Stratifiable Spaces

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Stratifiable 공간에 관한 연구

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SUMMARY

Borges has already proved some properties of stratifiable spaces different from new definition.

In this paper, we will prove the properties of stratifiable spaces by using new definition, and some open problems are given.

Introduction

In this paper, we begin by defining a statifiable space in the different way frotm Borges'definition. Then I have proved the properties of stratifiable spaces according to its new defintion.

Main Theorems

Definition

A T_1-space is stratifiable iff to each closed set $A\subset X$, one can assign a sepuence $G_1(A)$, $G_2(A)$,..., of open sets such that

(1) $A=\bigcap_{n=1}^{\infty} G_n(A)=\bigcap_{n=1}^{\infty}\overline{G_n(A)}$

(2) if $A\subset B$ closed, then for each n, we have $G_n(A)\subset G_n(B)$.

Note that we can assume the sequence to be decreasing.

Theorem 1 Stratifiable spaces are normal.

Prrf: Let X be stratifiable and U a nbd of A.

Then $X-U$ is closed in X and there exist decreasing sequences $G_n(A)$, $G_n(X-U)$ such that

$$A=\bigcap_{n=1}^{\infty} G_n(A)=\bigcap_{n=1}^{\infty}\overline{G_n(A)} \text{ and } X-U=\bigcap_{n=1}^{\infty} G_n(X-U)$$

$$=\bigcap_{n=1}^{\infty}\overline{G_n(X-U)}.$$

Let $U(A)=(X-\overline{G_n(X-U)})\cap G_n(A)$.

Then $U(A)$ is open in X, and $A\subset U(A)$ is clear. If $x\notin U$, then $x\notin A$ and pick n so that $x\notin\overline{G_n(A)}$,

Then $x\in X-\overline{G_n(A)}$ and $x\in G_n(X-U)$.

So, $(X-\overline{G_n(A)})\cap G_n(X-U)$ is a *nbd* of x.

Let $p\in U(A)$ and choose m so that $p\in(X-\overline{G_m(X-U)})\cap G_m(A)$.

Then $p\in X-\overline{G_m(X-U)}$ and $p\in G_m(A)$.

If $m>n$, then $G_m(A)\subset G_n(A)$ implies $p\in G_n(A)$.

So, $p\notin X-G_n(A)$ and hence $p\notin X-\overline{G_n(A)}$.

If $m<n$, then $X-\overline{G_m(X-U)}\subset X-\overline{G_n(X-U)}$ implies $p\in X-\overline{G_n(X-U)}$ and hence $p\notin\overline{G_n(X-U)}$. So $p\notin G_n(X-U)$.

In either case, $p\notin(X-\overline{G_n(A)})\cap G_n(X-U)$.

Thus, $(X-\overline{G_n(A)})\cap G_n(X-U)\cap U(A)=\phi$.

Therefore, X is normal.

Note that X is perfectly normal.

Thorem 2

AT_1-space X is stratifiable iff for each $x\in X$. One can assign a sequence $g_1(x)$, $g_2(x)$,.... of open nbds of x such that

(1) $x=\bigcap_{n=1}^{\infty} g_n(x)$,

(2) if $y\in g_n(x_n)$, $n=1,2,\dots$, then $x_n\to y$,

(3) if $y\notin A$, where A is closed, then there exists n such that $y\notin\overline{U\{g_n(x):x\in A\}}$.

Proof: Suppose a T_1-space is stratifiable. Then each $\{x\}$ is closed and so there exists a sequence $g_1(x),\dots$, of nbds of x such that $\{x\}=\bigcap_{n=1}^{\infty} g_n(x)$ and if $y\in g_n(x_n)$, $n=1,2,\dots$ then $x_n\to y$.

For each $x\in X$ and each $n\in Z^+$, define $h_n(x)=G_n(\{x\})\cap g_n(x)$.

Then $h_1(x)$, $h_2(x),\dots$ is a sepuence of nbds of x.

If $y\in h_n(x_n)$, $n=1,2,\dots$, then $y\in g_n(x_n)$, $n=1,2,\dots$ and so $x_n\to y$.

If A is closed and $y\notin A$, then there exists n such that $y\notin\overline{G_n(A)}$.

For all $x\in A$, $\{x\}\subset A$ and so $G_n(\{x\})\subset G_n(A)$ giving $h_n(x)\subset G_n(A)$.

Thus $U\{h_n(x): x\in A\}\subset G_n(A)$ and $y\notin\overline{U\{h_n(x): x\in A\}}$.

For the converse, suppose A is closed in X.

For each $n\in Z^+$, define $G_n(A)=U\}g_n(x): y\in A\}$

Then $G_1(A)$, $G_2(A),\dots$ is a sepuence of open sets, and $A\subset G_n(A)$.

If $a\in\bigcap_{n=1}^{\infty} G_n(A)$, then for each n, $a\in g_n(x_n)$, $x_n\in A$ and so $x_n\to a$.

Thus, $a\in\overline{A}=A$ implies $\bigcap_{n=1}^{\infty} G_n(A)\subset A$.

Therefore, $A=\bigcap_{n=1}^{\infty} G_n(A)$.

Now, if $y\in\bigcap_{n=1}^{\infty}\overline{G_n(A)}$, then $y\in\overline{U\{g_n(x): x\in A\}}$ for each n, so $y\in\overline{A}=A$.

Hence, $A=\bigcap_{n=1}^{\infty}\overline{G_n(A)}$.

Therefore, $A=\bigcap_{n=1}^{\infty} G_n(A)=\bigcap_{n=1}^{\infty}\overline{G_n(A)}$.

If $A\subset B$ closed and $z\in G_n(A)$ for any fixed n, then there exists an $x\in A$ such that $z\in G_n(x)$, so since $x\in B$; $z\in U\ \{g_n(y): y\in B\}=G_n(B)$.

Hence $G_n(A)\subset G_n(B)$.

Therefore, X is stratifiable.

Theorem 3

Every subspace of a stratifiable space is stratifiable.

Proof : Let X be stratifiable and A a subspace of X. Let B be any closed subset of A. Then there exists a closed set F in X such that $B=F\cap A$.

Since X is stratifiable space, then there exists a sequence $G_1(F)$, $G_2(F),\dots$ of open sets such that

(1) $F=\bigcap_{n=1}^{\infty} G_n(F)=,_{n=1}^{\infty}\overline{G_nF)}$,

(2) if $F\subset K$, then $G_n(F)\subset G_n(K)$ for each n.

For each n, let $G_n(B)=G_n(F)\ \cap A$.

Then $G_1(B)$, $G_2(B)$, ... is a sequence of open sets in A. Now,

$$B=F\cap A=(\bigcap_{n=1}^{\infty} G_n(F))\cap A=\bigcap_{n=1}^{\infty}(G_n(F)\cap A)$$

$$=\bigcap_{n=1}^{\infty} G_n(B).$$

If $x\in\bigcap_{n=1}^{\infty}\overline{G_n(B)}$, then $x\in\overline{G_n(B)}$ and hence for every nbd U of x, $U\cap G(B)=\phi$, $n=1,2,\dots$.

Hence $x\in\overline{B}(A)=B$ and $B=\bigcap_{n=1}^{\infty}\overline{G_n(B)}$

If $B\subset C$ closed in A and for any ffixed $n, x\in G_n(F)\cap A$ and so$x\in G_n(F)$, $x\in A$.

Let $C=K\cap A$ where $K$is closed in X.

Since $B\subset C$ and X is stratifiable, then $F\subset K$ and for each n, $G_n(F)\subset G_n(K)$.

Hence for each n, $G_n(B)B\subset G_n(C)$. Therefore X is stratifiable.

Theorem 4

A countable product of stratifiable spaces is stratifiable.

Proof : Let X_n be stratifiable for each n and $X=\Pi X_n$. For each n,

Let $x_n\in X_n$ and suppose $g_{n1}(x_n)$, $g_{n2}(x_n),\dots$ is an sequence of nbds of x_n.

Let $g_1(x)=g_n(x_1)x\prod_{n\neq 1} X_n$

$\vdots$

$g_n(x)=g_{1n}(x_1)\times xg_{nn}(x_n)\times \prod_{k=n+1} X_k$

$\vdots$

Then $\{x\}=\bigcap_{n=1}^{\infty} g_n(x)$ since $\{x_n\}=\bigcap_{m=1}^{\infty} g_{nm}(x_n)$ for each n,

and if $y\in g_n(x^n)$, $n=1,2,\ldots$, then $x^n\to y$.

Since if for each n, $y_n\in g_{nm}\left(x^n_m\right)$, $m=1,2,\ldots$, then $x^n_m\to y^n$.

Suppose $y\notin A$ and $A\subset\prod X_n$ is closed.

Let $\prod U_n$ be basic open containing y such that $\prod U_n\subset X-A$.

Let I be the the finte subset of N such that if $n\in$I, then $U_n\neq X_n$.

For every $n\in$I, $y_n\notin X_n-U_n$, so

let $m_n\geq n$ such that $y_n\in\overline{U\{g_{nm_n}(x_n):x_n\in X-U_n\}}$.

Let $k=\max\{m_n : n\in I\}$.

Then, for each $n\in I$, $y_n\in U\{g_{nk}(x):x_n\in X-U_n\}$

For each $n\in I$, let $v_n=X_n-\overline{U\{g_{nk}(x):x_n\in X-U_n\}}$.

and for each $n\notin I$, let $v_n=X_n$.

To show $y\notin\overline{U\{g_k(x):x\in A\}}$, let $y\in \overline{U\{g_k(x):x\in A\}}$.

Supose $p\in U\{g_k(x):x\in A\}\cap U_n$ since $U_n$is basic open containing y.

. Let $z\in A$ such that $p\in g_k(z)\cap\prod U_n$.

Suppose $n\in I$.

Then $p_n\in g_{nk}(z_n)\cap U_n$ implies $p_n\in g_{nk}(z_n)\cap (X-\overline{U\{g_{nk}(x_n):x_n\in X-U_n\}})$.

Hence $z_n\notin X-U_n$ and $z_n\in U_n$.

So $z\in\prod U_n$ and$z\in X-A$, a contradiction. Hence, $y\notin\overline{U\{g_k(x):x\in A\}}$.

Therefore, $X=\prod X_n$ is stra ifiable.

Propositions :

1. Every stratifiable is monotonically normal.
2. Every monotonically normal space is collection-wise normal.
3. Every stratifiable space is semi-stratifiable space.
4. Evry semi-stratifiable space is subparacompact.
5. A T_2-space X is paracompact iff X is subparacompact and collectionwisenormal.

Theorem 5

Every stratifiable space is paracompact.

Proof : It follows from propositions 1, 2, 3, 4 and 5, that evey stratifiable space is paracompact.

Definition 2

A space X is said to have the countable chain condition, abbreviated by ccc, iff every collection of pairwise-disjoint open sets is countable.

Theorem 6

The following are equivalent for a stratifiable space X.

(1) X is Lindelöf.

(2) X is separable.

(3) X has ccc.

Proof : (1)⇒(2).

Let X be stratifiable and for each $x\in X$, let $g_n(x)$ be an assigned sequence of open sets cotaining x satisfying Theorem 2, (1), (2) and (3). Then for each n, $O_n=\{g_n(x):x\in X\}$ is an open over of X, so let $g_n(x_{n1}),\ldots$ be a countable subcover of O_n

Let $D=\{X_{nm}: n,\ m\in N\}$.

Suppose $y\in X-D$

For each n, let $m_n\in N$ such that $y\in g_n(x_nm_n)$.

Then$x_nm_n\to y$. $y\in\overline{D}$. Therefore, $X=D$ and X is separable.

(2)⇒(3).

Let $\{U_\alpha:\alpha\in I\}$ be a collection of pairwise-disjoint open sets and D a countable dense set in X. Then each U_α meets D and so for each, choose an x_α such that $x_\alpha\in U_\alpha\cap D$.

Since the U are pairwise disjoint, the x_α are pairwise-distinct.

Let $D_1=\{x_\alpha:\alpha\in I\}$, then $D_1\subset D$ and so D_1 is countable.

Hence I is countable and $\{U_\alpha:\alpha\in I\}$ is countable.

Therefore, X has ccc.

(3)⇒(1).

Suppose X has ccc. Let O be an open cover of X. It follows from Theorem 5 that there exists an nbd-finite open cover V of O.

Let $W=\{W_\alpha : \alpha\in I\}$ be an irreducible subcover of V. If I is countable. For each $\alpha\in I$,

let $p_\alpha\in W_\alpha-\bigcup_{\alpha\neq\beta} W_\beta$. Then $\{p_\alpha : \alpha\in I\}$ has the limit point since X is normal. Furthermore, for $\alpha\neq\beta$, there exists nbds U_α, U_β such that $p_\alpha\in U_\alpha$, $p_\beta\in U_\beta$ and $U_\alpha\cap U_\beta=\phi$.

Hence $\{U_\alpha : \alpha\in I\}$ is a collection of pairwise disjoint open sets and countable. Thus I is countable, a contradiction.

Therefore, X is Lindelöf.

Conclusion

Borges' definition follows from Definition I that a space X is stratifiable iff each open set $U\subset X$, one can assign a sequence U_n, $n=1,2,\ldots$ of open sets such that

(1) for each n, $\bar{U}_n\subset U$

(2) $\bigcup_{n=1}^{\infty} U_n=U$

(3) $U\subset V$ open implies $U_n\subset V_n$.

We have open questions:

(1) Is every stratifiable space metrizable?

(2) Is every stratifiable space M_1?

But we have known that some stratifiable space with some properties is metrizable; that is,

(3) A stratifiable W^Δ-space is metrizabtle. W^Δ-space is metrizable.

(4) A straifiable locally compact space is metrizable.

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〈抄 錄〉

Stratifiable 공간에 관한 연구

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Borges가 일찌기 Stratifiable Space에 대하여 본 논문과 다른 방법으로 이 공간에 대한 여러가지 성질을 증명하였다.

본 논문에서 새로운 정의에 입각하여 이들 공간의 성질을 증명하였고, 반면 미해결의 문제를 제시하면서 끝을 맺었다.