration and Y be a space. Consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C(X,Y) \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle u^*} \\ A\times I & \xrightarrow[H]{} & C(B,Y) \end{array}$$

By the exponential law, we have the following commutative diagram

$$\begin{array}{ccc} B \times A & \xrightarrow{H^\circ} & C(I,Y) \\ {\scriptstyle u\times 1_A}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ X \times A & \xrightarrow[f^\circ]{} & Y \end{array}$$

Since $u \times 1_A$ is a cofibration, there exists a morphism $G^\circ : X \times A \to C(I,Y)$ such that $\rho_0 \circ G^\circ = f^\circ$ and $G^\circ \circ (u \times 1_A) = H^\circ$. By the exponential law, we have $G : A \times I \to C(X,Y)$ which is defined by $G(a,t)(x) = G^\circ(x,a)(t)$. Note that $G \circ \sigma_0 = f$ and $u^* \circ G = H$.

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