On Some Relations Between a Cauchy Sequence with respect to Convergence in Probability and a Weak Cauchy Sequence

Ik-chan Kim · Bong-soo Ko

確率收斂에 관한 Cauchy數列과 Weak Cauchy數列에 대한 몇가지 關係

金 益 贊 · 高 鳳 秀

Summary

In this papre we shall study additional properties for a Cauchy sequence with respect to convergence in probadility, completeness of the space of all the random variables with respect to convergence in probability, and some relations between the Cauchy sequence and a weak Cauchy sequence.

0. Introduction

Let $\mathfrak{X}$ be the space of all random variables on a probability space (Ω, β, P), where two random variables which are equal almost everywhere are identified. The space $\mathfrak{X}$ is a linear space with respect to the algebraic operations defined by

$$(X+Y)(\omega)=X(\omega)+Y(\omega)$$
$$(aX)(\omega)=aX(\omega)$$

for all real numbers all $\omega\in\Omega$, and every $X, Y\in\mathfrak{X}$

A sequence $\{X_n\}$ in $\mathfrak{X}$ is said to converge in probability if there is a X_o in $\mathfrak{X}$, determined up to a set of probability measure 0, with the following property: For every $\varepsilon>0$ there is an integer N such that

$$P\{\omega\in\Omega : |X_n(\omega)-X_o(\omega)|\geqq\varepsilon\}\leqq\varepsilon \text{ for } n\geqq N.$$

In this case we say that the sequence $\{X_n\}$ converges in probability to X_o, and we write $P\text{-}lim_{n\to\infty} X_n=X_o$. And we say that a sequence $\{X_n\}$ in $\mathfrak{X}$ is a Cauchy sequnce with respect to convergence in probability if for every $\varepsilon>0$ there exists N such that

$$P\{\omega\in\Omega : |X_m(\omega)-X_n(\omega)|\geqq\varepsilon\}<\varepsilon$$
$$\text{for } m, n\geqq N.$$

It is well known that $\mathfrak{X}$ s complete with respect to convergencei in probability [4], [9] in the sense that if a sequence $\{X_n\}$ is a Cauchy sequence with respect to convergence in probability in $\mathfrak{X}$ then there exists a X_o in $\mathfrak{X}$ such that

$$X_o=P\text{-}lim_{n\to\infty} X_n.$$

In section I we will show that two limit random variables of two subsequences of a Cauchy sequence with respect to convergence in probability are identified.

In section II we will prove that in view of the completeness of the space $\mathfrak{X}$ with respect to convergence in probability the space $\mathfrak{X}$ is complete with respect to some metric.

In section III we will investigate some relations between a Cauchy sequence with respect to convergence in probability and a weak Cauchy sequence.

I. A Property for a Cauchy Sequence with respect to Convergence in Proability

Proposition. I−1. If a sequence $\{X_n\}$ is a Cauchy sequence with respect to convergence in probability, and if $\{X_{n_i}\}$ and $\{X_{m_j}\}$ are subsequences of $\{X_n\}$ which converge almost everywhere to the limit random variables X and Y on (Ω, β, P) respectively, then $X=Y$.

Proof. Since $\{X_n\}$ is a Cauchy sequence with respect to convergence in probability, given $\eta>0$, we can choose an integer N such that for every $\varepsilon>0$

$$P\{\omega\epsilon\Omega:\ |X_n(\omega)-X_m(\omega)|\geqslant\frac{\varepsilon}{3}\}<\frac{\eta}{3}$$

for $n,\ m\geqslant N$.

For sufficiently small $\varepsilon>0$ we also can choose integers N_{k1} and N_{k2} greater then N such that

$$P\{\omega\epsilon\Omega:\ |X_{n_i}(\omega)-X(\omega)|\geqslant\frac{\varepsilon}{3}\}<\frac{\eta}{3}\ \text{for}\ n_i\geqslant N_{k1}$$

and

$$P\{\omega\epsilon\Omega:\ |X_{m_j}(\omega)-Y(\omega)|\geqslant\frac{\varepsilon}{3}\}<\frac{\eta}{3}\ \text{for}\ m_j\geqslant N_{k2},$$

since subsequences $\{X_{n_i}\}$ and $\{X_{m_j}\}$ of $\{X_n\}$ converge almost everywhere to X and Y respectively. Let $N^*=\max\{N_{k1},\ N_{k2}\}$. Then we obtain the following result:

$$P\{\omega\epsilon\Omega:\ |X(\omega)-Y(\omega)|\geqslant\varepsilon\}\leqslant P\{\omega\epsilon\Omega:\ |X_{n_i}(\omega)-X(\omega)|\geqslant\frac{\varepsilon}{3}\}+P\{\omega\epsilon\Omega:\ |X_{n_i}(\omega)-X_{m_j}(\omega)|\geqslant\frac{\varepsilon}{3}\}+P\{\omega\epsilon\Omega:\ |X_{m_j}(\omega)-Y(\omega)|\geqslant\frac{\varepsilon}{3}\}\text{for}\ n_i m_j\geqslant N^*$$

this implies

$$P\{\omega\epsilon\Omega:\ |X(\omega)-Y(\omega)|\geqslant\varepsilon\}<\eta\quad\text{for every}\ \varepsilon>0$$

Since η is arbitrary,

$$P\{\omega\epsilon\Omega:\ |X(\omega)-Y(\omega)|\geqslant\varepsilon\}=0\quad\text{for every}\ \varepsilon>0$$

II. Metrization of Convergence in Probability

The real-valued function $\|X\|$ defined on the linear space $\mathfrak{X}$ of all random variables X on a probability space(Ω, β, P) by

$$\|X\|=\int_\Omega \min\{1,\ |X(\omega)|\}dp\quad X\epsilon\mathfrak{X}$$

is a quasi norm on $\mathfrak{X}$, *i.e.*

(i) $0\leqslant\|X\|<\infty$ for every X in $\mathfrak{X}$ and $\|X\|=0$ if and only if $X=0$

(ii) $\|-X\|=\|X\|$ for every X in $\mathfrak{X}$

(iii) $\|X+Y\|\leqslant\|X\|+\|Y\|$ for every $X,\ Y$ in $\mathfrak{X}$

and consequently $d(X,\ Y)=\|X-Y\|(X,\ Y\epsilon\mathfrak{X})$ is a meiric on $\mathfrak{X}$

Theorem. II−2. A sequence $\{X_n\}$ in $\mathfrak{X}$ is a Cauchy sequence with respect to the metric d if and only if it is a Cauchy sequence with respect to convenrice in probaility. For X_o in $\mathfrak{X}$, $lim_{n\to\infty}d(X_n,\ X)=0$ if and only if $P\text{-}lim_{n\to\infty}X_n=X$. Thus in view of the completeness of $\mathfrak{X}$ with respect to convergence in probability the space $\mathfrak{X}$ is complete with respect to the metric d.

Proof. For $X\ \epsilon\ \mathfrak{X}$ and $\eta>0$, let

$$A_\eta=\{\omega\epsilon\Omega:\ |X(\omega)|\geqslant\eta\},$$

(1) $min\ \{1,\ \eta\}\ P(A_\eta)\leqslant\int_{A_\eta}min\{1,\ |X|\}\ dP\leqslant\|X\|$

and

(2) $\|X\| = \int_{A_\eta} \min\{1, |X|\} dP + \int_{A_\eta^c} \min\{1, |X|\} dP \leqslant P(A_\eta) + \min\{1, \eta\}\ P(A_\eta^c)$.

It follows from (1) that

(3) $\min\{1, \eta\}\ P\{\omega\epsilon\Omega : |X_m(\omega) - X_n(\omega)| \geqslant \eta\} \leqslant \|X_m - X_n\|$

so that if $\{X_n\}$ is a Cauchy sequence with respect to the metric d then it is a Cauchy sequence with respect to convergence in probability.

It follows from (2) that

(4) $\|X_m - X_n\| \leqslant P\{\omega\epsilon\Omega : X_m(\omega) - X_n(\omega)| \geqslant \eta\} + \min\{1, \eta\}$

If $\{X_n\}$ is a Caucy sequence with respect to convergence in probability then for every $\varepsilon > 0$ let $\eta > 0$ be such that $\min\{1, \eta\} < \frac{\varepsilon}{2}$ and let N be such that the first term on the right side of (4) is less than $\frac{\varepsilon}{2}$ for $m, n \geqslant N$. Then $\|X_m - X_n\| < \varepsilon$ for $m, n \geqslant N$, so that the sequence is a Cauchy sepuence with respect to the metric d.

Consequently, it follows from (3) and (4) that for X_0 in $\mathcal{X}$, $\lim_{n\to\infty} d(X_n, X_0) = 0$ if and only if $P\text{-}\lim_{n\to\infty} X_n = X_0$ also follows from (3) and (4).

Ⅲ. Some Relations between a Probability Cauchy seauence and a weak Cauchy Sequence

We say that a sequence $\{X_n\}$ in $\mathcal{X}$ is a weak Cauchy sequence if for every $\varepsilon > 0$ there exists an integer N such that

$$|P\{\omega\epsilon\Omega : X_n(\omega)\epsilon F\} - P\{\omega\epsilon\Omega : X_m(\omega)\epsilon F\}| < \varepsilon \quad \text{for } m, n \geqslant N$$

and all the closed subsets F of the real numbers.

Theorem. Ⅲ-3. A sequence $\{X_n\}$ in $\mathcal{X}$ is a weak Cauchy sequence if and only if for every $\varepsilon > 0$ there exists an integer N such that

$$\left| \int_\Omega f(X_n) dP - \int_\Omega f(X_m) dP \right| < \varepsilon \quad \text{for } m, n \geqslant N$$

and all the bounded real-valued uniformly continuous functions f defined on the real numbers.

Proof. We first show the sufficiently condition. Let f be a bounded real-valued uniformly continuous function. If $|f(x)| < M$ for all real numbers x, we construct a partition of a closed interval $[-M, M]$, say

$$-M = t_0 < t_1 < t_2 < \cdots < t_j = M.$$

If $B_i = \{x\epsilon R : t_i \leqslant f(x) \leqslant t_{i+1}\}$ $\quad i = 0, 1, \cdots, j-1$, then B_i is a closed subset of the real numbers, and it follows from the hypothesis that

$$\left\{ \sum_{i=0}^{j-1} t_i P\{\omega\epsilon\Omega : X_n(\omega)\epsilon B_i\} - \sum_{i=0}^{j-1} t_i P\{\omega\epsilon\Omega : X_m(\omega)\epsilon B_i\} \right\} \to 0$$

as $n, m \to \infty$.

Now, we note that

$$(1)\ \left| \int_\Omega f(X_n) dP - \int_\Omega f(X_m) dP \right| \leqslant \left| \int_\Omega f(X_n) dP - \sum_{i=0}^{j-1} t_i P\ \{\omega\epsilon\Omega : X_n(\omega)\epsilon B_i\} \right| + \left| \sum_{i=0}^{j-1} t_i P\ \{\omega\epsilon\Omega : X_n(\omega)\epsilon B_i\} - \sum_{i=0}^{j-1} t_i P\ \{\omega\epsilon\Omega : X_m(\omega)\epsilon B_i\} \right| + \left| \int_\Omega f(X_m) dp - \sum_{i=0}^{j-1} t_i P\ \{\omega\epsilon\Omega : X_m(\omega)\epsilon B_i\} \right|.$$

The first term of the right side of (1) may be written as

$$\left| \sum_{i=0}^{j-1} \int_{\{\omega\in\Omega\,:\,X_n(\omega)\in B_i\}} \{f(X_n)-t_i\}\, dP \right|$$

and this is bounded by $\max_i (t_{i+1}-t_i)$, which can be made arbitrary small by choice of the partition. The third term on the right side of (1) is bounded by $\max_i(t_{i+1}-t_i)$, which can also be made arbitrary small. The second term approaches 0 as n, $m\to\infty$. Hence, this proves the condition.

We second prove the necessary condition. Let F be a closed subset of the real numbers. Suppose $\delta>0$ is given. For sufficiently small $\varepsilon>0$, We can choose a subset $G=\{x\in F : (\sup_{y\in F}|x-y|) < \varepsilon\}$ which satisfies

$$P\{\omega\in\Omega : X_n(\omega)\in G\} < P\{\omega\in\Omega : X_n(\omega)\in F\}+\delta$$

and

$$P\{\omega\in\Omega : X_m(\omega)\in G\} < P\{\omega\in\Omega : X_m(\omega)\in F\}+\delta,$$

since the sets of this form decrease to F as $\varepsilon\downarrow 0$.

Let f be the function defined by

$$f(x)=\varphi(\frac{1}{\varepsilon}\ sup\ \{|x-y| : |x-y|<\varepsilon,\ y\in F\})$$

where

$$\varphi(t)=\begin{cases} 1 & if\ t\leqslant 0 \\ 1-t & if\ 0\leqslant t\leqslant 1 \\ 0 & if\ 1\leqslant t. \end{cases}$$

Then f is a bounded uniformly continuous on the real numbers, $f(x)=1$ on F, $f(x)=0$ on the complement G^c of G and $0\leqslant f(x)\leqslant 1$ for all real numbers. By the hypothesis, we have

(2) $\left| \int_\Omega f(X_n)dP-\int_\Omega f(X_m)dP \right| <\varepsilon$ for n, $m\geqslant N$,

which, together with the relations,

$$P\{\omega\in\Omega : X_n(\omega)\in F\}=\int_{\{\omega\in\Omega\,:\,X_n(\omega)\in F\}} f(X_n)dP\leqslant \int_\Omega f(X_n)dP$$

and

$$\int_\Omega f(X_m)dP=\int_{\{\omega\in\Omega\,:\,X_m(\omega)\in F\}} f(X_m)dp$$
$$\leqslant P\{\omega\in\Omega : X_m(\omega)\in G\} < P\{\epsilon\omega\Omega : X_m(\omega)\in F\}+\delta,$$

or

$$P\{\omega\in\Omega : X_m(\omega)\in F\}\leqslant\int_\Omega f(X_m)dP$$

and

$$\int_\Omega f(X_n)dP\leqslant P\{\omega\in\Omega : X_n(\omega)\in F\}+\delta.$$

we have, then,

$$\left| P\{\omega\in\Omega : X_n(\omega)\in F\} - P\{\omega\in\Omega : X_m(\omega)\in F\} \right|$$
$$\leqslant \left| \int_\Omega f(X_n)dP - \int_\Omega f(X_m)dP \right|$$

From (2) we have

$$\left| |P\{\omega\in\Omega : X_n(\omega)\in F\} - P\{\omega\in\Omega : X_m(\omega)\in F\} \right| <\varepsilon$$

for n, $m\geqslant N$.

Since F is an arbitrary subest of the real numbers, this result completes the proof.

Theorem. III$-$4. If a sequence $\{X_n\}$ is a Cauchy sequence with respect to convergence in probability in $\mathfrak{X}$, then it is a weak Cauchy sequence.

Proof. Suppose $\delta>0$ is given. Let $\{X_n\}$ be a Cauchy sequence with respect to convergence in probability in $\mathfrak{X}$. then

(1) $lim_{n,m\to\infty}P\{\omega\in\Omega : |X_n(\omega)-X_m(\omega)|\geqslant\varepsilon\}=0$ for each $\varepsilon>0$.

For any bounded real-valued uniformly continuous function f defined on the real numbers, there exists $\varepsilon > 0$ such that

(2) $|f(x)-f(y)| < \delta$ whenever $|x-y| < \varepsilon$
for each x, y.
Now, let we consider that

$$(3)\quad \left|\int_\Omega f(X_n)dP-\int_\Omega f(X_m)dP\right| \leqslant \int_{\{\omega\in\Omega\,:\,|X_n(\omega)-X_m(\omega)|\geqslant\varepsilon\}} [f(X_n)-f(X_m)]dP+\int_{\{\omega\in\Omega\,:\,|X_n(\omega)-X_m(\omega)|\leqslant\varepsilon\}} f(X_n)-f(X_m)dP.$$

From (2), we note that the second term of the right side of (3) is less than δ. And from (1) we also note that the first term of the right side of (3) converges to 0. We obtain, therefore, the result that

$$lim_{n,m\to\infty}\left|\int_\Omega f(X_n)dP-\int_\Omega f(X_m)dP\right| < \delta.$$

But δ is arbitrary. Hence

$$lim_{n,m\to\infty}\left|\int_\Omega f(X_n)dP-\int_\Omega f(X_m)dP\right| = 0$$

for all f.

Thus, Theorem Ⅲ−3 shows that the sequence $\{X_n\}$ is a weak Cauchy sequence in $\mathfrak{X}$.

The converse of Theorem Ⅲ−4 is not true.

Example Ⅲ−5. Let $\Omega=\{\omega_0, \omega_1, \cdots, \omega_m, \cdots\}$, β a collection of all the subsets of Ω, and P a measure defined on β such that

$$P(A)=\begin{cases} 1 & \text{if } \omega_0 \in A \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } A\in\beta.$$

Now, let we consider a sequence $\{X_n\}$ of random variables defined on (Ω, β, P) which is defined by, for each n

$$X_n(\omega_i)=\begin{cases} \frac{1}{n} & \text{if } i=0 \\ 0 & \text{otherwise.} \end{cases}$$

We note that the real sequence $\{X_n(\omega_0)\}$ is a Cauchy sequence. And we also note that $\{f(X_n(\omega_0))\}$ is a real Cauchy sequence for every bounded real-valued uniformly continuous functions f defined on the real numbers. Thus

$$lim_{n,m\to\infty}\left\{\int_\Omega f(X_n)dP-\int_\Omega f((X_m)dP\right\}$$
$$=lim_{n,m\to\infty}\left\{f(X_n)(\omega_0)-f(X_m)(\omega_0)\right\}=0,$$

this implies that $\{X_n\}$ is a weak Cauchy sequence in $\mathfrak{X}$. But, given $\varepsilon>0$, we can choose sufficiently large integers N and M such that

$$0<\frac{M-N}{M\cdot N}<\varepsilon.$$

Then

$$\left\{\omega_i\in\Omega : |X_n(\omega_i)-X_m(\omega_i)|\geqslant\varepsilon\right\}\supset$$
$$\left\{\omega_i\in\Omega : |X_n(\omega_i)-X_m(\omega_i)|\geqslant\frac{M-N}{M\cdot N}\right\}\supset\left\{\omega_0\right\},$$

this implies that

$$P\{\omega_i\in\Omega : |X_n(\omega_i)-X_m(\omega_i)\geqslant\varepsilon\}=1.$$

Since ε is arbitrary, the sequence $\{X_n\}$ is not a Cauchy sequence with respect to convergence in probability.

Theorem. Ⅲ−6. If a sequence $\{X_n\}$ is an increasing weak Cauchy sequence (in the sense that if $n\leqslant m$, $X_n(\omega)\leqslant X_m(\omega)$ for $\omega\in\Omega$) in $\mathfrak{X}$ and each X_n is integrable, then $\{X_n\}$ is a Cauchy sequence with respect to convergence in probability.

Proof. Let each X_n be integrable. We note that $|X_n-X_m|$ is integrable for every integers m, n. Apply Chebyshev's inequality to $|X_n-X_m|$. we have

$$(1)\quad P\{\omega\in\Omega : |X_n(\omega)-X_m(\omega)|\geqslant\varepsilon\}\leqslant\frac{1}{\varepsilon}\int_\Omega|X_n-X_m|dp$$

for every $\varepsilon>0$. If $n>m$, the right side of (1) may be written as

$$(2)\quad \frac{1}{\varepsilon}\left[\int_{\Omega} X_n\, dP - \int_{\Omega} X_m\, dP\right]$$

and it converges to 0 as $n\to\infty$, $m\to\infty$, since the sequence $\{X_n\}$ is a weak Cauchy sequence. The consequence follows from (2),

$$\lim_{n,m\to\infty} P\{\omega\in\Omega : |X_n(\omega) - X_m(\omega)| \geqq \varepsilon\} = 0$$
for every $\varepsilon>0$.

This result completes the proof.

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國 文 抄 錄

確率收斂에 관한 Cauchy數列과 Weak Cauchy 數列에 대한 몇가지 關係

本 論文에서는 確率收斂에 관한 Cauchy 數列의 性質, 確率變數들의 空間에 대한 距離化, 完備性 및 確率收斂에 관한 Cauchy 數列과 Weak Cauchy 數列의 關係를 證明하였다.