On ideals in a Pseudo-Bezout domain

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Pseudo-Bezout 整域의 構造에 關한 研究

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Summary

In this paper, we find some characterizations of t-ideal in a pseudo-Bezout domain, and we prove that the structure of pseudo-Bezout domain is the intersection of the localized valuation rings at prime t-ideals.

I. Introduction

Let D be any commutative integral domain and K its field of fractions, then the fractional principal ideals form a partially ordered group G. If G is actually lattice-ordered, D is said to be a pseudo-Bezout domain [3]. This is equivalent to the condition that any two nonzero elements of D have a greatest common divisor. (In this fact, Sheldon refers to such a ring as GCD-domain [5] and Cohn refers to such a ring as HCF-ring [4]).

The purpose of this paper is to focuse upon the t-ideals and investigate the structure of a pseudo-Bezout domain using prime t-ideals. In section II, we have some characterizations of t-dieal in a pseudo-Bezout domain. Section III shows the existence of the smallest t-ideal containing a given ideal, and the structure of a pseudo-Bezout domain with prime t-ideals.

Throughout this paper, the word "domain" will always mean a commutative integral domain with identity. An "ideal" of D cannot be equal to D itself, and a "proper ideal" is a nonzero ideal.

II. Definition and Characterizations

Definition 2.1. Let D be a pseudo-Bezout domain. The ideal of D is a t-ideal if whenever a and b are nonzero elements of I, gcd(a,b) is in I as well. ([5] and [7])

Among the examples of t-ideals of D are all principal ideals of D.

Proposition 2.2. Let D be a domain. Then D is a pseudo-Bezout domain if and only if every finitely generated t-ideal is principal.

Proof. Let I be a finitely generated t-ideal. Let $x_1, x_2, \ldots, x_n$ be generators for I. Then there exists $d = \gcd(x_1, x_2, \ldots, x_n) \in I$. Hence $I = (d)$ is principal. Conversely, let $x, y \in D-\{0\}$ Then the ideal generated by x and y is a principal ideal of D. Hence there exists $d = \gcd(x,y) \in D$. Therefore D is a pseudo-Bezout domain.

Corollary 2.3. If a pseudo-Bezout domain D is Noetherian, then every t-ideal is principal.

Theorem 2.4. Let D be a pseudo-Bezout domain and I be a prime ideal. Then these are equivalent;

(a) I is a t-ideal

(b) D_I is a valuation ring

(c) For any $x,y \in I$, there exists $z \in D\text{-}I$ such that $xz \in (y)$ or $yz \in (x)$.

Proof. (a) implies (b); Assume that I is a t-ideal. Take a nonzero element x/y in the fraction field of D such that $x,y \in D$ and $\gcd(x,y) = 1$. Then either $x \notin I$ or $y \notin I$ since $1 \notin I$. Hence $x/y \in D_I$ or $y/x \in D_I$. Thus D_I is a valuation ring. (b) implies (c); Now suppose that D_I is a valuation ring. Let $x,y \in I$. If $x/y \in D_I$, then $x/y = u/z$ where $u,z \in D$ and $z \notin I$. Hence $xz = yu \in (y)$. Otherwise $y/x \in D_I$, and there exists $z \in D\text{-}I$ such that $yz \in (x)$. (c) implies (a); Let I satisfy (c). If I is not a t-ideal, then there exists some $x,y \in I$ such that $d = \gcd(x,y) \notin I$. Letting $x = ud$ and $y = vd$, we have $u \in I$ and $v \in I$ since I is prime. Hence by (c), there exists $z \in D\text{-}I$ such that $uz \in (v)$ or $vz \in (u)$. Since $\gcd(u,v) = 1$, v must divide z or u must divide z. Then $z \in (v) \subset I$ or $z \in (u) \subset I$. But this is impossible because $z \in D\text{-}I$. Hence I is a t-ideal.

Corollary 2.5. Let D be a pseudo-Bezout domain. Then in D, every prime ideal contained in a prime t-ideal is again a prime t-ideal.

Proof. Let P be a prime t-ideal and Q be a prime ideal contained in P. Then $D_P \subset D_Q \subset K$ (=the fraction field of D) and D_P is a valuation ring by Theorem 2.4. Since D_Q is also a valuation ring by the above inclusion, Q is a prime t-ideal by Theorem 2.4.

III. The Structure of pseudo-Bezout Domain using t-ideals

In this section, let D denote a pseudo-Bezout domain. Let I be an ideal of D. We shall construct the smallest t-ideal which contains I. Note that while D is not a prime t-ideal, D is a t-ideal.

Let $J \subseteq D$. Define

$J' = \{ x \in D\text{: there exist } a,b \in J \text{ such that } x = \gcd(a,b) \}$.

Then $J \subseteq J'$ since $a = \gcd(a,a)$ for any $a \in J$. Let N denote the set of nonnegative integers. Define $I^0 = I$ and for each $n \in N$, $I^{n+1} = (I^n)'$. Let $\bar{I} = \bigcup_{n \in N} I^n$. We shall use the following lemmas to show that $\bar{I}$ is the promised smallest t-ideal of D which contains I.

Lemma 3.1. Let $n \in N$. Then I^n is closed under multiplication with elements in D.

Proof. We prove this by induction. For $n = 0$, $I^0 = I$ is an ideal of D and hence $ID \subset I$. Assume that this lemma holds for $n = k$. Let $x \in I^{k+1}$, $y \in D$. Then there exist $a,b \in I^k$ such that $x = \gcd(a,b)$. By assumption, $ay, by \in I^k$, and $xy = \gcd(ay,by) \in (I^k)' = I^{k+1}$. Hence $I^n D \subset I^n$ for all $n \in N$.

Lemma 3.2. Let $x,y \in I^n$. Then $x\text{-}y \in I^{2n}$.

Proof. We prove this by induction. For $n = 0$, $I^0 = I$ is an ideal of D, and hence $x\text{-}y \in I \subset I' = I^2$. Assume the result for $n = k$. Let $x,y \in I^{k+1}$. Then there exist $a,b,c,d \in I^k$ such that $x = \gcd(a,b)$, $y = \gcd(c,d)$. Putting $\gcd(x,y) = d$, we have x/d, $y/d \in D$. Let $x' = x/d$, $y' = y/d$. Then ax', ay', bx', by', cx', cy', dx', $dy' \in I^k$ by Lemma 3.1. Hence by assumption, $ax'\text{-}ay' = a(x'\text{-}y') \in I^{2k}$. Similarly $b(x'\text{-}y')$, $c(x'\text{-}y')$, $d(x'\text{-}y') \in I^{2k}$. Hence $\gcd(a(x'\text{-}y'), b(x'\text{-}y')) = x(x'\text{-}y') \in I^{2k+1}$, and $\gcd(c(x'\text{-}y'), d(x'\text{-}y')) = y(x'\text{-}y') \in I^{2k+1}$. Thus $\gcd(x(x'\text{-}y'), y(x'-y')) \in I^{2k+2}$. We have $\gcd(x(x'-y'), y(x'-y')) = d(x'\text{-}y') = x\text{-}y \in I^{2(k+1)}$. The lemma follows by induction.

Proposition 3.3. If I is an ideal of D, then $\bar{I}$ is the smallest t-ideal of D which contains I.

Proof. From lemma 3.1 and lemma 3.2, we have that $\bar{I}$ is an ideal. And the construction of $\bar{I}$ guarantees that it is a t-ideal containing I. If H is a t-ideal of D, and J is an ideal of D contained in H, then by definition, $J' \subset H$. Hence if $I \subseteq H$, then $I^n \subseteq H$ for all n, and therefore $\bar{I} \subseteq H$.

Lemma 3.4. Let J be a t-ideal and I be an ideal of D. If there exist $x \in \bar{I}$, $y \in D$ such that $xy \notin J$, then there exists $a_0 \in I$ such that $a_0 y \notin J$.

Proof. Suppose that there exist $x \in \bar{I}$, $y \in D$ such that $xy \notin J$. Then there exists $m \in N$ such that $x = a_m \in I^m$. Now there exist $b_{m-1}, c_{m-1} \in I^{m-1}$ such that $a_m = \gcd(b_{m-1}, c_{m-1})$. If $b_{m-1}y \in J$ and $c_{m-1}y \in J$, then $y \in J$ since J is a t-ideal. Since $xy \notin J$, this is impossible. Hence $b_{m-1}y \notin J$ or $c_{m-1}y \notin J$; that is, there exists $a_{m-1} \in I^{m-1}$ such that $a_{m-1}y \notin J$ (Taking $a_{m-1} = b_{m-1}$ or $a_{m-1} = c_{m-1}$). Again, there exist $b_{m-2}; c_{m-2} \in I_{m-2}$ such that $a_{m-1} = \gcd(b_{m-2}, c_{m-2})$. If $b_{m-2}y, c_{m-2}y \in J$, then $y \in J$ since J is a t-ideal. Since $a_{m-1}y \notin J$, this is impossible. Hence $b_{m-2}y \notin J$ or $c_{m-2}y \notin J$; that is, there exists $a_{m-2} \in I^{m-2}$ such that $a_{m-2}y \notin J$ (Taking $a_{m-2} = b_{m-2}$ or $a_{m-2} = c_{m-2}$). If we proceed this method, then we have that there exists $a_0 \in I^0 = I$ such that $a_0 y \notin J$.

Theorem 3.5. Let S be a multiplicatively closed subset of D. Let P be maximal in the set of t-ideals of D disjoint from S. Then P is a prime t-ideal of D.

Proof. Let I and H be ideals of D which properly contain P Then $\bar{I}$ and $\bar{H}$ meet S. Let $x \in \bar{I} \cap S$ and $y \in \bar{H} \cap S$. Then $xy \notin P$. Hence by lemma 3.4, there exists $a_0 \in I$ such that $a_0 y \notin P$ and again by lemma 3.4, there exists $c_0 \in H$ such that $a_0 c_0 \notin P$. Hence $IH \not\subseteq P$. Therefore P is a prime ideal of D, which completes the proof.

Corollary 3.6. Every proper t-ideal of D is contained in a prime t-ideal of D.

Proof. It is easily checked that the union of a chain of proper t-ideals of D is a proper t-ideal of D. By Zorn's Lemma, there exists a t-ideal P which is maximal in the set of t-ideals of D. Since P is contained in some maximal ideal M of D, P is the t-ideal which is maximal in the set of t-ideals such that $P \cap (D\text{-}M) = \phi$. By Theorem 3.5, P is a prime t-ideal of D.

Theorem 3.7. Let S be a saturated multiplicatively closed subset of D. Then complement of S is a union of prime t-ideals of D.

Proof. Let $x \in D\text{-}S$. Then $(x) \subseteq D\text{-}S$ since S is saturated. Since (x) is a t-ideal of D, then by Zorn's lemma, (x) is contained in a t-ideal P_x which is maximal in the set of t-ideals of D disjoint from S. By Theorem 3.5, P_x is a prime t-ideal of D. Hence $D\text{-}S = \bigcup_{x \in D\text{-}S} P_x$.

Recall that if A and B are ideals of D, then the set $B{:}A = \{x \in D \mid xA \subseteq B\}$ is an ideal of D.

Proposition 3.8. Let Q be a set of prime ideals of D which satisfies the following property; if $a,b \in D$ such that $a \notin (b)$, then there exists $P \in Q$ such that $(b){:}(a) \subseteq P$. Then $\bigcap_{P \in Q} D_P = D$.

Proof. Now $D \subseteq \bigcap_{P \in Q} D_P$ since $D \subseteq D_P$ for each $P \in Q$. Let $a,b \in D$ such that $a/b \in \bigcap_{P \in Q} D_P$. Let $P \in Q$. Then there exist $c,d \in D$ such that $a/b = c/d$ and $d \notin P$. Now $ad \in (b)$, and so $(b){:}(a) \not\subseteq P$ since $d \in (b){:}(a)$. Thus $a \in (b)$ by assumption, and so $a/b \in D$. Hence $D = \bigcap_{P \in Q} D_P$.

Lemma 3.9. Let $a,b \in D$ such that $a \notin (b)$. Then $(b){:}(a)$ is a proper t-ideal of D.

Proof. Since $a \notin (b)$, $(b){:}(a)$ is a proper ideal of D. Let $x,y \in (b){:}(a)$ and d be the gcd of x and y. Then we have $ax, ay \in (b)$ and $ax = bu$, $ay = bv$ for some $u,v \in D$. Putting $x = dx'$, $y = dy'$, we have $dx'av = xav = buv = yau = dy'au$ and $x'v = y'u$. Since $\gcd(x',y') = 1$, and $y'u \in (x'v) \subseteq (x')$, we have $u \in (x')$. Thus it follows that $u = wx'$ for some $w \in D$. From above $ax = bu$, we have $x'da = bwx'$ and hence $da = bw \in (b)$, and so $d \in (b){:}(a)$. Therefore $(b){:}(a)$ is a proper t-ideal of D.

Theorem 3.10. Let $\bar{Q}$ be the set of prime t-ideals of a pseudo-Bezout domain D. Then $D = \bigcap_{P \in \bar{Q}} D_P$.

Proof. By lemma 3.9, corollary 3.6 and proposition 3.8, it holds.

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國 文 抄 錄

Pseudo-Bezout 整域의 構造에 關한 硏究

本 論文에서는 Pseudo-Bezout 整域內에 t-ideal을 導入하여 먼저 그의 特性을 몇가지 찾았고, 다음으로 이 整域의 構造가 素 t-ideal들에서 所屬化된 付値環들의 交集合으로 나타남을 證明하였다.