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GENERAL FORMS OF PRIME MATRICES

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Abstract

In this paper, we consider three questions on Boolean prime matrices which were given by D. Richman and H. Schneider [5]. We study the methods that decide the primeness of a given matrix. Using these methods, we decide whether a given matrix is prime or not.

Keywords: Boolean matrix, monomial matrix, prime matrix, fully indecomposable matrix.

AMS Subject Classifications: 15A12, 15A23.

1 Introduction

There are many papers on the study of binary Boolean matrices([1]-[6]). In the number theory, one of the most important subjects is that of prime number. Moreover the concept of "prime" has been applied in various subjects of mathematics such as matrix theory. Many authors have defined a prime matrix and studied on it (see [1], [2], [5]). In this paper, we use the definition of prime matrix which was defined by D. Richman and H. Schneider [5].

Let $M_n(\Re^+)$ be the semigroup of $n \times n$ nonnegative real matrices.

In section II, we give some definitions containing prime matrix and introduce some basic results and known results.

In Section III, we introduce three questions on prime matrices which were given in [5]. That is,

Question 1) Are there only 3 primes in $M_4(\mathcal{B})$?

Question 2) Does every prime matrix A in $M_n(\Re^+)$ satisfy $(a_i^*)^T a_k^* \leq 1$ for all $1 \leq i, k \leq n, i \neq k$?

Question 3) If A is prime in $M_n(\Re^+)$ if and only if A^* is prime in $M_n(\mathcal{B})$?

Let's consider these three questions. We also obtain some prime matrices.

2 Preliminaries

Definition 2.1. A matrix $A \in M_n(\Re^+)$ is called *monomial* if it has exactly one positive element in each row and column.

For $A \in M_n(\Re^+)$, a_j denotes the jth column of A. By A^*, we denote the $(0,1)$-matrix defined by $a_{ij}^* = 1$ if $a_{ij} > 0$ and $a_{ij}^* = 0$ if $a_{ij} = 0$.

Further we define a_j^* to be the jth column of A^*. The transpose of a_j will be denoted by $(a_j)^T$. We call $A \in M_n(\Re^+)$ itinvertible if A is non-singular and $A^{-1} \in M_n(\Re^+)$.

Lemma 2.1. $A \in M_n(\Re^+)$ is invertible if and only if it is monomial.

Proof). Suppose A is monomial. Then there exists permutation matrices P and Q such that $PAQ = D$, where D is a diagonal matrix with nonzero diagonal entries. Since P and Q are invertible, $A = P^{-1}DQ^{-1}$ and hence $A^{-1} = QD^{-1}P$. Therefore A is invertible.

Conversely, suppose A is invetible. Then there exists $B = (b_{ij})$ such that $AB = I$. If A is not monomial, then there exists some $i\,(1 \le i \le n)$ such that $a_{i,s_1}, a_{i,s_2}, \cdots, a_{i,s_k}$ are nonzero and the others are zero with $2 \le k \le n$. Then

$$(I)_{ij} = \sum_{l=1}^{n} a_{il}b_{lj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Therefore

$$\sum_{l=1}^{n} a_{il}b_{lj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Since a_{i,s_l} are not zero, we have $b_{s_1 j} = \cdots = b_{s_k j} = 0$ for $i \neq j$. Hence ${b_{s_1}}^T \cdots {b_{s_k}}^T$ are lineary dependent, that is, B is not nonsingular, it is a contradiction. Hence A is monomial. ∎

Definition 2.2. A matrix $P \in M_n(\Re^+)$ is *prime* if i) P is not monomial, and ii) $P = BC$, with $B, C \in M_n(\Re^+)$ implies that either B is monomial or C is monomial.

If P is neither prime nor monomial, then P is called *factorizable.*

r, $1 \le r \le n$, and a fully indecomposable prime $P \in M_n(\Re^+)$ such that

$$MAN = P \oplus D,$$

where M, N are permutation matrices in $M_n(\Re^+)$ and D a nonsingular diagonal matrix in $M_{n-r}(\Re^+)$. ■

Theorem 2.4. [5] If P is prime in $M_r(\Re^+)$ and Q is monomial in $M_{n-r}(\Re^+)$, where $1 \le r \le n$, then $P \oplus Q$ is prime in $M_n(\Re^+)$. ■

Theorem 2.5. [5] $A \in M_n(\Re^+)$ is prime if and only if there exists a fully indecomposable prime $P \in M_r(\Re^+)$ such that

$$MAN = P \oplus D,$$

where M, $N$are permutation matrices in $M_n(\Re^+)$ and D is nonsingular diagonal matrix in $M_{n-r}(\Re^+)$. ■

Definition 2.3. We say that A, $A' \in M_n$ are *equivalent* if there are permutation matrices M, N such that $A' = MAN$.

3 Questions on Prime Matrices

Now, let's consider the three questions on prime matrices. We denote the $(0,1)$ Boolean algebra by $\mathcal{B}$.

Question 1) Are there only 3 primes in $M_4(\mathcal{B})$?

Answer) Yes

For $A \in M_n(\Re^+)$, A has *column(or row)-containment* if there exists distinct i and j such that $a_{ki} \neq 0$ implies $a_{kj} \neq 0$ for all $1 \leq k \leq n$.

Theorem 2.1. [5] Let $1 \leq i, k \leq n$ with $i \neq k$ and $A \in M_n(\Re^+)$. If $a_i{}^* \geq a_k{}^*$(entrywise), then A is factorizable.

Similary, if$(a_i{}^*)^T \geq (a_k{}^*)^T$, then A is also factorizable. ■

A matrix $A \in M_n(\Re^+)$ is called *fully indecomposable* if there do not exist permutation matrices M, N such that

$$MAN = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is a square matrix

A matrix A is *completely decomposable* if there exist permutation matrices M, N such that

$$MAN = A_1 \oplus \cdots \oplus A_s$$

, where A_i is fully indecomposable, $i = 1, 2, \cdots, s$ and $s \geq 1$.

Theorem 2.2. [5] Let $n > 1$. If

i) $A \in M_n(\Re^+)$ is fully indecomposable, and

ii) $(a_i{}^*)^T a_k{}^* \leq 1$ for all i, k such that $1 \leq i,\ k \leq n$, and $i \neq k$,

then A is prime. ■

Theorem 2.3. [5] Let $A \in M_n(\Re^+)$ be prime. Then there exist an

Proof) Up to equivalence, the only fully indecomposable matrices without row or column containment are

$$P_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

For P_1 and P_2, we have $(a_i{}^*)^T a_k{}^* \leq 1$ for all distinct i, k. Thus they are prime by Theorem 2.2. But P_3 is not prime. (see Theorem 4.3.)

Up to equivalence, the prime but not fully indecomposable matrices in $M_4(B)$ without row or column containment must have the form of $P \oplus D$, where P is a fully indecomposable prime in $M_r(B)$ and D is nonsingular diagonal matrix in $M_{4-r}(B)$.

Note that when $n \leq 3$ the only prime is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Up to equivalence, we have a prime matrix in $M_4(B)$ such that $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$ by Theorem 2.4. ∎

Lemma 3.1. [1] Let $A \in M_n(\mathcal{B})$ be fully indecomposable. Replace all the ones in A that are in J_2's of A by zeros. If the resulting matrix contains an $n \times n$ permutation matrix, then A is prime. ([1]). ■

Question 2) Does every prime A in $M_n(\Re^+)$ satisfy $(a_i{}^*)^T a_k{}^* \le 1$ for all $1 \le i,\ k \le n,\ i \ne k$?

Answer) No.

For, consider a matrix $A = \left[\begin{array}{cc|c|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$, which is prime by Lemma 3.1.

But $(a_4{}^*)^T a_5{}^* = 2 > 1$. ■

Lemma 3.2. For A, B, $C \in M_n(\Re^+)$ if $A = BC$, then $A^* = B^*C^*$.

Proof) Define

$$(a_{ij})^* = \begin{cases} 1 & \text{if } a_{ij} > 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

Then, since $A = BC$, $(a_{ij})^* = (\sum_{h=1}^n b_{ih}c_{hj})^*$

Case 1) $(a_{ij})^* = 0 \iff (\sum_{h=1}^n b_{ih}c_{hj})^* = 0$

$\iff b_{ih} = 0$ or $c_{hj} = 0$ for all $h = 1, \cdots, n$.

$\iff b_{ih}{}^* = 0$ or $c_{hj}{}^* = 0$ for all $h = 1, \cdots, n$.

$$\Longleftrightarrow \quad \sum_{h=1}^{n} (b_{ih}{}^{*} c_{hj}{}^{*} = 0$$

Case 2) Similarly, $(a_{ij})^{*} = 1$ if and only if $(\sum_{h=1}^{n} b_{ih} c_{hj})^{*} = 1$. Hence $A^{*} = B^{*}C^{*}$. ■

Question 3) If $A \in M_n(\Re^{+})$ is a prime if and only if A^{*} is prime?

Answer) We know that it is a necessary condition as follows:

Let A^{*} be a prime matrix. Suppose A is not prime. Then A is monomial or $A = BC$ with $B, C \in M_n(\Re^{+})$, not monomial.

If A is monomial then A^{*} is also monomial. Then A^{*} is not prime, which is a contradiction. If $A = BC$, then $A^{*} = B^{*}C^{*}$ by Lemma 3.2 and B^{*}, C^{*} are not monomial. Hence A^{*} is not prime, which is also a contradiction.

Hence A is prime. ■

4 General forms of prime matrices in $M_n(\Re^{+})$

Lemma 4.1. Let $A \in M_n(\Re^{+})$. Assume that A is fully indecomposable and $(a_i{}^{*})^{T} a_k{}^{*} \leq 1$, for any distinct i and k, and A has one column with $(n-1)$ positive entries. Then A^{*} is isomorphic to

$$\left[\begin{array}{c|c} & 1 \\ I_{n-1} & \vdots \\ & 1 \\ \hline 1 \ \cdots \ 1 & 0 \end{array}\right]$$

Proof) Since A has one column with $(n-1)$ positive entries and that column does not contain other columns, we can write a matrix $C = \left[\begin{array}{ccc|c} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ \hline 1 & \cdots & 1 & 0 \end{array}\right]$,

which is equivalent to A^*.

And each column of B has at least one 1 since A is fully indecomposable. But if some column B_i of B has two 1's, then $(a_i^*)^T a_n^* = 2$, which is not the case by assumption. Therefore each column of B has only one 1. Since A cannot have row or column containment, B must be equivalent to I_{n-1}. Thus A^* is equivalent to $\left[\begin{array}{ccc|c} & & & 1 \\ & I_{n-1} & & \vdots \\ & & & 1 \\ \hline 1 & \cdots & 1 & 0 \end{array}\right]$. ■

Theorem 4.1. Let $A \in M_n(\Re^+)$. If A^* is isomorphic to $\left[\begin{array}{ccc|c} & & & 1 \\ & I_{n-1} & & \vdots \\ & & & 1 \\ \hline 1 & \cdots & 1 & 0 \end{array}\right]$, then A is prime.

Proof) We know that is fully indecomposable and $(a_i^*)^T a_k^* = 1$, for any distinct i, k $(1 \leq i, k \leq n)$. This condition satisfies the assumption of Theorem 2.2. Hence the given matrix and A are prime. ■

Theorem 4.2. If A^* is isomorphic to $\begin{bmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 1 \end{bmatrix}$, then A is prime.

Proof) Clearly, A is fully indecomposable and $(a_i{}^*)^T a_k{}^* \leq 1$ for any distinct i, k. This condition satisfies the assumption of Theorem 2.2. Hence the given matrix and A are prime. ∎

Theorem 4.3. If $A^*$is isomorphic to $J_n - I_n \quad (n \geq 4)$, then A^* is not prime, where J_n is the $n \times n$ matrix all of whose entries are one.

Proof) We have this factorization:

$J_n - I_n =$

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}_{n \times n}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & 0 & 0 \end{bmatrix}.$$

Therefore for $n \geq 4$, $J_n - I_n$ is not a prime matrix. And hence A^* is not a prime matrix. ■

New questions : 1. Can we prove the sufficient condition on the Question 3)?

2. How many prime matrices are there in $M_n(B)$?

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